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On Solutions to the Diophantine Equation $3^x + q^y = z^2$

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Abstract. In this paper, we consider the equation $3^x + q^y = z^2$ in which q is an odd prime, x, y, z are positive integers and $x + y = 2, 3, 4$. When $q > 3$, the cases of infinitely many solutions, of a unique solution and of no-solutions are determined. The case $q = 3$ with particular values x, y is also discussed. Various solutions for $x + y =$ 2, 3, 4, and also for $x + y > 4$ are exhibited. Sroysang [5] raised the Open Problem "Let q be a positive odd prime number. Now, we questions that what is the set of all solutions (x, y, z) for the Diophantine equation $3^x + q^y = z^2$ where x, y and z are non-negative integers." Based on our findings, a set of all solutions for the equation does not exist.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [1, 3, 4, 6].

In this paper, we consider the equation

$$3^x + q^y = z^2$$

in which q is an odd prime and x, y, z are positive integers.
All other values introduced are also positive integers.

Our main objective is to determine solutions to the equation when $q > 3$ and $x + y =$ 2, 3, 4. All six possibilities are investigated. It is shown that the equation has infinitely many solutions, a unique solution, and also no-solution cases.

Sroysang [5] investigated the equation $3^x + 17^y = z^2$ and proved it has no solutions in positive integers. He also raised the problem as to what is the set of all solutions (x, y, z) for the equation $3^x + q^y = z^2$. Although a formal proof is not given here, the results obtained imply that the answer to his problem is negative.

2. On solutions to the equation $3^x + q^y = z^2$

In this section, we first determine solutions to $3^x + q^y = z^2$ when $q > 3$ and $x + y = 2, 3, 4$. Several solutions are exhibited. This is done in **Theorem 2.1**. Secondly, we discuss the case $q = 3$ with particular values x, y. Finally, we demonstrate some solutions to the equation when $x + y > 4$.

Theorem 2.1. Suppose that $3^x + q^y = z^2$ where $q > 3$ is prime, and x, y, z are positive integers. If x, y satisfy $x + y = 2, 3, 4$, then:

(a) The equation $3^1 + q^1 = z^2$ has infinitely many solutions.
(b) The equation $3^1 + q^2 = z^2$ has no solutions.
(c) The equation $3^2 + q^1 = z^2$ has a unique solution.
(d) The equation $3^1 + q^3 = z^2$ has no solutions when $3^1 + q^3 \leq 234885116$.
(e) The equation $3^2 + q^2 = z^2$ has no solutions.
(f) The equation $3^3 + q^1 = z^2$ has infinitely many solutions.

Proof: The six possible equations are considered separately, each of which is self-contained.

The case $x + y = 2$.

For $x + y = 2$, we have $x = y = 1$.

(a). $x = 1$ and $y = 1$.
We have

$$3^1 + q^1 = z^2. \qquad (1)$$

In (1), z^2 is even and denote $z = 2T$. Since $z^2 = 4T^2$, therefore $q = 4N + 1$ where $N + 1 = T^2$ and $N = T^2 - 1$. Thus, $q = 4N + 1 = 4(T^2 - 1) + 1$ and

$$q = 4T^2 - 3 \qquad q \text{ prime.} \qquad (2)$$

When $T = 3a$, then q is not prime. Therefore we have in (2) that $T = 3a + 1$, $T = 3a + 2$.

(i) If $T = 3a+ 1$, then $4T^2 - 3 = 4(3a + 1)^2 - 3 = 36a^2 + 24a + 1$ provided

$$q = 36a^2 + 24a + 1 \qquad \text{is prime.} \qquad (3)$$

(ii) If $T = 3a+ 2$, then $4T^2 - 3 = 4(3a + 2)^2 - 3 = 36a^2 + 48a + 13$ provided

$$q = 36a^2 + 48a + 13 \qquad \text{is prime.} \qquad (4)$$

We now demonstrate some solutions of (1) using (3) and (4).
If (3), then the first two solutions for which q is prime are:

Solution 1. $3^1 + 61^1 = 8^2$ $\quad a = 1$,

Solution 2. $3^1 + 193^1 = 14^2$ $\quad a = 2$.

If (4), then the first two solutions for which q is prime are:

Solution 3. $3^1 + 13^1 = 4^2$ $a = 0$,

Solution 4. $3^1 + 97^1 = 10^2$ $a = 1$.

There are infinitely many primes of the form $4N + 1$. There are also infinitely many primes q of the form q in (3) as well as of the form q in (4). For our purposes, it certainly suffices that only one of these forms contain infinitely many primes.
The equation $3^1 + q^1 = z^2$ has infinitely many solutions.

The case $x + y = 3$.

The case $x + y = 3$, has the two possibilities **(b)** and **(c)**.

(b). $x = 1$ and $y = 2$.
We have

$$3^1 + q^2 = z^2$$

which yields $3 = z^2 – q^2 = (z – q)(z + q)$. Hence, $z – q = 1$ and $z + q = 3$. Then $z = q + 1$ implying that $2q + 1 = 3$ or $q = 1$ which is impossible.

The equation $3^1 + q^2 = z^2$ has no solutions.

