Annals of Pure and Applied Mathematics
Vol. 25, No. 1, 2022, 17-40
ISSN: 2279-087X (P), 2279-0888(online)
Published on 13 March 2022
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v25n1a03858

Annals of Pure and Applied Mathematics

Solution of the Quadratic Integral Equation by Homotopy Analysis Method

***R.K.Bairwa*[1*] and *Ajay Kumar*[2]**

[1,2]Department of Mathematics, University of Rajasthan,
Jaipur - 302004, Rajasthan, India.
[1]email: dr.rajendra.maths@gmail.com
[2]email: jangir.kmrajay@gmail.com
[*]Corresponding author

Received 2 February 2022; accepted 12 March 2022

Abstract. In the present paper, we derive an approximate solution of the quadratic integral equation by using the homotopy analysis method (HAM). This approach provides a solution in the form of a rapidly converging series, and it includes an auxiliary parameter that controls the series solution's convergence. We compare the HAM solution with the exact solution graphically. Additionally, an absolute error comparison between the exact and HAM solutions is performed. The findings indicate that HAM is a very straightforward and attractive approach for computation.

Keywords: Quadratic integral equation; Homotopy analysis method; Auxiliary parameter.

AMS Mathematics Subject Classification (2010): 35C10, 45G10

1. Introduction

The quadratic integral equation is a particular form of a nonlinear integral equation. It is a very important and applicable equation in various fields such as mathematical physics, biology, economics, engineering and technology. The quadratic integral equation is also applicable in the radiative transformer's theory, kinetic theory of gases, the theory of neutron transport and traffic theory [6,8,11, 12, 20,] and some other applications, which are given in references [5, 7, 15, 17].

In recent years, various analytical and numerical methods, such as the Adomian decomposition method (ADM) [18], the variational homotopy perturbation method (VHPM) [3], the differential transform method (DTM) [29], the homotopy perturbation method (HPM)[9], modified Laplace adomian decomposition method (MLADM) [19], etc., have been developed to solve integral equations and integro-differential equations. The homotopy analysis method (HAM) is a powerful and reliable analytical technique for solving different kinds of linear and nonlinear integral and differential equations. It is based on homotopy, a fundamental concept in topology and differential geometry [24, 26, 28]. It is a computational approach that produces analytical solutions and offers a number of benefits over more traditional numerical methods. It is free from rounding off errors since it does not need discretization, and it does not require a great amount of computer memory or power to implement. The method introduces the solution in the form of a convergent series with elegantly computable terms. This approach has been employed effectively to solve linear integral equations [2], linear and nonlinear integro-difference equations [1, 5, 30], nonlinear differential equations [25, 27], and linear and nonlinear fractional differential equations [22, 31, 32] etc.

In the present work, we consider the following quadratic integral equation of the type

$$x(t)=a(t)+g\left(t,x(t)\right)\int_0^t f\left(s,x(s)\right)ds, \tag{1}$$

where $a(t)$ is a specific continuous function, f and g have partial derivatives of arbitrary order with respect to their second arguments.

The key goal of this work is to apply the Homotopy analysis method for solving quadratic integral equations and to demonstrate the suggested method's correctness, efficiency, and simplicity.

2. Basic idea of homotopy analysis method

To explain the fundamental concept of the Homotopy analysis method (HAM) [24], we consider the following operator equation as follows

$$N\left[x(t)\right]=0, \tag{2}$$

where N is a non-linear operator, $x(t)$ is an unknown function and t is an independent variable. For simplicity, we ignore all boundary and initial conditions, which can be treated in a similar way. By means of generalizing the traditional homotopy method, Liao [26] constructed the so-called zero-order deformation equation as

$$(1-q)L\left[\phi(t;q)-x_0(t)\right]=q\hbar\, H(t)N\left[\phi(t;q)\right], \tag{3}$$

where $q\in[0,1]$ is the embedding parameter, $\hbar\neq 0$ is called the convergence control auxiliary parameter, $H(t)\neq 0$ is an auxiliary function, L is an auxiliary linear operator with the property $L[0]=0$, $x_0(t)$ is an initial guess of $x(t)$ and $x(t;q)$ is an unknown function. It is important that one has great freedom to choose the auxiliary things in HAM.

When, $q=0$, due to the property $L[0]=0$, the deformation equation (3) becomes

$$\phi(t;0)=x_0(t), \tag{4}$$

and when $q=1$, (since $\hbar\neq 0$ and $H(t)\neq 0$ almost everywhere) the zero-order deformation equation (3) on using equation (2) gives

$$\phi(t;1)=x(t). \tag{5}$$

Thus according to (4) and (5), as the embedding parameter q increases from 0 to 1, $x(t;q)$ varies from the initial approximation $x_0(t)$ to the exact solution $x(t)$. Using the parameter q, we expand $x(t;q)$ in Taylor series as follows

$$\phi(t;p) = x_0(t) + \sum_{m=1}^{\infty} x_m(t)\, q^m, \tag{6}$$

where

$$x_m(t) = \frac{1}{m!}\left[\frac{\partial^m \phi(t;q)}{\partial q^m}\right]_{q=0}. \tag{7}$$

We assume that linear operator L, the initial guess $x_0(t)$, the convergence control parameter $\hbar$ and the auxiliary function $H(t)$ are properly chosen such that the series (6) is convergent at $q=1$. Now, taking $q=1$ in equation (6) and using equation (5), we get

$$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t) \tag{8}$$

Following Liao [24, 26, 28], differentiating equation (3) m times with respect to the embedding parameter q and then, setting $q=0$, and finally dividing by $m!$, we obtain the so-called m^{th}-order deformation equation as

$$L\left[x_m(t) - \kappa_m\, x_{m-1}(t)\right] = \hbar\, H(t) R_m\left[\vec{x}_{m-1}(t)\right], \tag{9}$$

where

$$R_m\left[\vec{x}_{m-1}\right] = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1} N\left[\phi(t;q)\right]}{\partial q^{m-1}}\right]_{q=0} \tag{10}$$

