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All the Solutions of the Diophantine Equations $p^x + (p + 1)^y + (p + 2)^z = M^3$ when p is Prime and $1 \le x, y, z \le 2$

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Abstract. In this paper, we consider the equations $p^x + (p + 1)^y + (p + 2)^z = M^3$ when p is prime and x, y, z are integers satisfying $1 \le x, y, z \le 2$. We establish: (i) A unique solution exists when $p = 2$. (ii) No solutions exist when $p = 4N + 1$. (iii) Infinitely many solutions exist when $p = 4N + 3$, and $x = y = z = 1$. No solutions exist for all other possibilities.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds. Among them are for example [3, 4, 8, 9].

In [2], in a preliminary step towards larger equations, we have extended the above equation to equations of the form $p^x + (p + 1)^y + (p + 2)^z = M^2$ for all primes $p \ge 2$ when $1 \le x, y, z \le 2$. In [1, 2], we have determined all the solutions for all primes $p \ge 2$ when $1 \le x, y, z \le 2$. In this paper, the equations $p^x + (p + 1)^y + (p + 2)^z = M^2$ are taken one step ahead, and we consider now equations of the form $p^x + (p + 1)^y + (p + 2)^z = M^3$ when $1 \le x, y, z \le 2$. For all primes $p \ge 2$, we establish all the solutions for $p^x + (p + 1)^y + (p + 2)^z = M^3$ when $1 \le x, y, z \le 2$. This is done in the respective Sections 2, 3 and 4 in which all theorems and all cases are considered separately and are self-contained.

2. All the solutions of $p^x + (p + 1)^y + (p + 2)^z = M^3$ when $p = 2$, $1 \le x, y, z \le 2$

In this section all the solutions of $2^x + 3^y + 4^z = M^3$ are determined.

Theorem 2.1. Let $1 \le x, y, z \le 2$. Then $2^x + 3^y + 4^z = M^3$ has a unique solution when $x = 1$, $y = z = 2$.

Proof: When $1 \le x, y, z \le 2$, the eight cases of $2^x + 3^y + 4^z = M^3$ are listed below.

(1)	2	$+$	3	$+$	4	$=$	9	$\ne$	M^3.
(2)	2	$+$	3	$+$	4^2	$=$	21	$\ne$	M^3.
(3)	2	$+$	3^2	$+$	4	$=$	15	$\ne$	M^3.
(4)	2^2	$+$	3	$+$	4	$=$	11	$\ne$	M^3.
(5)	2	$+$	3^2	$+$	4^2	$=$	3^3	$=$	M^3.
(6)	2^2	$+$	3	$+$	4^2	$=$	23	$\ne$	M^3.
(7)	2^2	$+$	3^2	$+$	4	$=$	17	$\ne$	M^3.
(8)	2^2	$+$	3^2	$+$	4^2	$=$	29	$\ne$	M^3.

It follows that case **(5)** when $x = 1$, $y = z = 2$ yields a solution for which $M = 3$, whereas in all other cases **(1) – (4)**, **(6) – (8)** no solutions exist.

This completes the proof of Theorem 2.1. □

3. All the solutions of $p^x + (p + 1)^y + (p + 2)^z = M^3$ when $p = 4N + 1$, $1 \le x, y, z \le 2$

Here we consider $p^x + (p + 1)^y + (p + 2)^z = M^3$ for all primes $p = 4N + 1$, when $1 \le x, y, z \le 2$. In Theorem 3.1 we establish that the equations have no solutions.

Theorem 3.1. Let $1 \le x, y, z \le 2$. If $p = 4N + 1$, then $p^x + (p + 1)^y + (p + 2)^z = M^3$ have no solutions.

