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All the Solutions of the Diophantine Equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$ when p, q are Distinct Primes

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Abstract. In this paper, we consider the two equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$ when p, q are distinct primes and y, z are positive integers. For all primes p, q we establish that both equations have no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation

$$p^x + q^y = z^2$$

has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this paper we consider the two equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$. By using elementary methods, we establish for all distinct primes p, q that both equations have no solutions. This is done in the respective Sections 2 and 3. Although existing similarities, we nevertheless consider both equations separately, in which all theorems and cases are self-contained.

2. All the solutions of $p^4 + q^y = z^4$ when p, q are distinct primes

In this section, we discuss the equation $p^4 + q^y = z^4$ and its solutions. This is done in the following theorem.

Theorem 2.1. Let y, z be positive integers. For all three possibilities

(a) $p = 2$ and q an odd prime,
(b) p an odd prime and $q = 2$,
(c) p, q distinct odd primes,

the equation $p^4 + q^y = z^4$ has no solutions.

Proof: All three cases are considered separately, and are self-contained.

(a) Suppose that $p = 2$ and q is an odd prime. If for some prime q, there exist values y and z satisfying

$$2^4 + q^y = z^4,$$

then $q^y = z^4 - 2^4$ or

$$q^y = (z^2 - 2^2)(z^2 + 2^2) = (z - 2)(z + 2)(z^2 + 2^2).$$

Denote

$$z - 2 = q^A, \quad z + 2 = q^B, \quad z^2 + 2^2 = q^C, \quad A < B < C, \quad q^y = q^{A+B+C}. \qquad (1)$$

From (1), $z = 2 + q^A$. If $A = 0$, then $z = 3$. Hence $z + 2 = 5 = q^B$ and $q = 5$, $B = 1$. The equation $2^4 + q^y = z^4$ then yields $2^4 + 5^y = 3^4$ which has no solutions. Thus $A \neq 0$. When $A > 0$, then from (1) we have $z = 2 + q^A$ and $4 + q^A = q^B$ implying that $q \mid 4$ which is impossible since q is an odd prime. Therefore $A \ngtr 0$, and the conditions in (1) are not satisfied. Hence, when $p = 2$ and q is an odd prime, then $2^4 + q^y \neq z^4$.
Part **(a)** is complete.

(b) Suppose that p is an odd prime and $q = 2$. If for some prime p, there exist values y and z for which

$$p^4 + 2^y = z^4,$$

then

$$2^y = z^4 - p^4 = (z^2 - p^2)(z^2 + p^2) = (z - p)(z + p)(z^2 + p^2).$$

Denote

$$z - p = 2^A, \quad z + p = 2^B, \quad z^2 + p^2 = 2^C, \quad A < B < C, \quad 2^y = 2^{A+B+C}, \qquad (2)$$

where all three conditions in (2) must be satisfied simultaneously.
The first two conditions $z - p = 2^A$ and $z + p = 2^B$ yield $2p + 2^A = 2^B$ or

$$2p = 2^B - 2^A = 2^A(2^{B-A} - 1).$$

The product $2p$ is a multiple of 2 only implying that $A = 1$. Thus $z = p + 2$. For the third condition we then obtain

$$z^2 + p^2 = 2p^2 + 4p + 4 = 2(p^2 + 2p + 2) = 2^C.$$

Since $(p^2 + 2p + 2)$ is odd for all primes p, it follows that

$$2(p^2 + 2p + 2) \neq 2^C.$$

The three conditions are not satisfied simultaneously, and therefore $p^4 + 2^y \neq z^4$.
This concludes case **(b)**.

(c) Suppose that p, q are distinct odd primes. If there exist primes p, q and values y, z which satisfy $p^4 + q^y = z^4$, then

$$q^y = z^4 - p^4 = (z^2 - p^2)(z^2 + p^2) = (z - p)(z + p)(z^2 + p^2).$$

Denote

$$z - p = q^A, \quad z + p = q^B, \quad z^2 + p^2 = q^C, \quad A < B < C, \quad q^y = q^{A+B+C}. \qquad (3)$$

The first two conditions in (3) namely $z - p = q^A$ and $z + p = q^B$ yield $2p + q^A = q^B$.
If $A = 0$, then $q^A = q^0 = 1$ and $z = p + 1$. Thus $2p + 1 = q^B$. The third condition yields

$$z^2 + p^2 = 2p^2 + 2p + 1 = 2p^2 + q^B = q^C,$$

and $0 = A < B < C$ then imply that $q \mid p$ which is impossible. Hence $A \neq 0$. When $A > 0$, it follows from the equality $2p + q^A = q^B$ that $q \mid p$ which is impossible. Thus $A \ngtr 0$. The conditions in (3) are not satisfied simultaneously. Therefore $p^4 + q^y \neq z^4$.