$\vec{x}_{m-1}$ stands for the vector $\vec{x}_{m-1} = \{x_0(t), x_1(t), x_2(t), ..., x_{m-1}(t)\}$,

and

$$\kappa_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}. \tag{11}$$

Applying the inverse operator L^{-1} on the both side of the equation (9), we have

$$x_m(t) = \kappa_m\, x_{m-1}(t) + \hbar\, L^{-1}\left[H(t) R_m\left[\vec{x}_{m-1}(t)\right]\right]. \tag{12}$$

In this way, it is easy to obtain $x_m(t)$ for $m \ge 1$, thus getting the solution from equation (8).

3. Solution of the quadratic integral equation by using homotopy analysis method

In this part, we have obtained the approximate solution of the quadratic integral equation by the application of the HAM method.

Example 1. Consider the following quadratic integral equation, given by

$$x(t) = t^2 - \frac{t^{10}}{35} + \frac{t}{5} x(t) \int_0^t s^2 x^2(s)\, ds \tag{13}$$

The equation has the exact solution

$$x(t) = t^2 . \tag{14}$$

Applying HAM Method to equation (13) and taking the linear operator as

$$L\left[\phi(t;q)\right] = \phi(t;q) \tag{15}$$

and nonlinear operator as

$$N\left[\phi(t;q)\right] = \phi(t;q) - \left(t^2 - \frac{t^{10}}{35}\right) - \frac{t}{5}\phi(t;q)\int_0^t s^2\phi^2(s;q)\,ds \tag{16}$$

For the above operator, with assumption $H(t)=1$, we construct the zeroth-order deformation equation from equation (13) as

$$(1-q)L\left[\phi(t;q) - x_0(t)\right] = q\,\hbar\,H(t)\,N\left[\phi(t;q)\right], \tag{17}$$

and the m^{th}-order deformation equation is given by

$$L\left[x_m(t) - \kappa_m\, x_{m-1}(t)\right] = \hbar R_m\left[\vec{x}_{m-1}(t)\right], \tag{18}$$

where

$$R_m\left[\vec{x}_{m-1}\right] = \frac{1}{(m-1)!}\left[\frac{\partial^{m-1} N\left[\phi(t;q)\right]}{\partial q^{m-1}}\right]_{q=0} \quad \text{and} \quad \kappa_m = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases} . \tag{19}$$

By using equation (19), we evaluate the value of $R_m\left[\vec{x}_{m-1}\right]$

for $m=1$

$$R_1\left[\vec{x}_0\right] = x_0(t) - \left(t^2 - \frac{t^{10}}{35}\right) - \frac{t}{5}x_0(t)\int_0^t s^2 x_0^{\,2}(s)\,ds , \tag{20}$$

for $m=2$

$$R_2\left[\vec{x}_1\right] = x_1(t) - \frac{t}{5}\frac{1}{1!}\frac{\partial}{\partial q}\left[\phi(t;q)\int_0^t s^2\phi^2(s;q)\,ds\right]_{q=0} , \tag{21}$$

and so on.

For $m \in N$ $(m \neq 1)$

$$R_m\left[\vec{x}_{m-1}\right] = x_{m-1}(t) - \frac{t}{5}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi(t;q)\int_0^t s^2\phi^2(s;q)\,ds\right]_{q=0} . \tag{22}$$

Hence, the value of $R_m\left[\vec{x}_{m-1}\right]$ for $m \in N$ is given by

$$\begin{aligned} R_m\left[\vec{x}_{m-1}\right] &= x_{m-1}(t) - (1-\kappa_m)\left(t^2 - \frac{t^{10}}{35}\right) \\ &\quad - \frac{t}{5}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi(t;q)\int_0^t s^2\phi^2(s;q)\,ds\right]_{q=0} . \end{aligned} \tag{23}$$

Appling the operator L^{-1} on both sides of equation (18), we have

$$x_m(t) = \kappa_m x_{m-1}(t) + \hbar L^{-1}\left[R_m(\vec{x}_{m-1})\right]. \tag{24}$$

Now, taking initial solution as

$$x_0(t) = t^2 - \frac{t^{10}}{35} \tag{25}$$

The equation (24) is a recurrence relation for $m \geq 1$, using equation (25) in equation (24), we get the value of $x_m(t)$ for $m = 1, 2, 3, \ldots$ as follows

$$x_1(t) = -\hbar\left[\frac{1}{35}t^{10} - \frac{29}{18375}t^{18} + \frac{61}{2113125}t^{26} - \frac{1}{4930625}t^{34}\right], \tag{26}$$

$$x_2(t) = -\hbar(1+\hbar)\left[\frac{1}{35}t^{10} - \frac{29}{18375}t^{18} + \frac{61}{2113125}t^{26} - \frac{1}{4930625}t^{34}\right] + \hbar^2\left[\frac{29}{18375}t^{18} - \frac{1927}{14791875}t^{26} + \frac{150434}{34391109375}t^{34} - \frac{400574}{5215984921875}t^{42} + \frac{387353}{545659325859375}t^{50} - \frac{93}{32646284453125}t^{58}\right], \tag{27}$$

and so on. The other rest of the components may be determined likewise.
Therefore, the HAM series solution of equation (13) is given by

$$x(t) = x_0(t) + \sum_{m=1}^{\infty} x_m(t) = x_0(t) + x_1(t) + x_2(t) +, \ldots,$$

$$x(t) = t^2 - \frac{1}{35}\left(1 + 2\hbar + \hbar^2\right)t^{10} + \frac{58}{18375}\hbar(1+\hbar)t^{18} - \left(\frac{122}{2113125}\hbar + \frac{2354}{14791875}\hbar^2\right)t^{26} + \left(\frac{2}{4930625}\hbar + \frac{22487}{4913015625}\hbar^2\right)t^{34} - \frac{400574}{5215984921875}\hbar^2 t^{42} + \frac{387353}{545659325859375}\hbar^2 t^{50} - \frac{93}{32646284453125}\hbar^2 t^{58} +, \ldots, \tag{28}$$

