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On Solutions to the Diophantine Equation $7^x + 10^y = z^2$ when x, y, z are Positive Integers

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Abstract. We establish that the equation $7^x + 10^y = z^2$ has no solutions in positive integers x, y, z.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [1, 2, 6, 8].

Articles of various authors have been written on the equation $p^x + (p + A)^y = z^2$ where $A = 4, 6, 8, 10$, and p, $q = p + A$ are primes. For instance [2, 3, 4, 5]. In this paper, we investigate the equation $p^x + (p + A)^y = z^2$ where p is prime and A is odd, namely $7^x + 10^y = z^2$.

The values x, y, z are positive integers.

2. On the equation $7^x + 10^y = z^2$

In the following theorem it is shown that $7^x + 10^y = z^2$ has no solutions.

Theorem 2.1. The equation

$$7^x + 10^y = z^2 \qquad (1)$$

has no solutions in positive integers x, y, z.

Proof: We shall assume that (1) has solutions with positive integers x, y, z, and reach a contradiction.

By our assumption z is always odd. The last digit of 7^x is one of 1, 3, 7, 9, whereas the last digit of 10^y is 0.

Suppose that $x = 2t + 1$ where t is an integer. It is then easily seen that
7^1 ends in 7, 7^3 ends in 3, 7^5 ends in 7, 7^7 ends in 3, and so on.
Thus, for all values y, $7^{2t+1} + 10^y = z^2$ ends either in the digit 3 or in the digit 7. Since no odd value z^2 ends in the digit 3 or in the digit 7, it follows that $x \neq 2t + 1$. Therefore, by our assumption x must be even. For y, we shall consider two cases, namely y even and y odd. The following values m, n are integers.

Suppose that $x = 2m$ is even, $y = 2n$ is even.
From (1) we obtain that $7^{2m} + 10^{2n} = z^2$ implying

$$7^{2m} = z^2 - 10^{2n} = (z - 10^n)(z + 10^n).$$

Denote

$$z - 10^n = 7^A, \qquad z + 10^n = 7^B, \qquad A < B, \qquad A + B = 2m,$$

where A, B are integers. Then $7^B - 7^A$ yields

$$2 \cdot 10^n = 7^A(7^{B-A} - 1). \tag{2}$$

The factor 7^A divides the right side of (2). If $A > 0$ then $7^A \nmid 2{\cdot}10^n$. Therefore $A = 0$ in (2), and hence $B = 2m$. This then implies

$$2{\cdot}10^n = 7^B - 1 = 7^{2m} - 1 = (7^m - 1)(7^m + 1). \tag{3}$$

It is easily seen for all values $m = 1, 2, 3, \ldots$, that $3 \mid (7^m - 1)$. Hence, the right side of (3) is a multiple of 3, whereas the left side of (3) $2{\cdot}10^n$ is not. Therefore, (3) is impossible and $y \neq 2n$.

Suppose that $x = 2m$ is even, and $y = 2n + 1$ is odd.
Then from (1) we have $7^{2m} + 10^{2n+1} = z^2$ or

$$10^{2n+1} = z^2 - 7^{2m} = z^2 - (7^m)^2 = (z - 7^m)(z + 7^m).$$

Denote

$$z - 7^m = 10^C, \qquad z + 7^m = 10^D, \qquad C < D, \qquad C + D = 2n + 1,$$

where C, D are integers. Then $10^D - 10^C$ results in

$$2{\cdot}7^m = 10^C(10^{D-C} - 1). \tag{4}$$

Since both values 7^m and $z = 7^m + 10^C$ are always odd, it follows that $C \neq 0$, and hence $C > 0$. This implies therefore that the right side of (4) is a multiple of 5, whereas the left side (4) $2{\cdot}7^m$ is not. Thus (4) is impossible, and $y \neq 2n + 1$.

We have shown that no value y satisfies the equation $7^x + 10^y = z^2$. Our assumption that (1) has solutions is therefore false.

The equation $7^x + 10^y = z^2$ has no solutions as asserted. □

3. Conclusion

It is observed that the equation $p^x + (p + 3)^y = z^2$ has solutions for various primes p when $x = y = 1$. The first five such solutions are:
$3^1 + 6^1 = 3^2$, $11^1 + 14^1 = 5^2$, $23^1 + 26^1 = 7^2$, $59^1 + 62^1 = 11^2$, $83^1 + 86^1 = 13^2$.

Two questions may now be raised.

Question 1. Are there infinitely many solutions of $p^x + (p + 3)^y = z^2$ in which p is an odd prime, and $x = y = 1$?

We presume that the answer is affirmative.

Question 2. Are there solutions of $p^x + (p + 3)^y = z^2$ in which $p > 3$ is prime, and at least one of x, y is larger than 1 ?

When $p = 3$, we have the solution $3^2 + (3 + 3)^3 = 15^2$, and when $p = 2$, we have the solution $2^2 + (2 + 3)^1 = 3^2$.

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