Annals of Pure and Applied Mathematics
Vol. 20, No.2, 2019, 79-83
ISSN: 2279-087X (P), 2279-0888(online)
Published on 27 December 2019
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.641v20n2a6

Annals of
Pure and Applied Mathematics

On Solutions of the Diophantine Equation $8^x + 9^y = z^2$ when x, y, z are Positive Integers

Nechemia Burshtein

117 Arlozorov Street, Tel – Aviv 6209814, Israel
Email: anb17@netvision.net.il

Received 2 December 2019; accepted 26 December 2019

Abstract. In this article, we establish in a very elementary manner that the equation $8^x + 9^y = z^2$ has no solutions when x, y, z are positive integers. Our results are achieved in particular by utilizing the last digits of the powers 8^x, 9^y.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [5, 6, 7].

In this article, we consider the equation $8^x + 9^y = z^2$ in which x, y, z are positive integers. In the determination of solutions to the equation, our main objective is simplicity of the process. We employ basic and very simple facts, and in particular our new method which uses the last digits of the powers involved. In [1, 2, 3], this method has already been applied to different equations of the form $A^x + B^y = z^2$ in which A, B are both primes, or at least one of A, B is prime. The results of this article are indeed achieved by these elementary methods.

2. The solutions of $8^x + 9^y = z^2$

We observe that for all values x, the power 8^x has a last digit which equals one of the values 2, 4, 6, 8, whereas for all values y, the power 9^y has a last digit which equals 1 or 9. This yields a total of 8 possibilities for the odd sum $8^x + 9^y$. All 8 cases, each of which makes use of the last digits of 8^x and 9^y are contained in the following Table 1.

Table 1.

case	last digit of 8^x	last digit of 9^y	last digit of $8^x + 9^y$	$8^x + 9^y = z^2$
1	2	1	3	no solutions
2	2	9	1	to be determined
3	4	1	5	to be determined
4	4	9	3	no solutions
5	6	1	7	no solutions
6	6	9	5	to be determined
7	8	1	9	to be determined
8	8	9	7	no solutions

If indeed the equation $8^x + 9^y = z^2$ has a solution, then for all values x, y the value z^2 is odd. An odd square z^2 does not have a last digit which is equal to 3 or to 7. Therefore it follows that the equation $8^x + 9^y = z^2$ has no solutions in cases 1, 4, 5, 8 as indicated in Table 1. The remaining cases 2, 3, 6, 7 will be investigated for solutions, discussed in the respective Theorems 2.1, 2.2, 2.3, 2.4 each of which is self-contained.

In the following Theorem 2.1, we consider case 2, in which the last digit of $8^x + 9^y$ equals 1.

Theorem 2.1. If $8^x + 9^y$ ends in 1, then the equation $8^x + 9^y = z^2$ has no solutions.

Proof: We shall assume that $8^x + 9^y = z^2$ has a solution and reach a contradiction.

In this case, 8^x ends in 2. Therefore $x = 3 + 4m$ and $m \geq 0$. The power 9^y ends in 9 when $y = 2n + 1$ and $n \geq 0$. The values m, n are integers.

When $8^x + 9^y = z^2$, then $2^{3x} + 3^{2y} = z^2$, and

$$2^{3x} = z^2 - 3^{2y} = z^2 - (3^y)^2 = (z - 3^y)(z + 3^y).$$

Denote

$$z - 3^y = 2^A, \qquad z + 3^y = 2^B, \qquad A < B, \qquad A + B = 3x = 3(3 + 4m),$$

where A, B are integers. Then $2^B - 2^A$ yields

$$2 \cdot 3^y = 2^A (2^{B-A} - 1). \tag{1}$$

From (1) it follows that $A = 1$, and hence $3^y = 2^{B-1} - 1$ or $3^y = 2^{7+12m} - 1$. It is a well known simple fact that for all $a \geq 0$, the value $(2^{2a+1} - 1)$ is not divisible by 3. Since $2^{7+12m} - 1$ is of this form, it follows that $3^y \neq 2^{7+12m} - 1$. This contradiction implies that our assumption is false, and when $8^x + 9^y$ ends in 1, the equation $8^x + 9^y = z^2$ has no solutions as asserted.

This concludes the proof of Theorem 2.1. □

We now consider case 3 in which $8^x + 9^y$ ends in 5.

Theorem 2.2. If 8^x ends in 4 and 9^y ends in 1, then the equation $8^x + 9^y = z^2$ has no solutions.

Proof: We shall assume that $8^x + 9^y = z^2$ has a solution and reach a contradiction.

The hypothesis yields that $x = 2 + 4m$ $(m \geq 0)$, and $y = 2n$ $(n \geq 1)$, where m, n are integers.

When $8^x + 9^y = z^2$, then $8^{2+4m} + 9^{2n} = z^2$, and

$$8^{2+4m} = z^2 - 9^{2n} = z^2 - (9^n)^2 = (z - 9^n)(z + 9^n).$$

Denote

$$z - 9^n = 8^C, \qquad z + 9^n = 8^D, \qquad C < D, \qquad C + D = 2 + 4m,$$

where C, D are integers. The above values result in

$$2z = 8^C + 8^D = 8^C(8^{D-C} + 1). \tag{2}$$

If $C = 0$ in (2), then $2z \neq (8^D + 1)$. Hence $C > 0$. In (2) the value $2z$ is a multiple of 2 only since z is odd, whereas the right side of (2) is a multiple of 8^C. Since this is impossible, therefore $2z \neq 8^C + 8^D$. This contradiction implies that our assumption is false. When 8^x ends in 4 and 9^y ends in 1, the equation $8^x + 9^y = z^2$ has no solutions.