If we put $\hbar = -1$, in equation (28), we have

$$x(t) = t^2 - 0 + 0 - \left(-\frac{122}{2113125} + \frac{2354}{14791875}\right)t^{26} + \left(-\frac{2}{4930625} + \frac{22487}{4913015625}\right)t^{34} - \frac{400574}{5215984921875}t^{42} + \frac{387353}{545659325859375}t^{50} - \frac{93}{32646284453125}t^{58} +, \ldots, \tag{29}$$

The result in (29) was obtained earlier by Fu *et al.* [18] using the method of ADM. It can be easily observed that for $\hbar = -1,$ the series solution converges to the exact solution $x(t) = t^2$ as $m \to \infty.$

In **Table 1**, we compare the approximate solution obtained by using HAM with the exact solution. **Figure 1** and **Figure 2** show the comparison graph of the approximate solution by the HAM with the exact solution and the absolute error graph between them, respectively. It is worth noting that the approximate solutions in all the figures and tables are evaluated using only three terms from the homotopy analysis solution.

Table 1: Numerical comparison of solution of the equation (13) by HAM with the exact solution at different values of t

t	**Exact Solution**	**HAM Solution**	$\lvert x_{\text{Exact}} - x_{\text{HAM}} \rvert$
0.1	1.000000000000000e-002	1.000000000000000e-002	0
0.2	4.000000000000001e-002	4.000000000000001e-002	0
0.3	9.000000000000002e-002	9.000000000000002e-002	0
0.4	1.600000000000000e-001	1.599999999999955e-001	4.579669976578771e-015
0.5	2.500000000000000e-001	2.499999999984892e-001	1.510819247485529e-012
0.6	3.600000000000000e-001	3.599999998271376e-001	1.728623910679517e-010
0.7	4.899999999999999e-001	4.899999905029860e-001	9.497013953030375e-009
0.8	6.400000000000001e-001	6.399996956247078e-001	3.043752923037602e-007
0.9	8.100000000000001e-001	8.099935631118501e-001	6.436888149941034e-006
1.0	1.000000000000000e+000	9.999026882850697e-001	9.731171493032598e-005

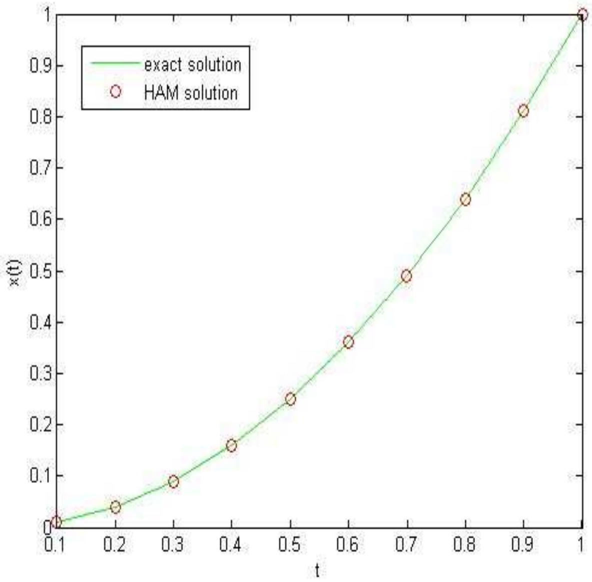

Figure: 1. Comparison Graph of HAM solution with exact solution of equation (13), when $\hbar = -1$

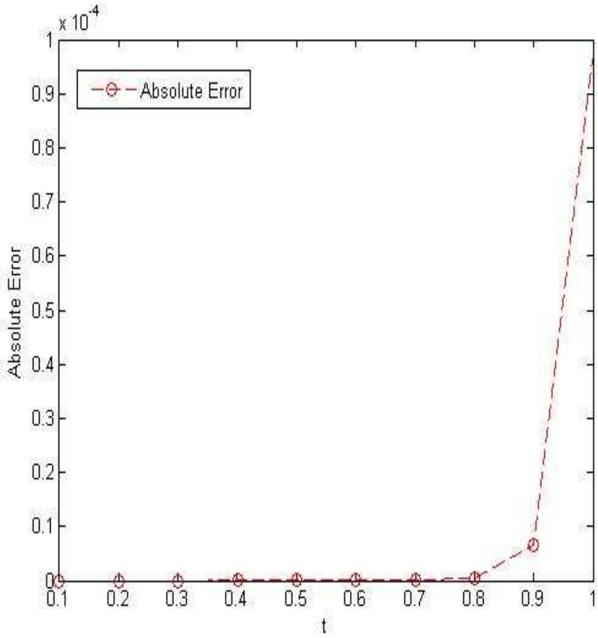

Figure 2: Graph of absolute error between HAM solution and exact solution of equation (13), when $\hbar = -1$

