The Diophantine Equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$

are Insolvable in Positive Integers $x, y, z$

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Abstract. In this article, the author has investigated the equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ with positive integers $x, y, z$. It was established that both equations have no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [2, 4, 6, 8].

In this article, we consider the two equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ in which $x, y, z$ are positive integers. It will be shown that both equations have no solutions. This is done in Sections 2 and 3. Although similarities exist, nevertheless, the theorems and all the cases within are self-contained. The results achieved are mainly and in particular based on our new method which utilizes the last digits of the powers involved.

2. The equation $2^x + 11^y = z^2$

Theorem 2.1. Let $x, y, z$ be positive integers. Then the equation $2^x + 11^y = z^2$ has no solutions.

Proof: Let $m \geq 0$ be an integer. For all values $x \geq 1$, four possibilities exist:

(a) $x = 4m + 1$, $y \geq 1$, $m \geq 0$.
(b) $x = 4m + 2$, $y \geq 1$, $m \geq 0$.
(c) $x = 4m + 3$, $y \geq 1$, $m \geq 0$.
(d) $x = 4m$, $y \geq 1$, $m \geq 1$.
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(a) Suppose that \( x = 4m + 1 \), \( y \geq 1 \).
For all values \( m \geq 0 \), the power \( 2^{4m+1} \) ends in the digit 2. For all values \( y \geq 1 \), the power \( 11^y \) ends in the digit 1. Hence \( 2^{4m+1} + 11^y \) ends in the digit 3. If for some value \( z \), \( 2^{4m+1} + 11^y = z^2 \), then \( z^2 \) is odd and ends in the digit 3. But, an odd square does not have a last digit which is equal to 3. Therefore it follows that \( 2^{4m+1} + 11^y \neq z^2 \).

Case (a) is complete.

(b) Suppose that \( x = 4m + 2 \), \( y \geq 1 \).
We shall now consider two cases, namely when the equation has a solution, and hence \( 2^{4m+1} + 11^y \) has the form \( 4(v + 1) \) or \( 4v + 1 \) and hence \( (2) \) is impossible. This contradicts our assumption that when \( x = 4m + 2 \) the equation has a solution, and hence \( 2^{4m+1} + 11^y \neq z^2 \).

This concludes case (b).

(c) Suppose that \( x = 4m + 3 \), \( y \geq 1 \).
We shall assume that \( 2^{4m+3} + 11^y = z^2 \) has a solution, and reach a contradiction.

The sum \( 2^{4m+3} + 11^y \) is odd, hence by our assumption \( z^2 \) is odd. An odd number \( z \) is of the form \( 4N + 1 \) or \( 4N + 3 \). Thus, in any case \( z^2 \) has the form \( 4T + 1 \) where \( T \) is an integer. We shall now consider two cases, namely \( y \) is odd and \( y \) is even.

Suppose \( y \) is odd and \( y = 2n + 1 \) where \( n \geq 0 \) is an integer. Since \( 11 = 4N + 3 \) \((N = 2)\), then for all values \( n \), \( 11^{2n+1} \) is of the form \( 4U + 3 \) where \( U \) is an integer. The power \( 2^{4m+3} \) is of the form \( 4V \) where \( V \) is an integer. Thus, the sum \( 2^{4m+3} + 11^{2n+1} \) has the form \( 4(V + U) + 3 \neq 4T + 1 = z^2 \). Hence \( y \neq 2n + 1 \).

Suppose \( y \) is even and \( y = 2k \) where \( k \geq 1 \) is an integer. We have \( 2^{4m+3} + 11^{2k} = z^2 \) or \( 2^{4m+3} + 11^{2k} = z^2 - 11^{2k} = z^2 - (11^k)^2 \) and
\[
2^{4m+3} = (z - 11^k)(z + 11^k). \tag{3}
\]

