On Solutions to the Diophantine Equation $3^x + q^y = z^2$

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Abstract. In this paper, we consider the equation $3^x + q^y = z^2$ in which $q$ is an odd prime, $x, y, z$ are positive integers and $x + y = 2, 3, 4$. When $q > 3$, the cases of infinitely many solutions, of a unique solution and of no-solutions are determined. The case $q = 3$ with particular values $x, y$ is also discussed. Various solutions for $x + y = 2, 3, 4,$ and also for $x + y > 4$ are exhibited. Sroysang [5] raised the Open Problem "Let $q$ be a positive odd prime number. Now, we questions that what is the set of all solutions $(x, y, z)$ for the Diophantine equation $3^x + q^y = z^2$ where $x, y$ and $z$ are non-negative integers." Based on our findings, a set of all solutions for the equation does not exist.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation $p^x + q^y = z^2$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [1, 3, 4, 6].

In this paper, we consider the equation $3^x + q^y = z^2$

in which $q$ is an odd prime and $x, y, z$ are positive integers.

All other values introduced are also positive integers.

Our main objective is to determine solutions to the equation when $q > 3$ and $x + y = 2, 3, 4$. All six possibilities are investigated. It is shown that the equation has infinitely many solutions, a unique solution, and also no-solution cases.

Sroysang [5] investigated the equation $3^x + 17^y = z^2$ and proved it has no solutions in positive integers. He also raised the problem as to what is the set of all
solutions \((x, y, z)\) for the equation \(3^x + q^y = z^2\). Although a formal proof is not given here, the results obtained imply that the answer to his problem is negative.

2. **On solutions to the equation** \(3^x + q^y = z^2\)

In this section, we first determine solutions to \(3^x + q^y = z^2\) when \(q > 3\) and \(x + y = 2, 3, 4\). Several solutions are exhibited. This is done in Theorem 2.1. Secondly, we discuss the case \(q = 3\) with particular values \(x, y\). Finally, we demonstrate some solutions to the equation when \(x + y > 4\).

**Theorem 2.1.** Suppose that \(3^x + q^y = z^2\) where \(q > 3\) is prime, and \(x, y, z\) are positive integers. If \(x, y\) satisfy \(x + y = 2, 3, 4\), then:

(a) The equation \(3^1 + q^1 = z^2\) has infinitely many solutions.
(b) The equation \(3^1 + q^2 = z^2\) has no solutions.
(c) The equation \(3^2 + q^1 = z^2\) has a unique solution.
(d) The equation \(3^1 + q^3 = z^2\) has no solutions when \(3^1 + q^3 \leq 234885116\).
(e) The equation \(3^2 + q^2 = z^2\) has no solutions.
(f) The equation \(3^3 + q^1 = z^2\) has infinitely many solutions.

**Proof:** The six possible equations are considered separately, each of which is self-contained.

**The case** \(x + y = 2\).

For \(x + y = 2\), we have \(x = y = 1\).

(a) \(x = 1\) and \(y = 1\).
We have
\[3^1 + q^1 = z^2.\]
In (1), \(z^2\) is even and denote \(z = 2T\). Since \(z^2 = 4T^2\), therefore \(q = 4N + 1\) where \(N + 1 = T^2\) and \(N = T^2 - 1\). Thus, \(q = 4N + 1 = 4(T^2 - 1) + 1\) and
\[q = 4T^2 - 3\]
\(q\) prime.  \(2\)
When \(T = 3a\), then \(q\) is not prime. Therefore we have in \(2\) that \(T = 3a + 1, T = 3a + 2\).

(i) If \(T = 3a + 1\), then \(4T^2 - 3 = 4(3a + 1)^2 - 3 = 36a^2 + 24a + 1\) provided
\[q = 36a^2 + 24a + 1\]
is prime. \(3\)
(ii) If \(T = 3a + 2\), then \(4T^2 - 3 = 4(3a + 2)^2 - 3 = 36a^2 + 48a + 13\) provided
\[q = 36a^2 + 48a + 13\]
is prime. \(4\)

We now demonstrate some solutions of (1) using (3) and (4).
If (3), then the first two solutions for which \(q\) is prime are:

**Solution 1.** \(3^1 + 61^1 = 8^2\) \(a = 1\),

**Solution 2.** \(3^1 + 193^1 = 14^2\) \(a = 2\).
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If (4), then the first two solutions for which $q$ is prime are:

**Solution 3.** $3^1 + 13^1 = 4^2 \quad a = 0,$

**Solution 4.** $3^1 + 97^1 = 10^2 \quad a = 1.$

There are infinitely many primes of the form $4N + 1$. There are also infinitely many primes $q$ of the form $q$ in (3) as well as of the form $q$ in (4). For our purposes, it certainly suffices that only one of these forms contain infinitely many primes.

The equation $3^1 + q^1 = z^2$ has infinitely many solutions.

**The case $x + y = 3$.**

The case $x + y = 3$, has the two possibilities (b) and (c).

(b). $x = 1$ and $y = 2$.

We have $3^1 + q^2 = z^2$

which yields $3 = z^2 - q^2 = (z - q)(z + q)$. Hence, $z - q = 1$ and $z + q = 3$. Then $z = q + 1$ implying that $2q + 1 = 3$ or $q = 1$ which is impossible.

The equation $3^1 + q^2 = z^2$ has no solutions.

(c). $x = 2$ and $y = 1$.