Proof: When $1 \le x, y, z \le 2$ and $p = 4N + 1$ is prime, eight cases exist:

(1)	$(4N + 1)$	$+$	$(4N + 2)$	$+$	$(4N + 3)$	$=$	M^3.
(2)	$(4N + 1)$	$+$	$(4N + 2)$	$+$	$(4N + 3)^2$	$=$	M^3.
(3)	$(4N + 1)$	$+$	$(4N + 2)^2$	$+$	$(4N + 3)$	$=$	M^3.
(4)	$(4N + 1)^2$	$+$	$(4N + 2)$	$+$	$(4N + 3)$	$=$	M^3.
(5)	$(4N + 1)$	$+$	$(4N + 2)^2$	$+$	$(4N + 3)^2$	$=$	M^3.
(6)	$(4N + 1)^2$	$+$	$(4N + 2)$	$+$	$(4N + 3)^2$	$=$	M^3.
(7)	$(4N + 1)^2$	$+$	$(4N + 2)^2$	$+$	$(4N + 3)$	$=$	M^3.
(8)	$(4N + 1)^2$	$+$	$(4N + 2)^2$	$+$	$(4N + 3)^2$	$=$	M^3.

These eight cases each of which is self-contained are considered separately.

(1) The case $(4N + 1) + (4N + 2) + (4N + 3) = M^3$.

The left side of the equation yields

$$(4N + 1) + (4N + 2) + (4N + 3) = 12N + 6 = 6(2N + 1).$$

The prime 2 in the factor 6 has an odd exponent equal to 1. Since $(2N + 1)$ is odd, it follows that $6(2N + 1)$ is not equal to M^3.

The equation $(4N + 1) + (4N + 2) + (4N + 3) = M^3$ has no solutions.

(2) The case $(4N + 1) + (4N + 2) + (4N + 3)^2 = M^3$.

The left side of the equation yields

$$(4N + 1) + (4N + 2) + (16N^2 + 24\text{N} + 9) = 4(4N^2 + 8N + 3).$$

The prime 2 in the factor 4 has an even exponent equal to 2. Since $(4N^2 + 8N + 3)$ is odd for all values N, it follows that $4(4N^2 + 8N + 3) \ne M^3$.

The equation $(4N + 1) + (4N + 2) + (4N + 3)^2 = M^3$ has no solutions.

(3) The case $(4N + 1) + (4N + 2)^2 + (4N + 3) = M^3$.

Rewriting in terms of p the left side of the equation, we obtain

$$p + (p + 1)^2 + (p + 2) = p^2 + 4p + 3 = (p + 2)^2 - 1.$$

If $(p + 2)^2 - 1 = M^3$, then $(p + 2)^2 - M^3 = 1$. All four values $(p + 2)$, 2, M and 3 satisfy the conditions of Catalan's Conjecture which states that $3^2 - 2^3 = 1$ is the only solution of the above equation. Since this is impossible, it follows that $(p + 2)^2 - 1 \ne M^3$.

The equation $(4N + 1) + (4N + 2)^2 + (4N + 3) = M^3$ has no solutions.

(4) The case $(4N + 1)^2 + (4N + 2) + (4N + 3) = M^3$.

The left side of the equation yields

$$(16N^2 + 8N + 1) + (4N + 2) + (4N + 3) = 2(8N^2 + 8N + 3).$$

The prime 2 has an odd exponent equal to 1, and the factor $(8N^2 + 8N + 3)$ is odd for all values N. Hence $2(8N^2 + 8N + 3) \ne M^3$.

The equation $(4N + 1)^2 + (4N + 2) + (4N + 3) = M^3$ has no solutions.

(5) The case $(4N + 1) + (4N + 2)^2 + (4N + 3)^2 = M^3$.

The left side of the equation yields

$$(4N + 1) + (16N^2 + 16N + 4) + (16N^2 + 24N + 9) = 2(16N^2 + 22N + 7).$$

The prime 2 has an odd exponent equal to 1, and the factor $(16N^2 + 22N + 7)$ is odd for all values N. Thus $2(16N^2 + 22N + 7) \ne M^3$.

The equation $(4N + 1) + (4N + 2)^2 + (4N + 3)^2 = M^3$ has no solutions.

(6) The case $(4N + 1)^2 + (4N + 2) + (4N + 3)^2 = M^3$.

Rewriting in terms of p the left side of the equation, we obtain

$$p^2 + (p + 1) + (p + 2)^2 = 2p^2 + 5p + 5 = (2p^2 + 5p + 3) + 2.$$

We shall assume that for some prime p, $(2p^2 + 5p + 3) + 2 = M^3$ has a solution and reach a contradiction.