Example 2. Consider the following quadratic integral equation, given by

$$x(t)=t^3-\frac{t^{19}}{100}-\frac{t^{20}}{110}+\frac{t^3}{10}x^2(t)\int_0^t(s+1)x^3(s)\,ds \tag{30}$$

The equation has the exact solution

$$x(t)=t^3. \tag{31}$$

Applying HAM Method to equation (31) and taking the linear operator as

$$L[\phi(t;q)]=\phi(t;q), \tag{32}$$

and nonlinear operator as

$$N[\phi(t;q)]=\phi(t;q)-\left(t^3-\frac{t^{19}}{100}-\frac{t^{20}}{110}\right)-\frac{t^3}{10}\phi^2(t;q)\int_0^t(s+1)\phi^3(s;q)\,ds \tag{33}$$

For the above operator, with assumption $H(t)=1$, we construct the zeroth-order deformation equation from equation (13) as

$$(1-q)L[\phi(t;q)-x_0(t)]=q\,\hbar\,H(t)\,N[\phi(t;q)]. \tag{34}$$

and the m^th^-order deformation equation is given by

$$L[x_m(t)-\kappa_m\,x_{m-1}(t)]=\hbar R_m[\vec{x}_{m-1}(t)], \tag{35}$$

where

$$R_m[\vec{x}_{m-1}]=\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}N[\phi(t;q)]}{\partial q^{m-1}}\right]_{q=0} \text{ and } \kappa_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}. \tag{36}$$

By using equation (36), we evaluate the value of $R_m[\vec{x}_{m-1}]$

for $m=1$

$$R_1[\vec{x}_0]=x_0(t)-\left(t^3-\frac{t^{19}}{100}-\frac{t^{20}}{110}\right)-\frac{t^3}{10}x_0^{\,2}(t)\int_0^t(s+1)x_0^{\,3}(s)\,ds, \tag{37}$$

for $m=2$

$$R_2[\vec{x}_1]=x_1(t)-\frac{t^3}{10}\frac{1}{(1)!}\frac{\partial}{\partial q}\left[\phi^2(t;q)\int_0^t(s+1)\,\phi^3(s;q)\,ds\right]_{q=0}, \tag{38}$$

and so on.

For $m\in N$ $(m\neq 1)$

$$R_m[\vec{x}_{m-1}]=x_{m-1}(t)-\frac{t^3}{10}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi^2(t;q)\int_0^t(s+1)\,\phi^3(s;q)\,ds\right]_{q=0}. \tag{39}$$

Hence, the value of $R_m[\vec{x}_{m-1}]$ for $m\in N$ is given by

$$R_m\left[\vec{x}_{m-1}\right]=x_{m-1}(t)-(1-\kappa_m)\left(t^3-\frac{t^{19}}{100}-\frac{t^{20}}{110}\right)$$
$$-\frac{t^3}{10}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi^2(t;q)\int_0^t s^2\phi^3(s;q)\,ds\right]_{q=0}. \quad (40)$$

Appling the operator L^{-1} on both sides of equation (35), we have

$$x_m(t)=\kappa_m x_{m-1}(t)+\hbar L^{-1}\left[R_m\left(\vec{x}_{m-1}\right)\right]. \quad (41)$$

Now, taking initial solution as

$$x_0(t)=t^3-\frac{t^{19}}{100}-\frac{t^{20}}{110}. \quad (42)$$

The equation (41) is a recurrence relation for $m\geq 1$, using equation (42) in equation (41),we get the value of $x_m(t)$ for $m=1,2,3,\ldots$ as follows

$$x_1(t)=-\hbar\left[\frac{1}{100}t^{19}+\frac{1}{110}t^{20}-\frac{41}{130000}t^{35}-\frac{19}{33000}t^{36}-\frac{89}{338800}t^{37}\right.$$
$$+\frac{183}{45500000}t^{51}+\frac{10177}{922350000}t^{52}+\frac{5639}{559020000}t^{53}+\frac{859}{279510000}t^{54}$$
$$-\frac{727}{26390000000}t^{67}-\frac{768101}{7618611000000}t^{68}-\frac{10807001}{78123045000000}t^{69}$$
$$-\frac{527271}{6251707000000}t^{70}-\frac{320851}{16211580000000}t^{71}+\frac{43}{406000000000}t^{83}$$
$$+\frac{2742959}{5665121000000000}t^{84}+\frac{20924839}{23637229000000000}t^{85}+\frac{102302983}{1580788700000000}t^{86}$$
$$+\frac{1688987}{4568211615000000}t^{87}+\frac{23}{340403250000}t^{88}-\frac{1}{5800000000000}t^{99}$$
$$-\frac{1779}{1882100000000000}t^{100}-\frac{89391}{41406200000000000}t^{101}-\frac{125959}{47902690000000000}t^{102}$$
$$\left.-\frac{99741}{55372262000000000}t^{103}-\frac{501}{761368602500000}t^{104}-\frac{1}{9985162000000}t^{105}\right], \quad (43)$$

and so on. The other rest of components may be derived accordingly.
Therefore, the HAM series solution of problem (30) is given by

$$x(t)=t^3-(\hbar+1)\left(\frac{1}{100}t^{19}+\frac{1}{110}t^{20}\right)-\hbar\left[-\frac{41}{130000}t^{35}-\frac{19}{33000}t^{36}-\frac{89}{338800}t^{37}\right.$$
$$+\frac{183}{45500000}t^{51}+\frac{10177}{922350000}t^{52}+\frac{5639}{559020000}t^{53}+\frac{859}{279510000}t^{54}$$
$$-\frac{727}{26390000000}t^{67}-\frac{768101}{7618611000000}t^{68}-\frac{10807001}{78123045000000}t^{69}$$
$$-\frac{527271}{6251707000000}t^{70}-\frac{320851}{16211580000000}t^{71}+\frac{43}{406000000000}t^{83}$$
$$+\frac{2742959}{5665121000000000}t^{84}+\frac{20924839}{23637229000000000}t^{85}+\frac{102302983}{158078870000000}t^{86}$$
$$+\frac{1688987}{4568211615000000}t^{87}+\frac{23}{340403250000}t^{88}-\frac{1}{5800000000000}t^{99}$$
$$-\frac{1779}{18821000000000000}t^{100}-\frac{89391}{41406200000000000}t^{101}-\frac{125959}{479026900000000000}t^{102}$$
$$\left.-\frac{99741}{553722620000000000}t^{103}-\frac{501}{761368602500000}t^{104}-\frac{1}{9985162000000}t^{105}\right]+,\ldots, \tag{44}$$