Denote in (3) \( z - 11^k = 2^C \), \( z + 11^k = 2^D \), \( C < D \), \( C + D = 4m + 3 \), where \( C, D \) are non-negative integers. Then \( 2^D - 2^C \) yields
\[
2^{2k} = 2^D(2^D - 2^C - 1). \tag{4}
\]
It follows from (4) that \( C > 0 \), and \( C = 1 \) is the only such possibility. When \( C = 1 \) then \( D = 4m + 2 \), and (4) after simplification results in
\[
11^k = 2^{4m+1} - 1. \tag{5}
\]
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When $k = 2a + 1$ where $a$ is an integer, one could verify that for all $a \geq 0$ the sum $11^{2a+1} + 1$ is a multiple of 3. Hence, if $k = 2a + 1$ in (5), then $11^{2a+1} \neq 2^{4m+1} - 1$, implying that $k \neq 2a + 1$. When $k = 2a$, then $11^{2a} + 1 = 2b$ where for all values $a$, the value $b$ is an odd integer. Thus, $11^{2a} + 1 = 2b \neq 2^{4m+1}$ and $k \neq 2a$. It now follows that there does not exist a value $y$ which satisfies the equation $2^{4m+3} + 11^y = z^2$. This is a contradiction, and our assumption is therefore false.

This completes case (c).

(d) Suppose that $x = 4m$, $y \geq 1$.

For all values $m \geq 1$, the power $2^{4m}$ ends in the digit 6. For all $y \geq 1$, the power $11^y$ ends in the digit 1. Therefore $2^{4m} + 11^y$ ends in the digit 7. If for some value $z$, the sum $2^{4m} + 11^y$ equals $z^2$, then $z^2$ is odd ends in the digit 7. An odd square does not have a last digit which is equal to 7, therefore $2^{4m} + 11^y \neq z^2$. This concludes case (d). The equation $2^x + 11^y = z^2$ has no solutions.

The proof of Theorem 2.1 is complete.

□

Remark 2.1. The equivalent equation for (2) is $1 = 11^y - 2^{4m+2}$, whereas for (5) the equivalent equation is $1 = 2^{4m+1} - 11^y$. In each of the equivalent equations, the conditions of Catalan’s Conjecture are satisfied. As a consequence of Catalan’s Conjecture, it follows that each equivalent equation has no solutions. In another manner, this reaffirms what we have shown earlier in a different way that equations (2) and (5) have no solutions. We have not used Catalan’s Conjecture earlier, since we have a preference for the elementary way.

3. The equation $19^x + 29^y = z^2$

Theorem 3.1. Let $x, y, z$ be positive integers. Then the equation $19^x + 29^y = z^2$ has no solutions.

Proof: For all values $x \geq 1$, the power $19^x$ ends in the digits 9 and 1. For all values $y \geq 1$, the power $29^y$ ends in the digits 9 and 1. Let $m, n$ be non-negative integers. We shall consider the four existing possibilities as follows:

(a) $x = 2m + 1, \quad y = 2n + 1, \quad m \geq 0, \quad n \geq 0$.
(b) $x = 2m + 1, \quad y = 2n, \quad m \geq 0, \quad n \geq 1$.
(c) $x = 2m, \quad y = 2n + 1, \quad m \geq 1, \quad n \geq 0$.
(d) $x = 2m, \quad y = 2n, \quad m \geq 1, \quad n \geq 1$.

(a) Suppose that $x = 2m + 1, y = 2n + 1$.

For all values $m \geq 0, n \geq 0$, each of the powers $19^{2m+1}$ and $29^{2n+1}$ has a last digit equal to 9. If for some value $z$, $19^{2m+1} + 29^{2n+1} = z^2$, then $z^2$ is even and ends in the digit 8. An even square does not have a last digit equal to 8, therefore $19^{2m+1} + 29^{2n+1} \neq z^2$. Case (a) is complete.
(b) Suppose that $x = 2m + 1$, $y = 2n$.

We shall assume that $19^{2m+1} + 29^{2n} = z^2$ has a solution, and reach a contradiction.

For all values $m \geq 0$, the power $19^{2m+1}$ has a last digit equal to 9. For all values $n \geq 1$, the power $29^{2n}$ has a last digit equal to 1. By our assumption, we have $19^{2m+1} + 29^{2n} = z^2$ implying that $19^{2m+1} = z^2 - 29^{2n} = z^2 - (29^n)^2$ or $19^{2m+1} = (z - 29^n)(z + 29^n)$.