The value $(2p^2 + 5p + 3)$ is even for all primes p. Our assumption that $(2p^2 + 5p + 3) + 2 = M^3$ implies that $(2p^2 + 5p + 3) + 2$, M are even, and $M^3 - (2p^2 + 5p + 3) = 2$ is the smallest possible difference of two consecutive even integers. We shall consider both possibilities of $p = 4N + 1$, namely when N is even and when N is odd.

It is easily seen that $2p^2 + 5p + 5 = 4(8N^2 + 9N + 3)$. If N is even, then $(8N^2 + 9N + 3)$ is odd. The factor $4 = 2^2$ then implies that $4(8N^2 + 9N + 3) \ne M^3$ contrary to our assumption. Therefore, N is not even, and by our assumption N is odd.

When N is odd, say $N = 2n + 1$ (n an integer), then $p = 4N + 1 = 8n + 5$. In the following Table 1, the first seven such primes are presented. The even values M are taken as the smallest possible values (min M) for which min M^3 exceeds $(2p^2 + 5p + 3)$ for the first time, in order to achieve the smallest possible difference $M^3 – (2p^2 + 5p + 3) = t = 2$.

Table 1.

$p = 4N + 1 = 8n + 5$	$2p^2 + 5p + 3$	min M	min M^3	min $M^3 – (2p^2 + 5p + 3) = t$
5	78	6	216	138
13	406	8	512	106
29	1830	14	2744	914
37	2926	16	4096	1170
53	5886	20	8000	2114
61	7750	20	8000	250
101	20910	28	21952	1042

In Table 1, for each prime p, the respective data is self-explanatory. All values min $M^3 – (2p^2 + 5p + 3) = t$ are even. The smallest possible number t is equal to 106 and has 3 digits. If D denotes the number of digits of each number t, then $D \geq 3$. It is clearly seen that $D = 1$, i.e., $t = 2$ is never attained. We can now state that our assumption is false when N is odd.

The equation $(4N + 1)^2 + (4N + 2) + (4N + 3)^2 = M^3$ has no solutions.

(7) The case $(4N + 1)^2 + (4N + 2)^2 + (4N + 3) = M^3$.

Rewriting in terms of p the left side of the equation, we obtain

$$p^2 + (p + 1)^2 + (p + 2) = 2p^2 + 3p + 3 = (2p^2 + 3p + 1) + 2.$$

We shall assume that for some prime p, $(2p^2 + 3p + 1) + 2 = M^3$ and reach a contradiction.

The value $(2p^2 + 3p + 1)$ is even for all primes p. Our assumption that $(2p^2 + 3p + 1) + 2 = M^3$ implies that $(2p^2 + 3p + 1) + 2$, M are even, and $M^3 – (2p^2 + 3p + 1) = 2$ is the smallest possible difference of two consecutive even integers. We shall now consider both possibilities of $p = 4N + 1$, namely N odd and N even.

Denote $M = 2m$. When $N = 2n + 1$, then $p = 4N + 1 = 8n + 5$. Our assumption that $M^3 – (2p^2 + 3p + 1) = 2$ yields

$$8m^3 – \big(2(8n+5)^2+3(8n+5)+1\big) = 8m^3 – (128n^2+184n+66) = 8(m^3 – 16n^2 – 23n – 8) – 2 = 2.$$

But, for all values m, n, $8(m^3 – 16n^2 – 23n – 8) – 2 \neq 2$. Thus, our assumption is false when N is odd.

When $N = 2n$ is even, then $p = 4N + 1 = 8n + 1$. In the following Table 2, the first seven such primes are presented. The even values M are taken as the smallest possible values (min M) for which min M^3 exceeds $(2p^2 + 3p + 1)$ for the first time, in order to achieve the smallest possible difference $M^3 – (2p^2 + 3p + 1) = t = 2$.

Table 2.

$p = 4N + 1 = 8n + 1$	$2p^2 + 3p + 1$	min M	min M^3	min $M^3 - (2p^2 + 3p + 1) = t$
17	630	10	1000	370
41	3486	16	4096	610
73	10878	24	13824	2946
89	16110	26	17576	1466
97	19110	28	21952	2842
113	25878	30	27000	1122
137	37950	34	39304	1354

In Table 2, the primes presented and the data obtained are self-evident. All values min $M^3 - (2p^2 + 3p + 1) = t$ are even. The smallest possible number t is equal to 370 and has 3 digits. If D denotes the number of digits of each number t, then $D \ge 3$. As p, min M are increasing, so are $(2p^2 + 3p + 1)$ and min M^3. Hence, the value $D = 1$, namely $t = 2$ which is one digit is never attained. Since the numbers in Table 2 quite clearly indicate this fact, we can therefore state that our assumption is false when N is even.

