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The Mathematical Description of Homogeneous Turbulence for Incompressible Fluids

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Abstract. The present article establishes the mathematical description of turbulence for incompressible fluids as a response to an important question: "Is a complete mathematical description of turbulence possible?" in ResearchGate.net website discussion started August 26, 2018. Since inhomogeneous turbulence is too complex, this article will limit the discussion to homogeneous turbulence. This article provides the necessary propositions for a mathematical description of homogeneous turbulence for incompressible fluids which is achievable via a probabilistic formulation, deterministic chaos, and the Kolmogorov's -5/3 law.

Keywords: Turbulence, Chaos, Kolmogorov's -5/3 law

AMS Mathematics Subject Classification (2010): 76F02

1. Introduction

Dr. Ilya M. Peshkov asked an important question: "Is a complete mathematical description of turbulence possible?" in ResearchGate.net website discussion started August 26, 2018 [1]. This article will limit the mathematical description to homogeneous turbulence since the complexity of inhomogeneous turbulence is dependent on the particular physical process generating the fluid inhomogeneous turbulence and its quite complex.

Additionally, it is conjectured the Navier-Stokes's momentum equation form a Lotka-Volterra system from the perspective of Sir Oliver Heaviside's operational calculus used in diffusion of electric displacement [11]. Sir Oliver Heaviside defined and used the generalize functions such as unit step, and the so-called Dirac delta function prior it was fashionable. He invented operational calculus back in the 19th century [11].

The Lotka-Volterra system representation of the Navier Stokes momentum equations compares the component velocities as if they were preys (3 different types of sheep populations) competing for part of the fluid energy (grass) since in this case there is no predator. An alternative point of view to Navier Stokes momentum interpretation would be a matrix Riccati equation but this would not be pursued in this article. The next section discusses the propositions for the mathematical description of incompressible fluid homogeneous turbulence.

2. Propositions for the mathematical description of incompressible fluid homogeneous turbulence

The following propositions may establish a mathematical description of homogeneous turbulence for incompressible fluids via the following:

1) The turbulence modelled by of the Navier-Stokes's momentum equations means that the actual realizable Navier Stokes momentum equation solution is of a probabilistic formulation [3,6].
2) The non-linear advective flow within the momentum Navier-Stokes's momentum equation is conjectured to form a Lotka-Volterra system of equations for the fluid velocity field, which provides a deterministic view to the chaotic vortices of turbulence [8, 9, 10].
3) The homogeneous turbulence power spectrum (or *Spectrum*), $E(k)$, due to the fluid's spatial mean kinetic energy per unit mass, is given by Kolmogorov's -5/3 Law [7] within the inertial range of the fluid
$$E(k) \propto \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$$
where ϵ is the ensemble mean kinetic energy per unit mass dissipation rate and k is the wavenumber.

The three propositions listed above provide the complete mathematical description of homogeneous turbulence provided by the Navier-Stokes's momentum equations. These propositions one and three are well-known in the academic literature and there are too many literature references which expand this concept to be included in this article. But a common literature point of view is the perspective in which randomness needs to be *added* to a deterministic mean flow, i.e., Reynolds's equations which are said to model homogeneous turbulence using the deterministic Navier-Stokes's equations. This linear additive idea or super position is not the perspective of this article since then the mathematical completeness of the nature of homogeneous turbulence is destroyed via the nonlinear advective property found in proposition number two above. According to proposition one, the nature of homogeneous turbulence is built-in into the velocity field solution of the Navier-Stokes within a probability certainty which is analogous to the commonly known perspective of the probability of an electron might be located somewhere within a hydrogen atom as described by Quantum Theory Schroeder equation. The proposition number two above implies the Navier-Stokes's equation could be represented as a nonlinear Lotka-Volterra system of differential equations with time constant parameters operators. The Laplacian operator, $\mathrm{q}^2 = \Delta = \sum_n q_n^2$, and spatial gradient, $q_n = \frac{\partial}{\partial x_n}$, are treated as scalars for a given dimension n with respect to partial time derivative operator as Sir Oliver Heaviside would have done.

$$\frac{\partial u_k}{\partial t} = \nu q^2 u_k - \sum\nolimits_n q_n u_n u_k - q_k\{\phi + \frac{p}{\rho_o}\}$$

The above Lotka-Volterra system of differential equations in time need to be solved as an integral equation which would involve the functional operators of the spatial derivatives along with the divergence of the velocity field which is zero, $\sum_n q_n u_n = 0$.

This perspective introduces the fluid velocity components taking part in a predator-prey type system in which each velocity component is a prey competing for homogeneous turbulence energy and the homogeneous turbulence could be seen as a dynamic set, maybe a Mandelbrot fractal, developed by the nonlinear Lotka-Volterra system of differential equations. This aspect of chaos introduces the vorticity nature of homogeneous turbulence which is not present in proposition one above. Even if we could solve the nonlinear Lotka-Volterra system of "differential" equations it would yield an integral equation involving spatial operators as "parameters". Currently, the Navier-Stokes's momentum equations full nontrivial solution remains unknown unless a simplifying assumption is made, for example zero initial conditions [3] which provides the linearizing condition, $\sum_n q_n u_n u_k = 0$. A specialized solution of the Navier-Stokes with zero initial conditions or null incompressible Navier Stokes equations given the input pressure and external forces are known has been solved in the reference article [3] shows the nature of the Greens function kernel $(e^{t\nu\Delta} - 1)$ is essentially related to the normal Gaussian distribution function. The exponential Laplacian operator, $e^{t\nu\Delta}$, operating on some function is equivalent to the solution of the heat equation with the initial condition being the function, according to Professor Terence Tao in reference [5] in page 39. By integrating by parts in the time integral specified in Theorem 1 of reference [3] and using the relationship provided by [5] to obtain the null velocity field in terms of Gaussian distribution expectations as shown below provided the null velocity solution, U_k, satisfies $\sum_n U_n \frac{\partial U_k}{\partial x_n} = 0$.