(c). $x = 2$ and $y = 1$.
We obtain

$$3^2 + q^1 = z^2$$

which yields $q = z^2 – 3^2 = (z – 3)(z + 3)$.Thus, $z– 3 = 1$ and $z + 3 = q$. Therefore $z = 4$ and $q = 7$.

The equation $3^2 + q^1 = z^2$ has the unique solution

Solution 5. $3^2 + 7^1 = 4^2$.

The case $x + y = 3$ is complete, and consists of exactly one solution.

The case $x + y = 4$.

The case $x + y = 4$ consists of three possibilities demonstrated in **(d) – (f)**.

(d). $x = 1$ and $y = 3$.
We have

$$3^1 + q^3 = z^2, \qquad z \text{ is even.} \tag{5}$$

The value z^2 is even, denoted $z^2 = 4T^2$. If $q = 4N + 3$, then $q^3 = 4M + 3$ and (5) is clearly impossible. Therefore $q = 4N + 1$. Each of the 54 primes $q = 4N + 1$ where $5 \le q \le 617$, and up to $3 + 617^3 = 234885116$ have been examined. No solutions to (5) have been found.

It is presumed therefore that $3^1 + q^3 = z^2$ has no solutions.

(e). $x = 2$ and $y = 2$.
We have

$$3^2 + q^2 = z^2. \tag{6}$$

From (6) $3^2 = z^2 - q^2 = (z - q)(z + q)$. Hence, $z - q = 1,\ 3,\ 3^2$, and then respectively $z + q = 3^2,\ 3,\ 1$. The last two possibilities are a priori eliminated. Thus, we have $z - q = 1$ and $z + q = 3^2$. The values $z - q = 1$ and $z + q = 3^2$ yield $2q + 1 = 3^2$ or $q = 4$ which is impossible.

The equation $3^2 + q^2 = z^2$ has no solutions.

(f). $x = 3$ and $y = 1$.
We have

$$3^3 + q^1 = z^2, \qquad z \text{ is even.} \tag{7}$$

Since z^2 is even and $z^2 = 4T^2$, it therefore follows that $q = 4N + 1$. All such primes q where $5 \le q \le 617$ have been examined.

The first five solutions of (7) are as follows:

Solution 6. $3^3 + 37^1 = 8^2$.

Solution 7. $3^3 + 73^1 = 10^2$.

Solution 8. $3^3 + 229^1 = 16^2$.

Solution 9. $3^3 + 373^1 = 20^2$.

Solution 10. $3^3 + 457^1 = 22^2$.

In view of the above solutions, it is presumed that the equation $3^3 + q^1 = z^2$ has infinitely many solutions.

Case **(f)** is complete, and concludes the proof of **Theorem 2.1**. □

So far, we have considered primes q where $q > 3$. When $q = 3$, an interesting fact stems from the following **Lemma 2.1**.

Lemma 2.1. Let $m = 1, 2, \ldots$, and suppose that $3^x + 3^y = z^2$, where x, y are consecutive integers.
(i) If $x = 2m$, $y = 2m - 1$, then for all $m \ge 1$, $3^{2m} + 3^{2m-1} = z^2$ has no solutions.
(ii) If $x = 2m + 1$, $y = 2m$, then for all $m \ge 1$, $3^{2m+1} + 3^{2m} = z^2$ has infinitely many solutions.

Proof: (i) Suppose $3^{2m} + 3^{2m-1} = z^2$. Here we shall apply the technique introduced in [1]. For all $m \ge 1$, any solution of $3^{2m} + 3^{2m-1} = z^2$ implies that z^2 is even. It is then easily seen that either z^2 ends in the digit 2 or ends in the digit 8. Since no even square ends either in the digit 2 or ends in the digit 8, it follows that $3^{2m} + 3^{2m-1} = z^2$ has no solutions.

(ii) Suppose that $3^{2m+1} + 3^{2m} = z^2$. Then

$$3^{2m+1} + 3^{2m} = 3^{2m}(3 + 1) = (3^m)^2 \cdot 2^2 = (2 \cdot 3^m)^2 = z^2,$$

where z is a positive integer. Thus, for each and every integer $m \geq 1$, the equation $3^{2m+1} + 3^{2m} = z^2$ has a unique solution.

This concludes our proof. □

We now demonstrate some solutions of $3^x + q^y = z^2$ in which $x + y > 4$.

Solution 11. $3^3 + 13^2 = 14^2$ $x + y = 5$.

Solution 12. $3^4 + 19^1 = 10^2$ $x + y = 5$.

Solution 13. $3^5 + 13^1 = 16^2$ $x + y = 6$.

Solution 14. $3^5 + 157^1 = 20^2$ $x + y = 6$.

Solution 15. $3^7 + 313^1 = 50^2$ $x + y = 8$.

Final remark. Finding all solutions (x, y, z) for the Diophantine equation $3^x + q^y = z^2$ where x, y, z are positive integers is beyond the scope of this paper. Moreover, a set of all solutions to the equation clearly does not exist. However, finding particular solutions, or all the solutions to a given pair of fixed values x, y is possible. This has been done in this paper for all the possibilities of $x + y = 2, 3, 4$, and for some particular values when $x + y > 4$. We mention that **Solutions 3**, **5**, **7**, **12**, **13** were already exhibited in [2].

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