The equation $(4N + 1)^2 + (4N + 2)^2 + (4N + 3) = M^3$ has no solutions.

(8) The case $(4N + 1)^2 + (4N + 2)^2 + (4N + 3)^2 = M^3$.
The left side of the equation yields
$(16N^2 + 8N + 1) + (16N^2 + 16N + 4) + (16N^2 + 24N + 9) = 2(24N^2 + 24N + 7)$.
The prime 2 has an odd exponent equal to 1. Since $(24N^2 + 24N + 7)$ is odd for all values N, it follows that $2(24N^2 + 24N + 7) \ne M^3$.

The equation $(4N + 1)^2 + (4N + 2)^2 + (4N + 3)^2 = M^3$ has no solutions.

This concludes the proof of Theorem 3.1. □

4. All the solutions of $p^x + (p + 1)^y + (p + 2)^z = M^3$ when $p = 4N + 3$, $1 \le x, y, z \le 2$

In this section we consider $p^x + (p + 1)^y + (p + 2)^z = M^3$ when $1 \le x, y, z \le 2$, and $p = 4N + 3$.

Theorem 4.1. Let $1 \le x, y, z \le 2$. Suppose that $p = 4N + 3$. Then $p^x + (p + 1)^y + (p + 2)^z = M^3$ has: (i) Infinitely many solutions when $x = y = z = 1$. (ii) No solutions for all other possibilities.

Proof: When $1 \le x, y, z \le 2$ and $p = 4N + 3$ is prime, eight cases exist:

(1) $(4N + 3) + (4N + 4) + (4N + 5) = M^3$.
(2) $(4N + 3) + (4N + 4) + (4N + 5)^2 = M^3$.
(3) $(4N + 3) + (4N + 4)^2 + (4N + 5) = M^3$.
(4) $(4N + 3)^2 + (4N + 4) + (4N + 5) = M^3$.
(5) $(4N + 3) + (4N + 4)^2 + (4N + 5)^2 = M^3$.
(6) $(4N + 3)^2 + (4N + 4) + (4N + 5)^2 = M^3$.
(7) $(4N + 3)^2 + (4N + 4)^2 + (4N + 5) = M^3$.
(8) $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^2 = M^3$.

Each case is considered separately, and is self-contained.

(1) The case $(4N + 3) + (4N + 4) + (4N + 5) = M^3$.
The left side of the equation yields

$$(4N + 3) + (4N + 4) + (4N + 5) = 12N + 12 = 12(N + 1).$$

The factor $12 = 2^2\cdot 3$. If $12(N + 1) = M^3$, it then follows that $(N + 1)$ is of the form $N + 1 = 2^{1 + 3a}\cdot 3^{2 + 3b}\cdot K^3$ where $a \geq 0$, $b \geq 0$ and $K \geq 1$ are integers. Evidently, $N + 1$ is always a multiple of 18. The value $N = 2^{1 + 3a}\cdot 3^{2 + 3b}\cdot K^3 - 1$ must satisfy

$$4N + 3 = 4(2^{1 + 3a}\cdot 3^{2 + 3b}\cdot K^3 - 1) + 3 = p. \qquad (1)$$

Some examples satisfying (1) which are solutions of the equation are demonstrated as follows:

Example 1.	If	$a = 0$,	$b = 0$,	$K = 1$,	then	$N = 17$,	$p = 71$,	$M = 6$.
Example 2.	If	$a = 1$,	$b = 1$,	$K = 1$,	then	$N = 3887$,	$p = 15551$,	$M = 36$.
Example 3.	If	$a = 1$,	$b = 0$,	$K = 5$,	then	$N = 17999$,	$p = 71999$,	$M = 60$.

Certainly, there exist infinitely many values a, b, K for which (1) is prime.
The equation $(4N + 3) + (4N + 4) + (4N + 5) = M^3$ has infinitely many solutions.