The proof of Theorem 2.2 is complete. □

Case 6 is contained in the following theorem.

Theorem 2.3. If 8^x ends in 6 and 9^y ends in 9, then the equation $8^x + 9^y = z^2$ has no solutions.

Proof: We shall assume that $8^x + 9^y = z^2$ has a solution and reach a contradiction.

The hypothesis yields that $x = 4m$ $(m \geq 1)$, and $y = 2n + 1$ $(n \geq 0)$, where m, n are integers.

When $8^x + 9^y = z^2$, then $8^{4m} + 9^{2n+1} = z^2$, and

$$8^{4m} = z^2 - 9^{2n+1} = z^2 - (3^2)^{2n+1} = z^2 - (3^{2n+1})^2 = (z - 3^{2n+1})(z + 3^{2n+1}).$$

Denote

$$z - 3^{2n+1} = 8^G, \qquad z + 3^{2n+1} = 8^H, \qquad G < H, \qquad G + H = 4m,$$

where G, H are integers. The above values imply

$$2z = 8^G + 8^H = 8^G(8^{H-G} + 1). \tag{3}$$

If $G = 0$ in (3), then $2z \neq (8^H + 1)$. Thus $G > 0$. In (3) the value $2z$ is a multiple of 2 only since z is odd, whereas the right side of (3) is a multiple of 8^G. But this is impossible, and hence $2z \neq 8^G + 8^H$. This contradiction implies that our assumption is false. Therefore, when 8^x ends in 6 and 9^y ends in 9, the equation $8^x + 9^y = z^2$ has no solutions.

This concludes the proof of Theorem 2.3. □

Case 7 is considered in the following theorem.

Theorem 2.4. If $8^x + 9^y$ ends in 9, then the equation $8^x + 9^y = z^2$ has no solutions.

Proof: We shall assume that $8^x + 9^y = z^2$ has a solution and reach a contradiction.

The last digit of 8^x is 8, and the last digit of 9^y is 1. Hence $x = 1 + 4m$ $(m \geq 0)$, $y = 2n$ $(n \geq 1)$, and m, n are integers.

When $8^x + 9^y = z^2$, then $2^{3x} + 3^{2y} = z^2$, and

$$2^{3x} = z^2 - 3^{2y} = z^2 - (3^y)^2 = (z - 3^y)(z + 3^y).$$

Denote

$$z - 3^y = 2^K, \qquad z + 3^y = 2^L, \qquad K < L, \qquad K + L = 3x = 3\,(1 + 4m),$$

where K, L are integers. Then $2^L - 2^K$ yields

$$2 \cdot 3^y = 2^K(2^{L-K} - 1). \tag{4}$$

From (4) it follows that $K = 1$, and therefore $3^y = 2^{L-1} - 1$ or $3^y = 2^{12m+1} - 1$. It is already known for all values $m \geq 0$ that $(2^{12m+1} - 1)$ is not divisible by 3. Hence $3^y \neq 2^{12m+1} - 1$. This contradiction implies that our assumption is false.
When $8^x + 9^y$ ends in 9, the equation $8^x + 9^y = z^2$ has no solutions.

The proof of Theorem 2.4 is complete. □

Final Remark. In the four Theorems 2.1 – 2.4 related to the four "undetermined" cases 2, 3, 6, 7 in Table 1, we have shown that these "undetermined" cases have been determined as "no solutions" cases of the equation $8^x + 9^y = z^2$. Together with cases 1, 4, 5, 8 in Table 1, it has now been established that the equation $8^x + 9^y = z^2$ has no solutions in positive integers x, y, z as mentioned earlier.

The results of this paper were indeed achieved by using basic and elementary arguments. It seems that the technique using last digits of the powers in the equation is quite useful, and may be applied in the process of finding solutions to equations.

REFERENCES

1. N.Burshtein, On solutions to the diophantine equation $7^x + 10^y = z^2$ when x, y, z are positive integers, *Annals of Pure and Applied Mathematics*, 20 (2) (2019) 75-77.
2. N.Burshtein, On solutions of the diophantine equation $11^x + 23^y = z^2$ with consecutive positive integers x, y, *Annals of Pure and Applied Mathematics*, 20 (2) (2019) 57-61.
3. N.Burshtein, A short note on solutions of the diophantine equations $6^x + 11^y = z^2$ and $6^x - 11^y = z^2$ in positive integers x, y, z, *Annals of Pure and Applied Mathematics*, 19 (2) (2019) 55-56.
4. N.Burshtein, All the solutions of the diophantine equations $(p + 1)^x - p^y = z^2$ and $p^y - (p + 1)^x = z^2$ when p is prime and $x + y = 2, 3, 4$, *Annals of Pure and Applied Mathematics*, 19 (1) (2019) 53-57.
5. N.Burshtein, A note on the diophantine equation $p^x + (p + 1)^y = z^2$, *Annals of Pure and Applied Mathematics*, 19 (1) (2019) 19-20.
6. S.Chotchaisthit, On the diophantine equation $4^x + p^y = z^2$ where p is a prime number, *Amer. J. Math. Sci.*, 1 (1) (2012) 191-193.

7. Md. A.-A.Khan, A.Rashid and Md.S.Uddin, Non-negative integer solutions of two diophantine equations $2^x + 9^y = z^2$ and $5^x + 9^y = z^2$, *Journal of Applied Mathematics and Physics,* 4 (2016) 762-765.
8. B.Poonen, Some Diophantine equations of the form $x^n + y^n = z^m$, *Acta Arith.*, 86 (1998) 193-205.