This concludes case **(c)** and the proof of Theorem 2.1. □

3. All the solutions of $p^4 - q^y = z^4$ when p, q are distinct primes

In this section, we consider in Theorem 3.1 the solutions of the equation $p^4 - q^y = z^4$.

Theorem 3.1. Let y, z be positive integers. For all three possibilities
(a) $p = 2$ and q an odd prime,
(b) p an odd prime and $q = 2$,
(c) p, q distinct odd primes,
the equation $p^4 - q^y = z^4$ has no solutions.

Proof: All three cases are considered separately, and are self-contained.
(a) Suppose that $p = 2$ and q is an odd prime. We have

$$2^4 - q^y = z^4.$$

One could easily see that the above equation has no solutions. Hence $2^4 - q^y \neq z^4$.

(b) Suppose that p is an odd prime and $q = 2$. We shall assume that

$$p^4 - 2^y = z^4 \tag{4}$$

has a solution, and reach a contradiction. For any solution of (4), the value z is odd. We then have

$$2^y = p^4 - z^4 = (p^2 - z^2)(p^2 + z^2) = (p - z)(p + z)(p^2 + z^2).$$

Denote

$$p - z = 2^A, \quad p + z = 2^B, \quad p^2 + z^2 = 2^C, \quad A < B < C, \quad 2^y = 2^{A+B+C}, \tag{5}$$

where all three conditions in (5) must be satisfied simultaneously. The first two conditions $p - z = 2^A$ and $p + z = 2^B$ yield $2p = 2^A + 2^B = 2^A(2^{B-A} + 1)$ implying that $A = 1$ since $2p$ is a multiple of 2 only. Hence $p = z + 2$. For the third condition we then have

$$p^2 + z^2 = 2z^2 + 4z + 4 = 2(z^2 + 2z + 2) = 2^C.$$

Since z is odd, the factor $(z^2 + 2z + 2)$ is odd. It then follows that

$$2(z^2 + 2z + 2) \neq 2^C.$$

The three conditions are not satisfied simultaneously, and the contradiction has been derived. Hence $p^4 - 2^y \neq z^4$.
This concludes case **(b)**.

(c) Suppose that p , q are distinct odd primes. If $p^4 - q^y = z^4$ has a solution, we have

$$q^y = p^4 - z^4 = (p^2 - z^2)(p^2 + z^2) = (p - z)(p + z)(p^2 + z^2).$$

Denote

$$p - z = q^A, \quad p + z = q^B, \quad p^2 + z^2 = q^C, \quad A < B < C, \quad q^y = q^{A+B+C}, \tag{6}$$

and all three conditions in (6) must be satisfied simultaneously.
The first two conditions in (6) $p - z = q^A$ and $p + z = q^B$ yield

$$2p = q^A + q^B = q^A(q^{B-A} + 1). \tag{7}$$

If $A \geq 1$ in (7), then $q \mid p$ which is impossible since $\gcd(p, q) = 1$. Thus $A \ngeq 1$. If $A = 0$ in (6), then $p - z = q^0 = 1$ or $p = z + 1$. The second condition then implies that $p + z = 2z + 1 = q^B$, whereas the third condition implies that $p^2 + z^2 = (z + 1)^2 + z^2 = 2z^2 + 2z + 1 = 2z^2 + q^B = q^C$. From $2z^2 + q^B = q^C$ and since $0 = A < B < C$, it then follows that $q \mid z$, contrary to the fact that $\gcd(q, z) = 1$. Therefore $2z^2 + q^B \neq q^C$, and $A \neq 0$. The three conditions are not satisfied simultaneously. Hence $p^4 - q^y \neq z^4$.

This concludes case **(c)** and the proof of Theorem 3.1. □

Final remark. We have shown for all distinct primes p, q and positive integers y, z that both equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$ have no solutions. The results were achieved in a simple and elementary manner.

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