Denote $z - 29^n = 19^A$, $z + 29^n = 19^B$, $A < B$, $A + B = 2m + 1$, where $A$, $B$ are non-negative integers. Then $19^B - 19^A$ results in $2 \cdot 29^n = 19^B(19^B - A - 1)$. (6)

If $A > 0$, the power $19^A$ does not divide the left side of (6), and therefore $A = 0$. When $A = 0$, then $B = 2m + 1$, and (6) implies $2 \cdot 29^n = 19^{2m+1} - 1$. (7)

The right side of (7) is equal to $19^{2m+1} - 1^{2m+1}$, which yields the identity $19^{2m+1} - 1^{2m+1} = (19 - 1)(19^{2m} + 19^{2m-1} \cdot 1 + 19^{2m-2} \cdot 1^2 + \cdots + 1^2m)$. (8)

In (8), the factor $(19 - 1) = 18 = 2 \cdot 3^2$. Since in (7) $3 \nmid 2 \cdot 29^n$, it follows that (7) is impossible. This contradiction therefore implies that our assumption is false, and $19^{2m+1} + 29^{2n} \neq z^2$.

This concludes case (b).

(c) Suppose that $x = 2m$, $y = 2n + 1$.

We shall assume that $19^{2m} + 29^{2n+1} = z^2$ has a solution, and reach a contradiction.

For all values $m \geq 1$, the power $19^{2m}$ has a last digit equal to 1. For all values $n \geq 0$, the power $29^{2n+1}$ has a last digit equal to 9. By our assumption, we have $19^{2m} + 29^{2n+1} = z^2$ implying that $29^{2n+1} = z^2 - 19^{2m} = z^2 - (19^n)^2$ or $29^{2n+1} = (z - 19^n)(z + 19^n)$.

Denote $z - 19^n = 29^C$, $z + 19^n = 29^D$, $C < D$, $C + D = 2n + 1$, where $C$, $D$ are non-negative integers. Then $29^D - 29^C$ yields $2 \cdot 19^n = 29^D(29^D - 29^C - 1)$. (9)

If $C > 0$, the power $29^C$ does not divide the left side of (9), and therefore $C = 0$. When $C = 0$, then $D = 2n + 1$, and (9) implies $2 \cdot 19^n = 29^{2n+1} - 1$. (10)

The right side of (10) is equal to $29^{2n+1} - 1^{2n+1}$, which yields the identity $29^{2n+1} - 1^{2n+1} = (29 - 1)(29^{2n} + 29^{2n-1} \cdot 1^1 + 29^{2n-2} \cdot 1^2 + \cdots + 1^{2n})$. (11)

In (11), the factor $(29 - 1) = 28 = 2^2 \cdot 7$. Since in (10) $2^2 \nmid 2 \cdot 19^n$, it follows that (10) is impossible. This contradiction therefore implies that our assumption is false, and $19^{2m} + 29^{2n+1} \neq z^2$.

Case (c) is complete.

(d) Suppose that $x = 2m$, $y = 2n$.

For all values $m \geq 1$, $n \geq 1$, each of the powers $19^{2m}$ and $29^{2n}$ has a last digit equal to 1. If for some value $z$, $19^{2m} + 29^{2n} = z^2$, then $z^2$ is even and has a last digit equal to 2. An even square does not have a last digit equal to 2, therefore $19^{2m} + 29^{2n} \neq z^2$.

Case (d) is complete. The equation $19^2 + 29^y = z^2$ has no solutions.

This concludes the proof of Theorem 3.1. □
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**Final remark.** We have established that both equations $2^x + 11^y = z^2$ and $19^x + 29^y = z^2$ have no solutions when $x, y, z$ are positive integers. Our new method of using the last digits of the powers involved has been a key factor in determining the solutions. We are quite confident that this method can also be used in finding solutions to other equations.

**REFERENCES**