We obtain $3^2 + q^1 = z^2$

which yields $q = z^2 - 3^2 = (z - 3)(z + 3)$. Thus, $z - 3 = 1$ and $z + 3 = q$. Therefore $z = 4$ and $q = 7$.

The equation $3^2 + q^1 = z^2$ has the unique solution

**Solution 5.** $3^2 + 7^1 = 4^2$.

The case $x + y = 3$ is complete, and consists of exactly one solution.

**The case $x + y = 4$.**

The case $x + y = 4$ consists of three possibilities demonstrated in (d) – (f).

(d). $x = 1$ and $y = 3$.

We have $3^1 + q^3 = z^2, \quad z \text{ is even.}$

The value $z^2$ is even, denoted $z^2 = 4T^2$. If $q = 4N + 3$, then $q^3 = 4M + 3$ and (5) is clearly impossible. Therefore $q = 4N + 1$. Each of the 54 primes $q = 4N + 1$ where $5 \leq q \leq 617$, and up to $3 + 617^3 = 234885116$ have been examined. No solutions to (5) have been found.

It is presumed therefore that $3^1 + q^3 = z^2$ has no solutions.
We have
\[ 3^2 + q^2 = z^2, \quad (6) \]
From (6) \( 3^2 = z^2 - q^2 = (z - q)(z + q) \). Hence, \( z - q = 1, 3, 3^2 \), and then respectively \( z + q = 3^2, 3, 1 \). The last two possibilities are a priori eliminated. Thus, we have \( z - q = 1 \) and \( z + q = 3^2 \). The values \( z - q = 1 \) and \( z + q = 3^2 \) yield \( 2q + 1 = 3^2 \) or \( q = 4 \) which is impossible.

The equation \( 3^2 + q^2 = z^2 \) has no solutions.

(f) \( x = 3 \) and \( y = 1 \).
We have
\[ 3^3 + q^1 = z^2, \quad z \text{ is even.} \quad (7) \]
Since \( z^2 \) is even and \( z^2 = 4T^2 \), it therefore follows that \( q = 4N + 1 \). All such primes \( q \) where \( 5 \leq q \leq 617 \) have been examined.

The first five solutions of (7) are as follows:

Solution 6. \[ 3^3 + 37^1 = 8^2. \]
Solution 7. \[ 3^3 + 73^1 = 10^2. \]
Solution 8. \[ 3^3 + 229^1 = 16^2. \]
Solution 9. \[ 3^3 + 373^1 = 20^2. \]
Solution 10. \[ 3^3 + 457^1 = 22^2. \]

In view of the above solutions, it is presumed that the equation \( 3^3 + q^1 = z^2 \) has infinitely many solutions.

Case (f) is complete, and concludes the proof of Theorem 2.1. □

So far, we have considered primes \( q \) where \( q > 3 \). When \( q = 3 \), an interesting fact stems from the following Lemma 2.1.

Lemma 2.1. Let \( m = 1, 2, \ldots, \) and suppose that \( 3^x + 3^y = z^2 \), where \( x, y \) are consecutive integers.
(i) If \( x = 2m, y = 2m - 1 \), then for all \( m \geq 1 \), \( 3^{2m} + 3^{2m-1} = z^2 \) has no solutions.
(ii) If \( x = 2m + 1, y = 2m \), then for all \( m \geq 1 \), \( 3^{2m+1} + 3^{2m} = z^2 \) has infinitely many solutions.

Proof: (i) Suppose \( 3^{2m} + 3^{2m-1} = z^2 \). Here we shall apply the technique introduced in \([1]\). For all \( m \geq 1 \), any solution of \( 3^{2m} + 3^{2m-1} = z^2 \) implies that \( z^2 \) is even. It is then easily seen that either \( z^2 \) ends in the digit 2 or ends in the digit 8. Since no even square ends either in the digit 2 or ends in the digit 8, it follows that \( 3^{2m} + 3^{2m-1} = z^2 \) has no solutions.

(ii) Suppose that \( 3^{2m+1} + 3^{2m} = z^2 \). Then
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$3^{2m+1} + 3^{2m} = 3^{2m}(3 + 1) = (3^m)^2 \cdot 2^2 = (2 \cdot 3^m)^2 = z^2,$

where $z$ is a positive integer. Thus, for each and every integer $m \geq 1$, the equation $3^{2m+1} + 3^{2m} = z^2$ has a unique solution.

This concludes our proof. \(\square\)

We now demonstrate some solutions of $3^x + q^y = z^2$ in which $x + y > 4$.

**Solution 11.** $3^3 + 13^2 = 14^2 \quad x + y = 5.$

**Solution 12.** $3^4 + 19^1 = 10^2 \quad x + y = 5.$

**Solution 13.** $3^5 + 13^1 = 16^2 \quad x + y = 6.$

**Solution 14.** $3^5 + 157^1 = 20^2 \quad x + y = 6.$

**Solution 15.** $3^7 + 313^1 = 50^2 \quad x + y = 8.$

**Final remark.** Finding all solutions $(x, y, z)$ for the Diophantine equation $3^x + q^y = z^2$ where $x, y, z$ are positive integers is beyond the scope of this paper. Moreover, a set of all solutions to the equation clearly does not exist. However, finding particular solutions, or all the solutions to a given pair of fixed values $x, y$ is possible. This has been done in this paper for all the possibilities of $x + y = 2, 3, 4,$ and for some particular values when $x + y > 4$. We mention that **Solutions 3, 5, 7, 12, 13** were already exhibited in [2].

**REFERENCES**