$$U_k\big(t, x_k(t, x_{ok})\big) = -W_k(0, x_{ok}) + \int_{\mathbb{R}^3} \frac{dy}{(4\pi\nu t)^{\frac{3}{2}}} e^{-\sum_{m=1}^{3} \frac{(x_m(t,x_{ok}) - y_m)^2}{4\nu t}} W_k(0, y_{ok}) - \int_0^t d\tau \int_{\mathbb{R}^3} dy \frac{e^{-\sum_{m=1}^{3} \frac{(x_m(t-\tau,x_{ok}) - y_m)^2}{4\nu(t-\tau)}}}{(4\pi\nu(t-\tau))^{\frac{3}{2}}} \frac{\partial}{\partial y_k}\left(\phi(y_k) + \frac{p(y_k)}{\rho_o}\right)$$

Where $\psi_k\big(\tau, x_k(\tau, x_{ok})\big)$ is set to zero in Eq. 7 of reference [3] and $W_k\big(\tau, x_k(\tau, x_{ok})\big)$ is given as follows:

$$W_k\big(\tau, x_k(\tau, x_{ok})\big) = \iiint\limits_{V(\tau)} \left\{\frac{\partial}{\partial Y_k}\left(\phi(Y_k) + \frac{p(Y_k)}{\rho_o}\right)\right\} \frac{-dY}{4\pi\nu\sqrt{\sum_n (x_n(\tau, x_{ok}) - Y_n)^2}}$$

Note that the units of W_k are in meter per second. The system is not closed because in reference [3] the mathematical assumption was made that the input pressure force was treated as a known mathematical input, i.e., the solution is given if the pressure function is known function of time and space which typically is not known. An interesting feature of reference [3] in footnote #7 is a back of the envelope calculation which approximates the time scale validity, $T=R^2/\nu$, of the solution as a function of a radius, R, and kinematic viscosity, ν. The implication is that after that the time scale, T, the Navier-Stokes's equation are not valid due to the singularity of the Laplace form of the kernel. In order to extend the validity of the solution complex analysis may be use since this singularity is a simple pole, i.e., order one, which was not done in this article. But this "not valid" interpretation is not the correct interpretation; the proper interpretation should be a loss of accuracy due to probabilistic interpretation instead of an invalid equation interpretation. Additionally, the incompressible Navier Stokes's momentum equations can be integrated to obtain the Bernoulli equation or first integral as demonstrated in reference [2] to obtain the Bernoulli equation of the given flow field which is valid along a given stream lines of the fluid velocity field. Reference [4] provides the Laplace equation of the classical Lagrangian, L, of the incompressible fluid as an additional equation (*Eq. 16* in [4]) which is valid for every instant of time. For a fixed time, the Laplace equation of the classical Lagrangian could be solved with harmonic solutions which can be added up in space due to linearity of the Laplace equation, but not across time due to nonlinearity of the time field derivative or time material derivative in the incompressible Navier-Stokes's momentum equations. Since the external potential forces, $\phi(Y_k)$, are assumed to be harmonic, $\Delta\phi = 0$, they can be eliminated from the Laplace equation without consequence and taking the ensemble mean, therefore the Laplace equation is given as $\nu\Delta_x\{0.5u_i^2 - p/\rho_o\} = 0$.[1] By taking the ensemble mean within the inertial range and the spatial Fourier transform of the ensemble mean Laplace equation, $< \nu\Delta_x\{0.5u_i^2 - p/\rho_o\} >_{\overrightarrow{r_n}} = 0$, assuming the wavenumber, k, is nonzero this implies the power spectrum of the spatial mean potential energy per unit mass equals the power spectrum of the spatial mean kinetic energy per unit mass which is given by proposition three to be Kolmogorov -5/3 Law.

3. Conclusion

As long as the external conservative forces potential are not included since their Laplacian is zero, this article concludes the spectrum of the ensemble mean spatial fluid pressure per unit mass follows Kolmogorov's -5/3 Law $\propto \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ within the inertial range. Additionally, in the articles [2,3,4] although do not actually mention turbulence directly, yet this does not imply that homogeneous turbulence does not exist within the Navier-Stokes's momentum equation for Newtonian fluids. This article concludes the Navier Stokes's momentum equation can model homogeneous turbulence for Newtonian fluids and the Navier-Stokes's equations need to be solved from a probabilistic and deterministic chaos point of view provided by the Lotka-Volterra system of equations. Therefore, the incompressible Navier-Stokes's equations have enough complexity and nonlinearity to be able to model Newtonian fluids in homogeneous turbulence state.

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[1] $\left\{0.5u_i^2 - \frac{p}{\rho_o}\right\} = \{0.5u_i^2(\vec{x}(t) + \vec{r}_n, t) - \frac{p(\vec{x}(t)+\vec{r}_n)}{\rho_o}\}$

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