All the Solutions of the Diophantine Equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$ when $p, q$ are Distinct Primes

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Abstract. In this paper, we consider the two equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$ when $p, q$ are distinct primes and $y, z$ are positive integers. For all primes $p, q$ we establish that both equations have no solutions.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions.

The famous general equation $p^x + q^y = z^2$ has many forms. The literature contains a very large number of articles on non-linear such individual equations involving particular primes and powers of all kinds.

In this paper we consider the two equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$. By using elementary methods, we establish for all distinct primes $p, q$ that both equations have no solutions. This is done in the respective Sections 2 and 3. Although existing similarities, we nevertheless consider both equations separately, in which all theorems and cases are self-contained.

2. All the solutions of $p^4 + q^y = z^4$ when $p, q$ are distinct primes

In this section, we discuss the equation $p^4 + q^y = z^4$ and its solutions. This is done in the following theorem.

Theorem 2.1. Let $y, z$ be positive integers. For all three possibilities
(a) $p = 2$ and $q$ an odd prime,
(b) $p$ an odd prime and $q = 2$,
(c) $p, q$ distinct odd primes,
the equation $p^4 + q^y = z^4$ has no solutions.

Proof: All three cases are considered separately, and are self-contained.
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(a) Suppose that \( p = 2 \) and \( q \) is an odd prime. If for some prime \( q \), there exist values \( y \) and \( z \) satisfying
\[
2^4 + q^y = z^4,
\]
then \( q^y = z^4 - 2^4 \) or
\[
q^y = (z^2 - 2^2)(z^2 + 2^2) = (z - 2)(z + 2)(z^2 + 2^2).
\]
Denote
\[
\begin{align*}
z - 2 &= q^4, \\
z + 2 &= 2q^4, \\
z^2 + 2^2 &= q^4, \\
A < B < C, \quad q^y &= q^{A+B+C}.
\end{align*}
\] (1)

From (1), \( z = 2 + q^4 \). If \( A = 0 \), then \( z = 3 \). Hence \( z + 2 = 5 = q^4 \) and \( q = 5 \), \( B = 1 \). The equation \( 2^4 + q^y = z^4 \) then yields \( 2^4 + 5^4 = 3^4 \) which has no solutions. Thus \( A \neq 0 \). When \( A > 0 \), then from (1) we have \( z = 2 + q^4 \) and \( 4 + q^y = q^6 \) implying that \( q \mid 4 \) which is impossible since \( q \) is an odd prime. Therefore \( A \nmid 0 \), and the conditions in (1) are not satisfied. Hence, when \( p = 2 \) and \( q \) is an odd prime, then \( 2^4 + q^y \neq z^4 \).

Part (a) is complete.

(b) Suppose that \( p \) is an odd prime and \( q = 2 \). If for some prime \( p \), there exist values \( y \) and \( z \) for which
\[
p^4 + 2^y = z^4,
\]
then
\[
2^y = z^4 - p^4 = (z^2 - p^2)(z^2 + p^2) = (z - p)(z + p)(z^2 + p^2).
\]
Denote
\[
\begin{align*}
z - p &= 2^A, \\
z + p &= 2^B, \\
z^2 + p^2 &= 2^C, \\
A < B < C, \quad 2^y &= 2^{A+B+C}.
\end{align*}
\] (2)

where all three conditions in (2) must be satisfied simultaneously.

The first two conditions \( z - p = 2^A \) and \( z + p = 2^B \) yield \( 2p + 2^A = 2^B \) or
\[
2p = 2^B - 2^A = 2^B(2^{B-A} - 1).
\]
The product \( 2p \) is a multiple of \( 2 \) only implying that \( A = 1 \). Thus \( z = p + 2 \). For the third condition we then obtain
\[
z^2 + p^2 = 2p^2 + 4p + 4 = 2(p^2 + 2p + 2) = 2^C.
\]
Since \( (p^2 + 2p + 2) \) is odd for all primes \( p \), it follows that
\[
2(p^2 + 2p + 2) \neq 2^C.
\]
The three conditions are not satisfied simultaneously, and therefore \( p^4 + 2^y \neq z^4 \).

This concludes case (b).

(c) Suppose that \( p, q \) are distinct odd primes. If there exist primes \( p, q \) and values \( y, z \) which satisfy \( p^4 + q^y = z^4 \), then
\[
q^y = z^4 - p^4 = (z^2 - p^2)(z^2 + p^2) = (z - p)(z + p)(z^2 + p^2).
\]
Denote
\[
\begin{align*}
z - p &= q^4, \\
z + p &= q^6, \\
z^2 + p^2 &= q^4, \\
A < B < C, \quad q^y &= q^{A+B+C}.
\end{align*}
\] (3)

The first two conditions in (3) namely \( z - p = q^4 \) and \( z + p = q^6 \) yield \( 2p + q^4 = q^6 \).