If we put $\hbar=-1,$ in equation (44), we have

$$x(t)=t^3-0+\left[-\frac{41}{130000}t^{35}-\frac{19}{33000}t^{36}-\frac{89}{338800}t^{37}+\frac{183}{45500000}t^{51}+\frac{10177}{922350000}t^{52}+\frac{5639}{559020000}t^{53}\right.$$
$$+\frac{859}{279510000}t^{54}-\frac{727}{26390000000}t^{67}-\frac{768101}{7618611000000}t^{68}-\frac{10807001}{78123045000000}t^{69}-\frac{527271}{6251707000000}t^{70}$$
$$-\frac{320851}{16211580000000}t^{71}+\frac{43}{406000000000}t^{83}+\frac{2742959}{5665121000000000}t^{84}+\frac{20924839}{23637229000000000}t^{85}$$
$$+\frac{102302983}{158078870000000}t^{86}+\frac{1688987}{4568211615000000}t^{87}+\frac{23}{340403250000}t^{88}-\frac{1}{5800000000000}t^{99}$$
$$-\frac{1779}{18821000000000000}t^{100}-\frac{89391}{41406200000000000}t^{101}-\frac{125959}{479026900000000000}t^{102}$$
$$\left.-\frac{99741}{553722620000000000}t^{103}-\frac{501}{761368602500000}t^{104}-\frac{1}{9985162000000}t^{105}\right]+,\ldots, \tag{45}$$

The result in (45) was obtained earlier by Fu *et al.* [18] using the method of ADM. It can be easily observed that for $\hbar=-1,$ the series solution converges to the exact solution $x(t)=t^3$ as $m\to\infty$.

In **Table 2**, we compare the approximate solution obtained by using HAM with the exact solution. **Figure 3** and **Figure 4** show the comparison graph of the approximate solution by the HAM with the exact solution and the absolute error graph between them, respectively. It is worth noting that the approximate solutions in all the figures and tables are evaluated using only two terms from the homotopy analysis solution.

Table 2: Numerical comparison of solution of the equation (30) by HAM with the exact solution at different values of t

T	**Exact Solution**	**HAM Solution**	$\lvert x_{\text{Exact}} - x_{\text{HAM}} \rvert$
0.1	1.000000000000000e-003	1.000000000000000e-003	0
0.2	8.000000000000002e-003	8.000000000000002e-003	0
0.3	2.700000000000001e-002	2.700000000000001e-002	0
0.4	6.400000000000002e-002	6.400000000000002e-002	0
0.5	1.250000000000000e-001	1.249999999999805e-001	1.947053629436368e-014
0.6	2.160000000000000e-001	2.159999999870141e-001	1.298591789655745e-011
0.7	3.429999999999999e-001	3.429999967911216e-001	3.208878274207905e-009
0.8	5.120000000000001e-001	5.119996172603906e-001	3.827396095612556e-007
0.9	7.290000000000001e-001	7.289739210648457e-001	2.607893515438331e-005
1.0	1.000000000000000e+000	9.988740780250106e-001	1.125921974989397e-003

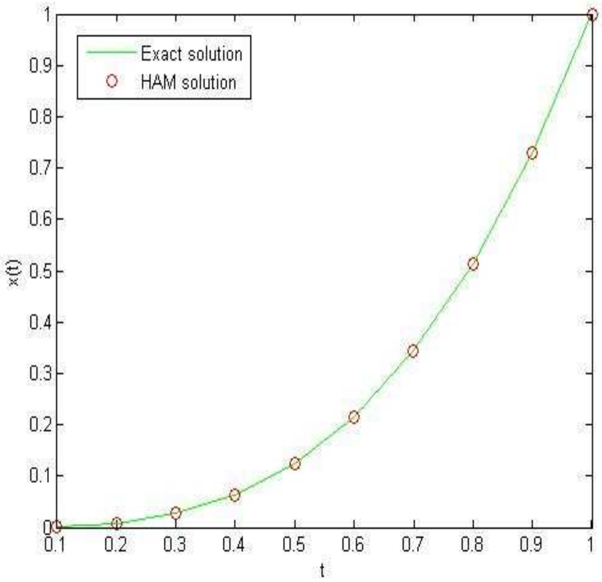

Figure 3: Comparison Graph of HAM solution with exact solution of equation (30), when $\hbar = -1$

