Annals of Pure and Applied Mathematics
Vol. 22, No. 2, 2020, 119-123
ISSN: 2279-087X (P), 2279-0888(Online)
Published on 1 December 2020
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v22n2a07799

Annals of Pure and Applied Mathematics

The Diophantine Equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ are Insolvable in Positive Integers x, y, z

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Received 4 November 2020; accepted 25 November 2020

Abstract. In this article, the author has investigated the equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ with positive integers x, y, z. It was established that both equations have no solutions.

Keywords: Diophantine equations

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [2, 4, 6, 8].

In this article, we consider the two equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ in which x, y, z are positive integers. It will be shown that both equations have no solutions. This is done in Sections 2 and 3. Although similarities exist, nevertheless, the theorems and all the cases within are self-contained. The results achieved are mainly and in particular based on our new method which utilizes the last digits of the powers involved.

2. The equation $2^x + 11^y = z^2$

Theorem 2.1. Let x, y, z be positive integers. Then the equation $2^x + 11^y = z^2$ has no solutions.

Proof: Let $m \geq 0$ be an integer. For all values $x \geq 1$, four possibilities exist:

(a) $x = 4m + 1$, $y \geq 1$, $m \geq 0$.
(b) $x = 4m + 2$, $y \geq 1$, $m \geq 0$.
(c) $x = 4m + 3$, $y \geq 1$, $m \geq 0$.
(d) $x = 4m$, $y \geq 1$, $m \geq 1$.

(a) Suppose that $x = 4m + 1$, $y \geq 1$.
For all values $m \geq 0$, the power 2^{4m+1} ends in the digit 2. For all values $y \geq 1$, the power 11^y ends in the digit 1. Hence $2^{4m+1} + 11^y$ ends in the digit 3. If for some value z, $2^{4m+1} + 11^y = z^2$, then z^2 is odd and ends in the digit 3. But, an odd square does not have a last digit which is equal to 3. Therefore it follows that $2^{4m+1} + 11^y \neq z^2$.
Case **(a)** is complete.

(b) Suppose that $x = 4m + 2$, $y \geq 1$.
We shall assume that $2^{4m+2} + 11^y = z^2$ has a solution, and reach a contradiction.

By our assumption, we have $2^{4m+2} + 11^y = z^2$ implying that $11^y = z^2 - 2^{4m+2} = z^2 - 2^{2(2m+1)}$ or

$$11^y = (z - 2^{2m+1})(z + 2^{2m+1}).$$

Denote

$$z - 2^{2m+1} = 11^A, \qquad z + 2^{2m+1} = 11^B, \qquad A < B, \qquad A + B = y,$$

where A, B are non-negative integers. Then $11^B - 11^A$ yields

$$2 \cdot 2^{2m+1} = 11^A(11^{B-A} - 1). \tag{1}$$

If $A > 0$, then $11^A \nmid 2 \cdot 2^{2m+1}$ in (1). Therefore $A \ngtr 0$, and $A = 0$. When $A = 0$, then $B = y$, and (1) results in

$$2^{2m+2} = 11^y - 1. \tag{2}$$

Since for all values y, the power 11^y ends in the digit 1, therefore in (2) the value $11^y - 1$ ends in the digit 0. This implies that $11^y - 1$ is a product of 5. But $5 \nmid 2^{2m+2}$, and hence (2) is impossible. This contradicts our assumption that when $x = 4m + 2$ the equation has a solution, and hence $2^{4m+2} + 11^y \neq z^2$.
This concludes case **(b)**.

(c) Suppose that $x = 4m + 3$, $y \geq 1$.
We shall assume that $2^{4m+3} + 11^y = z^2$ has a solution, and reach a contradiction.

The sum $2^{4m+3} + 11^y$ is odd, hence by our assumption z^2 is odd. An odd number z is of the form $4N + 1$ or $4N + 3$. Thus, in any case z^2 has the form $4T + 1$ where T is an integer. We shall now consider two cases, namely y is odd and y is even.

Suppose y is odd and $y = 2n + 1$ where $n \geq 0$ is an integer. Since $11 = 4N + 3$ ($N = 2$), then for all values n, 11^{2n+1} is of the form $4U + 3$ where U is an integer. The power 2^{4m+3} is of the form $4V$ where V is an integer. Thus, the sum $2^{4m+3} + 11^{2n+1}$ has the form $4(V + U) + 3 \neq 4T + 1 = z^2$. Hence $y \neq 2n + 1$.