(2) The case $(4N + 3) + (4N + 4) + (4N + 5)^2 = M^3$.
The left side of the equation yields

$$(4N + 3) + (4N + 4) + (16N^2 + 40N + 25) = 16(N^2 + 3N + 2) = 16(N + 1)(N + 2). \qquad (2)$$

We shall assume that for some value N, $16(N + 1)(N + 2) = M^3$ and reach a contradiction.

The factors $(N + 1)$, $(N + 2)$ are two consecutive integers. Therefore, either $(N + 1)$ is even and $(N + 2)$ is odd or vice versa. Without any loss of generality, we shall assume that $(N + 1)$ is even, and $(N + 2)$ is odd. Observe that if the even value $(N + 1)$ is not a multiple of 4, then since $(N + 2)$ is odd, it follows that $16(N + 1)(N + 2) \neq M^3$ contrary to our assumption. Therefore, by (2) and our assumption we have

$$N + 1 = 4A^3, \qquad N + 2 = 4A^3 + 1 = Q^3, \qquad 4^2(4A^3)(4A^3 + 1) = M^3,$$

where A assumes odd and even values, and Q is odd. We also note that the above values $N + 1$ and $N + 2$ must be satisfied simultaneously.

We will now show that $4A^3 + 1 = Q^3$, or $Q^3 - 4A^3 = 1$ is never achieved. In the following Table 3 we consider the first 10 values A. The values Q are taken as the smallest possible values Q denoted by min Q, for which min Q^3 exceeds $4A^3$ for the first time in order to achieve the smallest possible difference $Q^3 - 4A^3 = t$.

In Table 3, the numbers A,Q and the data obtained present a clear-cut view as to the behavior of the equality min $Q^3 - 4A^3 = t$. The smallest possible number t is $t = 17$. As A,Q are increasing, so does t. All numbers t in Table 3 consist of two and three digits. Evidently, the smallest one digit number which is equal to 1 is never attained. This implies that our assumption is false.

Table 3.

A	A^3	$4A^3$	min Q	min Q^3	min $Q^3 - 4A^3 = t$
1	1	4	3	27	23
2	8	32	5	125	93
3	27	108	5	125	17
4	64	256	7	343	87
5	125	500	9	729	229
6	216	864	11	1331	467
7	343	1372	13	2197	825
8	512	2048	13	2197	149
9	729	2916	15	3375	459
10	1000	4000	17	4913	913

The equation $(4N + 3) + (4N + 4) + (4N + 5)^2 = M^3$ has no solutions.

(3) The case $(4N + 3) + (4N + 4)^2 + (4N + 5) = M^3$.

The left side of the equation yields

$$(4N + 3) + (16N^2 + 32N + 16) + (4N + 5) = 8(2N^2 + 5N + 3) = 8(N + 1)(2N + 3). \qquad (3)$$

We shall assume that for some value N, $8(N + 1)(2N + 3) = M^3$ has a solution and reach a contradiction. The sum $2N + 3 = 2(N + 1) + 1$, and $\gcd(N + 1, 2(N + 1) + 1) = 1$. This fact together with our assumption imply that (3) must simultaneously satisfy the equalities

$$N + 1 = A^3, \qquad 2N + 3 = 2(N + 1) + 1 = 2A^3 + 1 = B^3, \qquad 8A^3B^3 = M^3.$$

We will now show that

$$2A^3 + v = B^3, \qquad v \ge 1 \qquad (4)$$

is false when $v = 1$.

In order to achieve the smallest possible value v in (4), we consider the largest possible value A so that the difference $B^3 - 2A^3$ yields the smallest possible value v. Set $A = B - 1$. It is easily seen when $A = 1, 2, 3$, that $B = 2, 3, 4$, and that the respective numbers v yield $v = 6, 11, 10$. For all values $A \ge 4$ and $B = A + 1$, then $B^3 - 2A^3 = v < 0$. Thus, the difference $B^3 - 2A^3 = v = 1$ is never attained. This implies that our assumption is false.

The equation $(4N + 3) + (4N + 4)^2 + (4N + 5) = M^3$ has no solutions.

(4) The case $(4N + 3)^2 + (4N + 4) + (4N + 5) = M^3$.

The left side of the equation yields

$$(16N^2 + 24N + 9) + (4N + 4) + (4N + 5) = 2(8N^2 + 16N + 9).$$

The prime 2 has an odd exponent equal to 1. The factor $(8N^2 + 16N + 9)$ is odd for all values N. It therefore follows that $2(8N^2 + 16N + 9) \ne M^3$.