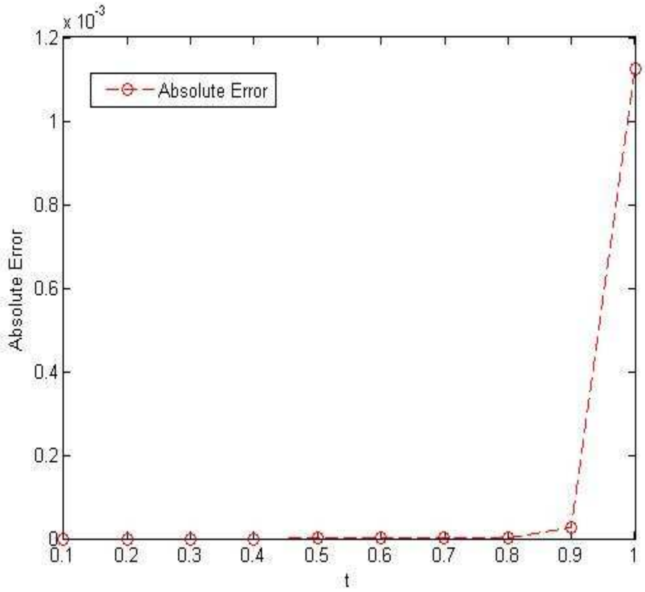

Figure 4: Graph of absolute error between HAM solution and exact solution of equation (30), when $\hbar = -1$

Example 3. Consider the following quadratic integral equation, given by

$$x(t)=t^3-\frac{t^{12}}{40}+\frac{t}{5}x(t)\int_0^t s\,x^2(s)\,ds \tag{46}$$

The equation has the exact solution

$$x(t)=t^2\,. \tag{47}$$

Applying HAM Method to equation (46) and taking the linear operator as

$$L\left[\phi(t;q)\right]=\phi(t;q) \tag{48}$$

and nonlinear operator as

$$N\left[\phi(t;q)\right]=\phi(t;q)-\left(t^2-\frac{t^{10}}{35}\right)-\frac{t}{5}\phi(t;q)\int_0^t s^2\phi^2(s;q)\,ds \tag{49}$$

For the above operator, with assumption $H(t)=1$, we construct the zeroth-order deformation equation from equation (13) as

$$(1-q)L\left[\phi(t;q)-x_0(t)\right]=q\,\hbar\,H(t)\,N\left[\phi(t;q)\right]. \tag{50}$$

and the m^{th}- order deformation equation is given by

$$L\left[x_m(t)-\kappa_m\,x_{m-1}(t)\right]=\hbar R_m\left[\vec{x}_{m-1}(t)\right], \tag{51}$$

where

$$R_m\left[\vec{x}_{m-1}\right]=\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}N\left[\phi(t;q)\right]}{\partial q^{m-1}}\right]_{q=0} \text{ and } \kappa_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}. \tag{52}$$

By using equation (52), we evaluate the value of $R_m\left[\vec{x}_{m-1}\right]$

for $m=1$

$$R_1\left[\vec{x}_0\right]=x_0(t)-\left(t^3-\frac{t^{12}}{40}\right)-\frac{t}{5}x_0(t)\int_0^t s x_0^{\,2}(s)\,ds\,, \tag{53}$$

for $m=2$

$$R_2\left[\vec{x}_1\right]=x_1(t)-\frac{t}{5}\frac{1}{1!}\frac{\partial}{\partial q}\left[\phi(t;q)\int_0^t s\,\phi^2(s;q)\,ds\right]_{q=0}\,, \tag{54}$$

and so on.

For $m\in N\ (m\neq 1)$

$$R_m\left[\vec{x}_{m-1}\right]=x_{m-1}(t)-\frac{t}{5}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi(t;q)\int_0^t s\,\phi^2(s;q)\,ds\right]_{q=0}\,. \tag{55}$$

Hence, the value of $R_m\left[\vec{x}_{m-1}\right]$ for $m\in N$ is given by

$$R_m\left[\vec{x}_{m-1}\right]=x_{m-1}(t)-(1-\kappa_m)\left(t^3-\frac{t^{12}}{40}\right)$$
$$-\frac{t}{5}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi(t;q)\int_0^t s\,\phi^2(s;q)\,ds\right]_{q=0}. \tag{56}$$

Appling the operator L^{-1} on both sides of equation (51), we have

$$x_m(t)=\kappa_m x_{m-1}(t)+\hbar L^{-1}\left[R_m\left(\vec{x}_{m-1}\right)\right]. \tag{57}$$

Now, taking initial solution as

$$x_0(t)=t^3-\frac{t^{12}}{40}. \tag{58}$$

The equation (57) is a recurrence relation for $m\geq 1$, using equation (58) in equation (57), we get the value of $x_m(t)$ for $m=1,2,3,\ldots$ as follows

$$x_1(t)=-\hbar\left[\frac{1}{40}t^{12}-\frac{33}{27200}t^{21}+\frac{69}{3536000}t^{30}-\frac{1}{8320000}t^{39}\right], \tag{59}$$

$$x_2(t)=-\hbar(\hbar+1)\left[\frac{1}{40}t^{12}-\frac{33}{27200}t^{21}+\frac{69}{3536000}t^{30}-\frac{1}{8320000}t^{39}\right]$$
$$+\hbar^2\left[\frac{33}{27200}t^{21}-\frac{249}{2828800}t^{30}+\frac{21867}{8415680000}t^{39}-\frac{148431}{3702899200000}t^{48}\right.$$
$$\left.+\frac{139521}{428789504000000}t^{57}-\frac{21}{18343936000000}t^{66}\right] \tag{60}$$

and so on. The other rest of components can be established likewise.
Therefore, the HAM series solution of problem (46) is given by

$$x(t)=x_0(t)+\sum_{m=1}^{\infty}x_m(t)=x_0(t)+x_1(t)+x_2(t)+,\ldots,$$

$$x(t)=t^3-\frac{1}{40}\left(1+2\hbar+\hbar^2\right)t^{12}+\frac{33}{13600}\hbar(1+\hbar)t^{21}-\left(\frac{69}{1768000}\hbar+\frac{117}{1088000}\hbar^2\right)t^{30}$$
$$+\left(\frac{1}{4160000}\hbar+\frac{45757}{16831360000}\hbar^2\right)t^{39}-\frac{148431}{3702899200000}\hbar^2t^{48}$$
$$+\frac{139521}{428789504000000}\hbar^2t^{57}-\frac{21}{18343936000000}\hbar^2t^{66}+,\ldots, \tag{61}$$