Suppose y is even and $y = 2k$ where $k \geq 1$ is an integer. We have $2^{4m+3} + 11^{2k} = z^2$ or $2^{4m+3} = z^2 - 11^{2k} = z^2 - (11^k)^2$ and

$$2^{4m+3} = (z - 11^k)(z + 11^k). \tag{3}$$

Denote in (3)

$$z - 11^k = 2^C, \qquad z + 11^k = 2^D, \qquad C < D, \qquad C + D = 4m + 3,$$

where C, D are non- negative integers. Then $2^D - 2^C$ yields

$$2 \cdot 11^k = 2^C(2^{D-C} - 1). \tag{4}$$

It follows from (4) that $C > 0$, and $C = 1$ is the only such possibility. When $C = 1$ then $D = 4m + 2$, and (4) after simplification results in

$$11^k = 2^{4m+1} - 1. \tag{5}$$

When $k = 2a + 1$ where a is an integer, one could verify that for all $a \geq 0$ the sum $11^{2a+1} + 1$ is a multiple of 3. Hence, if $k = 2a + 1$ in (5), then $11^{2a+1} \neq 2^{4m+1} - 1$, implying that $k \neq 2a + 1$. When $k = 2a$, then $11^{2a} + 1 = 2b$ where for all values a, the value b is an odd integer. Thus, $11^{2a} + 1 = 2b \neq 2^{4m+1}$ and $k \neq 2a$. It now follows that there does not exist a value y which satisfies the equation $2^{4m+3} + 11^y = z^2$. This is a contradiction, and our assumption is therefore false.
This completes case **(c)**.

(d) Suppose that $x = 4m$, $y \geq 1$.
For all values $m \geq 1$, the power 2^{4m} ends in the digit 6. For all $y \geq 1$, the power 11^y ends in the digit 1. Therefore $2^{4m} + 11^y$ ends in the digit 7. If for some value z, the sum $2^{4m} + 11^y$ equals z^2, then z^2 is odd and ends in the digit 7. An odd square does not have a last digit which is equal to 7. It therefore follows that $2^{4m} + 11^y \neq z^2$.
This concludes case **(d)**. The equation $2^x + 11^y = z^2$ has no solutions.

The proof of Theorem 2.1 is complete. □

Remark 2.1. The equivalent equation for (2) is $1 = 11^y - 2^{2m+2}$, whereas for (5) the equivalent equation is $1 = 2^{4m+1} - 11^k$. In each of the equivalent equations, the conditions of Catalan's Conjecture are satisfied. As a consequence of Catalan's Conjecture, it follows that each equivalent equation has no solutions. In another manner, this reaffirms what we have shown earlier in a different way that equations (2) and (5) have no solutions. We have not used Catalan's Conjecture earlier, since we have a preference for the elementary way.

3. The equation $19^x + 29^y = z^2$

Theorem 3.1. Let x, y, z be positive integers. Then the equation $19^x + 29^y = z^2$ has no solutions.

Proof: For all values $x \geq 1$, the power 19^x ends in the digits 9 and 1. For all values $y \geq 1$, the power 29^y ends in the digits 9 and 1. Let m, n be non-negative integers. We shall consider the four existing possibilities as follows:

(a)	$x = 2m + 1$,	$y = 2n + 1$,	$m \geq 0$,	$n \geq 0$.
(b)	$x = 2m + 1$,	$y = 2n$,	$m \geq 0$,	$n \geq 1$.
(c)	$x = 2m$,	$y = 2n + 1$,	$m \geq 1$,	$n \geq 0$.
(d)	$x = 2m$,	$y = 2n$,	$m \geq 1$,	$n \geq 1$.

(a) Suppose that $x = 2m + 1$, $y = 2n + 1$.
For all values $m \geq 0$, $n \geq 0$, each of the powers 19^{2m+1} and 29^{2n+1} has a last digit equal to 9. If for some value z, $19^{2m+1} + 29^{2n+1} = z^2$, then z^2 is even and ends in the digit 8. An even square does not have a last digit equal to 8, therefore $19^{2m+1} + 29^{2n+1} \neq z^2$.
Case **(a)** is complete.

(b) Suppose that $x = 2m + 1$, $y = 2n$.
We shall assume that $19^{2m+1} + 29^{2n} = z^2$ has a solution, and reach a contradiction.