The equation $(4N + 3)^2 + (4N + 4) + (4N + 5) = M^3$ has no solutions.

(5) The case $(4N + 3) + (4N + 4)^2 + (4N + 5)^2 = M^3$.

Rewriting in terms of p the left side of the equation yields

$$p + (p^2 + 2p + 1) + (p^2 + 4p + 4) = 2p^2 + 7p + 5.$$

We shall assume that for some prime p, $2p^2 + 7p + 5 = M^3$ has a solution and reach a contradiction.

The value $2p^2 + 7p + 5 = (2p^2 + 7p + 4) + 1 = M^3$ implies that M^3 is even for all primes p, and $2p^2 + 7p + 4$ is odd. In the following Table 4 we consider the first ten primes p. The values M are taken as the smallest possible values M denoted by $\min M$, for which $\min M^3$ exceeds $(2p^2 + 7p + 4)$ for the first time in order to obtain the smallest possible difference $\min M^3 - (2p^2 + 7p + 4) = t$. We will now show that $t = 1$ is not achieved.

Table 4.

$p = 4N + 3$	$2p^2 + 7p + 4$	$\min M$	$\min M^3$	$\min M^3 - (2p^2 + 7p + 4) = t$
3	43	4	64	21
7	151	6	216	65
11	323	8	512	189
19	859	10	1000	141
23	1223	12	1728	505
31	2143	14	2744	601
43	4003	16	4096	93
47	4751	18	5832	1081
59	7379	20	8000	621
67	9451	22	10648	1197

In Table 4, the primes p, $2p^2 + 7p + 4$, $\min M$ are increasing numbers. The numbers t decisively show that $t = 21$ is the smallest possible number. The number 21 has two digits. The other numbers t consist of 2, 3 and 4 digits. The smallest possible number $t = 1$ with one digit is never attained. Our assumption is therefore false.

The equation $(4N + 3) + (4N + 4)^2 + (4N + 5)^2 = M^3$ has no solutions.

(6) The case $(4N + 3)^2 + (4N + 4) + (4N + 5)^2 = M^3$.

The left side of the equation yields

$$(16N^2 + 24N + 9) + (4N + 4) + (16N^2 + 40N + 25) = 2(16N^2 + 34N + 19).$$

The prime 2 has an odd exponent equal to 1, and the factor $(16N^2 + 34N + 19)$ is odd for all values N. Hence $2(16N^2 + 34N + 19) \neq M^3$.

The equation $(4N + 3)^2 + (4N + 4) + (4N + 5)^2 = M^3$ has no solutions.

(7) The case $(4N + 3)^2 + (4N + 4)^2 + (4N + 5) = M^3$.

The left side of the equation yields

$$(16N^2 + 24N + 9) + (16N^2 + 32N + 16) + (4N + 5) = 2(16N^2 + 30N + 15).$$

The prime 2 has an odd exponent equal to 1, and the factor $(16N^2 + 30N + 15)$ is odd for all values N. Thus $2(16N^2 + 30N + 15) \neq M^3$.

The equation $(4N + 3)^2 + (4N + 4)^2 + (4N + 5) = M^3$ has no solutions.

(8) The case $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^2 = M^3$.

The left side of the equation yields

$$(16N^2 + 24N + 9) + (16N^2 + 32N + 16) + (16N^2 + 40N + 25) = 2(24N^2 + 48N + 25).$$

The prime 2 has an odd exponent equal to 1, and the factor $(24N^2 + 48N + 25)$ is odd for all values N. Therefore $2(24N^2 + 48N + 25) \ne M^3$.

The equation $(4N + 3)^2 + (4N + 4)^2 + (4N + 5)^2 = M^3$ has no solutions.

This concludes the proof of Theorem 4.1. □

Final remark. In this paper, we have considered the equations $p^x + (p+1)^y + (p+2)^z = M^3$ in which M is a positive integer, p is prime and p, $(p + 1)$, $(p + 2)$ are three consecutive integers. For all primes $p \ge 2$ and x, y, z satisfying $1 \le x, y, z \le 2$, we have established all the solutions of the above equations.

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