If we put $\hbar=-1,$ in equation (61), we have

$$x(t)=t^3-0+0-\left(-\frac{69}{1768000}+\frac{117}{1088000}\right)t^{30}+\left(-\frac{1}{4160000}+\frac{45757}{16831360000}\right)t^{39}$$
$$-\frac{148431}{3702899200000}t^{48}+\frac{139521}{428789504000000}t^{57}-\frac{21}{18343936000000}t^{66}+,...,\tag{62}$$

The result in (62) was obtained earlier by El-Sayed et *al.* [16] using the method of ADM and RTM. It can be easily observed that for $\hbar=-1,$ the series solution converges to the exact solution $x(t)=t^3$ as $m\to\infty.$

In **Table 3**, we compare the approximate solution obtained by using HAM with the exact solution. **Figure 5** and **Figure 6** show the comparison graph of the approximate solution by the HAM with the exact solution and the absolute error graph between them, respectively. It is worth noting that the approximate solutions in all the figures and tables are evaluated using only three terms from the homotopy analysis solution.

Table 3: Numerical comparison of solution of the equation (46) by HAM with the exact solution at different values of t

t	**Exact Solution**	**HAM Solution**	$\lvert x_{\text{Exact}}-x_{\text{HAM}}\rvert$
0.1	1.000000000000000e-003	1.000000000000000e-003	0
0.2	8.000000000000002e-003	8.000000000000002e-003	0
0.3	2.700000000000001e-002	2.700000000000001e-002	0
0.4	6.400000000000002e-002	6.399999999999993e-002	8.326672684688674e-017
0.5	1.250000000000000e-001	1.249999999999362e-001	6.381006834033087e-014
0.6	2.160000000000000e-001	2.159999999848598e-001	1.514016689796449e-011
0.7	3.429999999999999e-001	3.429999984580909e-001	1.541908989377561e-009
0.8	5.120000000000001e-001	5.119999156000681e-001	8.439993204323315e-008
0.9	7.290000000000001e-001	7.289971362433207e-001	2.863756679349905e-006
1.0	1.000000000000000e+000	9.999339287954754e-001	6.607120452462034e-005

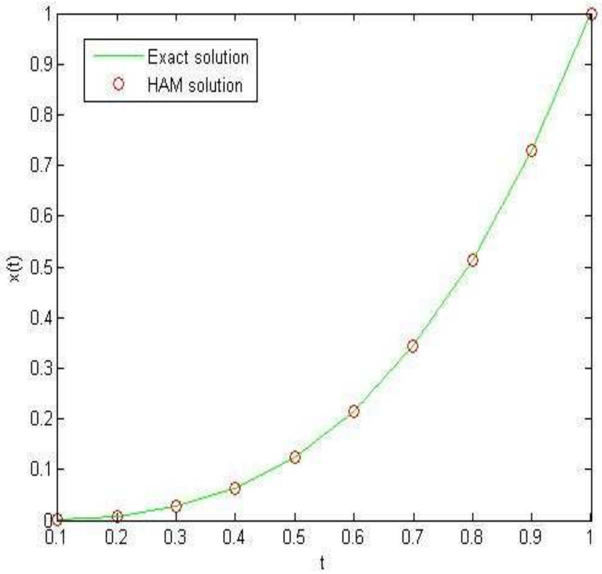

Figure 5: Comparison Graph of HAM solution with exact solution of equation (46), when $\hbar = -1$

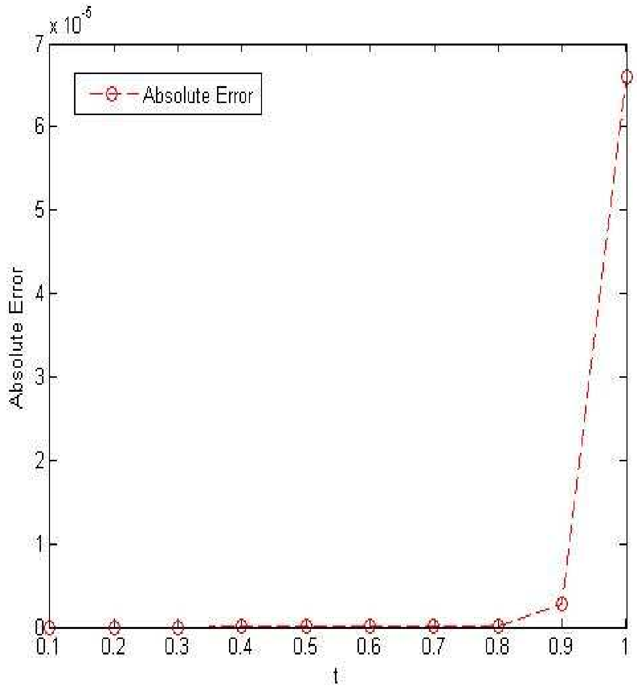

Figure 6: Graph of absolute error between HAM solution and exact solution of equation (46),when $\hbar = -1$

Example 4. Consider the following quadratic integral equation, given by

$$x(t)=t^2-\frac{t^{11}}{630}+\frac{1}{6}x^2(t)\int_0^t (t-s)^2 x^2(s)\,ds \tag{63}$$