If \( A = 0 \), then \( q^4 = q^6 = 1 \) and \( z = p + 1 \). Thus \( 2p + 1 = q^6 \). The third condition yields
\[
z^2 + p^2 = 2p^2 + 2p + 1 = 2p^2 + q^6 = q^y.
\]
and \( 0 = A < B < C \) then imply that \( q \mid p \) which is impossible. Hence \( A \neq 0 \). When \( A > 0 \), it follows from the equality \( 2p + q^4 = q^6 \) that \( q \mid p \) which is impossible. Thus \( A \nmid 0 \). The conditions in (3) are not satisfied simultaneously. Therefore \( p^4 + q^y \neq z^4 \).

This concludes case (c) and the proof of Theorem 2.1. \( \square \)
All the solutions of the Diophantine Equations $p^4 + q^4 = z^4$ and $p^4 - q^4 = z^4$ when $p, q$ are Distinct Primes

3. All the solutions of $p^4 - q^4 = z^4$ when $p, q$ are distinct primes

In this section, we consider in Theorem 3.1 the solutions of the equation $p^4 - q^4 = z^4$.

**Theorem 3.1.** Let $y, z$ be positive integers. For all three possibilities

(a) $p = 2$ and $q$ an odd prime,

(b) $p$ an odd prime and $q = 2$,

(c) $p, q$ distinct odd primes,

the equation $p^4 - q^4 = z^4$ has no solutions.

**Proof:** All three cases are considered separately, and are self-contained.

(a) Suppose that $p = 2$ and $q$ an odd prime. We have

$$2^4 - q^4 = (p^4 - z^4)(p^4 + z^4) = (p - z)(p + z)(p^2 + z^2).$$

One could easily see that the above equation has no solutions. Hence $2^4 - q^4 ≠ z^4$.

(b) Suppose that $p$ is an odd prime and $q = 2$. We shall assume that

$$p^4 - 2^4 = z^4$$

has a solution, and reach a contradiction. For any solution of (4), the value $z$ is odd. We then have

$$z^4 = (p^4 - z^4)(p^2 + z^2) = (p - z)(p + z)(p^2 + z^2).$$

Denote

$$p - z = 2^A, \quad p + z = 2^B, \quad p^2 + z^2 = 2^C, \quad \text{where all three conditions in (5) must be satisfied simultaneously.}$$

The first two conditions $p - z = 2^A$ and $p + z = 2^B$ yield $2p = 2^A + 2^B = 2^4(2^{B-A} + 1)$ implying that $A = 1$ since $2p$ is a multiple of 2 only. Hence $p = z + 2$. For the third condition we then have

$$p^2 + z^2 = 2^C.$$ 

Since $z$ is odd, the factor $(z^2 + 2z + 2)$ is odd. It then follows that

$$2(z^2 + 2z + 2) ≠ 2^C.$$ 

The three conditions are not satisfied simultaneously, and the contradiction has been derived. Hence $p^4 - 2^4 ≠ z^4$.

This concludes case (b).

(c) Suppose that $p, q$ are distinct odd primes. If $p^4 - q^4 = z^4$ has a solution, we have

$$q^4 = (p^4 - z^4)(p^2 + z^2) = (p - z)(p + z)(p^2 + z^2).$$

Denote

$$p - z = q^A, \quad p + z = q^B, \quad p^2 + z^2 = q^C, \quad \text{where all three conditions in (6) must be satisfied simultaneously.}$$

The first two conditions in (6) $p - z = q^A$ and $p + z = q^B$ yield

$$2p = q^A + q^B = q^C(q^{B-A} + 1).$$

If $A ≥ 1$ in (7), then $q | p$ which is impossible since $\gcd(p, q) = 1$. Thus $A ≤ 1$. If $A = 0$ in (6), then $p - z = q^0 = 1$ or $p = z + 1$. The second condition then implies that $p + z = 2z + 1 = q^B$, whereas the third condition implies that $p^2 + z^2 = (z + 1)^2 + z^2 = 2z^2 + 2z + 1 = 2z^2 + q^B = q^C$. From $2z^2 + q^B = q^C$ and since $0 = A < B < C$, it then follows that $q | z$, contrary to the fact that $\gcd(q, z) = 1$. Therefore $2z^2 + q^B ≠ q^C$, and $A ≠ 0$. The three conditions are not satisfied simultaneously. Hence $p^4 - q^4 ≠ z^4$.

This concludes case (c) and the proof of Theorem 3.1. □
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**Final remark.** We have shown for all distinct primes $p, q$ and positive integers $y, z$ that both equations $p^4 + q^y = z^4$ and $p^4 - q^y = z^4$ have no solutions. The results were achieved in a simple and elementary manner.

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**REFERENCES**