For all values $m \geq 0$, the power 19^{2m+1} has a last digit equal to 9. For all values $n \geq 1$, the power 29^{2n} has a last digit equal to 1. By our assumption, we have $19^{2m+1} + 29^{2n} = z^2$ implying that $19^{2m+1} = z^2 - 29^{2n} = z^2 - (29^n)^2$ or

$$19^{2m+1} = (z - 29^n)(z + 29^n).$$

Denote

$$z - 29^n = 19^A, \qquad z + 29^n = 19^B, \qquad A < B, \qquad A + B = 2m + 1,$$

where A, B are non-negative integers. Then $19^B - 19^A$ results in

$$2 \cdot 29^n = 19^A(19^{B-A} - 1). \tag{6}$$

If $A > 0$, the power 19^A does not divide the left side of (6), and therefore $A = 0$. When $A = 0$, then $B = 2\text{m} + 1$, and (6) implies

$$2 \cdot 29^n = 19^{2m+1} - 1. \tag{7}$$

The right side of (7) is equal to $19^{2m+1} - 1^{2m+1}$, which yields the identity

$$19^{2m+1} - 1^{2m+1} = (19 - 1)(19^{2m} + 19^{2m-1} \cdot 1^1 + 19^{2m-2} \cdot 1^2 + \cdots + 1^{2m}). \tag{8}$$

In (8), the factor $(19 - 1) = 18 = 2 \cdot 3^2$. Since in (7) $3 \nmid 2 \cdot 29^n$, it follows that (7) is impossible. This contradiction therefore implies that our assumption is false, and $19^{2m+1} + 29^{2n} \neq z^2$.
This concludes case **(b)**.

(c) Suppose that $x = 2m$, $y = 2n + 1$.
We shall assume that $19^{2m} + 29^{2n+1} = z^2$ has a solution, and reach a contradiction.

For all values $m \geq 1$, the power 19^{2m} has a last digit equal to 1. For all values $n \geq 0$, the power 29^{2n+1} has a last digit equal to 9. By our assumption, we have $19^{2m} + 29^{2n+1} = z^2$ implying that $29^{2n+1} = z^2 - 19^{2m} = z^2 - (19^m)^2$ or

$$29^{2n+1} = (z - 19^m)(z + 19^m).$$

Denote

$$z - 19^m = 29^C, \qquad z + 19^m = 29^D, \qquad C < D, \qquad C + D = 2n + 1,$$

where C, D are non-negative integers. Then $29^D - 29^C$ yields

$$2 \cdot 19^m = 29^C(29^{D-C} - 1). \tag{9}$$

If $C > 0$, the power 29^C does not divide the left side of (9), and therefore $C = 0$. When $C = 0$, then $D = 2n + 1$, and (9) implies

$$2 \cdot 19^m = 29^{2n+1} - 1. \tag{10}$$

The right side of (10) is equal $29^{2n+1} - 1^{2n+1}$, which yields the identity

$$29^{2n+1} - 1^{2n+1} = (29 - 1)(29^{2n} + 29^{2n-1} \cdot 1^1 + 29^{2n-2} \cdot 1^2 + \cdots + 1^{2n}). \tag{11}$$

In (11), the factor $(29 - 1) = 28 = 2^2 \cdot 7$. Since in (10) $2^2 \nmid 2 \cdot 19^m$, it follows that (10) is impossible. This contradiction therefore implies that our assumption is false, and $19^{2m} + 29^{2n+1} \neq z^2$.
Case **(c)** is complete.

(d) Suppose that $x = 2m$, $y = 2n$.
For all values $m \geq 1$, $n \geq 1$, each of the powers 19^{2m} and 29^{2n} has a last digit equal to 1. If for some value z, $19^{2m} + 29^{2n} = z^2$, then z^2 is even and has a last digit equal to 2. An even square does not have a last digit equal to 2, therefore $19^{2m} + 29^{2n} \neq z^2$.
Case **(d)** is complete. The equation $19^x + 29^y = z^2$ has no solutions.

This concludes the proof of Theorem 3.1. □

Final remark. We have established that both equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ have no solutions when x, y, z are positive integers. Our new method of using the last digits of the powers involved has been a key factor in determining the solutions. We are quite confident that this method can also be used in finding solutions to other equations.

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