The equation has the exact solution

$$x(t)=t^2 . \tag{64}$$

Applying HAM Method to equation (63) and taking the linear operator as

$$L\left[\phi(t;q)\right]=\phi(t;q), \tag{65}$$

and nonlinear operator as

$$N\left[\phi(t;q)\right]=\phi(t;q)-\left(t^2-\frac{t^{11}}{630}\right)-\frac{1}{6}\phi^2(t;q)\int_0^t (t-s)^2\phi^2(s;q)\,ds. \tag{66}$$

For the above operator, with assumption $H(t)=1$, we construct the zeroth-order deformation equation from equation (13) as

$$(1-q)L\left[\phi(t;q)-x_0(t)\right]=q\,\hbar\,H(t)\,N\left[\phi(t;q)\right]. \tag{67}$$

and the m^{th}- order deformation equation is given by

$$L\left[x_m(t)-\kappa_m\, x_{m-1}(t)\right]=\hbar R_m\left[\vec{x}_{m-1}(t)\right], \tag{68}$$

where

$$R_m\left[\vec{x}_{m-1}\right]=\frac{1}{(m-1)!}\left[\frac{\partial^{m-1}N\left[\phi(t;q)\right]}{\partial q^{m-1}}\right]_{q=0} \text{ and } \kappa_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases} . \tag{69}$$

By using equation (569), we evaluate the value of $R_m\left[\vec{x}_{m-1}\right]$

for $m=1$

$$R_1\left[\vec{x}_0\right]=x_0(t)-\left(t^2-\frac{t^{11}}{630}\right)-\frac{1}{6}x_0{}^2(t)\int_0^t (t-s)^2 x_0{}^2(s)\,ds, \tag{70}$$

for $m=2$

$$R_2\left[\vec{x}_1\right]=x_1(t)-\frac{1}{6}\frac{1}{1!}\frac{\partial}{\partial q}\left[\phi^2(t;q)\int_0^t (t-s)^2\,\phi^2(s;q)\,ds\right]_{q=0} , \tag{71}$$

and so on.

For $m\in N$ $(m\neq 1)$

$$R_m\left[\vec{x}_{m-1}\right]=x_{m-1}(t)-\frac{1}{6}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi^2(t;q)\int_0^t (t-s)^2\phi^2(s;q)\,ds\right]_{q=0} . \tag{72}$$

Hence, the value of $R_m\left[\vec{x}_{m-1}\right]$ for $m\in N$ is given by

$$R_m\left[\vec{x}_{m-1}\right]=x_{m-1}(t)-(1-\kappa_m)\left(t^2-\frac{t^{11}}{630}\right)-\frac{1}{6}\frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}}\left[\phi^2(t;q)\int_0^t (t-s)^2\phi^2(s;q)\,ds\right]_{q=0}. \tag{73}$$

Appling the operator L^{-1} on both sides of equation (68), we have

$$x_m(t)=\kappa_m x_{m-1}(t)+\hbar L^{-1}\left[R_m\left(\vec{x}_{m-1}\right)\right]. \tag{74}$$

Now, taking initial solution as

$$x_0(t)=t^2-\frac{t^{11}}{630}. \tag{75}$$

The equation (74) is a recurrence relation for $m\geq 1$, using equation (75) in (74), we get the value of $x_m(t)$ for $m=1,2,3,\ldots$ as follows

$$x_1(t)=-\hbar\left[\frac{1}{630}t^{11}-\frac{17}{3175200}t^{20}+\frac{61}{57510810000}t^{29}-\frac{143}{144927241200000}t^{38}+\frac{1}{6521725854000000}t^{47}\right] \tag{76}$$

$$\begin{aligned}x_2(t)=&-\hbar(1+\hbar)\left[\frac{1}{630}t^{11}-\frac{17}{3175200}t^{20}+\frac{61}{57510810000}t^{29}-\frac{143}{144927241200000}t^{38}+\frac{1}{6521725854000000}t^{47}\right]\\&+\hbar^2\left[\frac{17}{3175200}t^{20}-\frac{2519}{92017296000}t^{29}+\frac{16337737}{433622305670400000}t^{38}\right.\\&-\frac{388643}{2285809011319680000}t^{47}+\frac{18643430059}{21098314329652117958400000000}t^{56}\\&-\frac{176249640017}{7765857948315474463528800000000}t^{65}\\&+\frac{230762317}{596645183833993769758920000000000}t^{74}\\&\left.-\frac{547}{17355548333891221486200000000000}t^{83}\right]\end{aligned} \tag{77}$$

and so on. The other rest of components can be obtained accordingly.
Therefore, the HAM series solution of equation (63) is given by

$$x(t)=x_0(t)+\sum_{m=1}^{\infty}x_m(t)=x_0(t)+x_1(t)+x_2(t)+,\ldots,$$

$$x(t)=t^2-\frac{1}{630}\left(1+2\hbar+\hbar^2\right)t^{11}+\frac{17}{1587600}\hbar(1+\hbar)t^{20}$$
$$-\left(\frac{61}{28755405000}\hbar+\frac{89}{3129840000}\hbar^2\right)t^{29}$$
$$+\left(\frac{143}{72463620600000}\hbar+\frac{5588531}{1445407768556800000}\hbar^2\right)t^{38}$$
$$-\left(\frac{1}{3260862927000000}t+\frac{4575059}{2666777179872960000000}\hbar^2\right)t^{47}$$
$$+\frac{18643430059}{2109831432965211795840000000}\hbar^2t^{56}$$
$$-\frac{176249640017}{776585794831547446352880000000000}\hbar^2t^{65}$$
$$+\frac{230762317}{596645183833993769758920000000000}\hbar^2t^{74}$$
$$-\frac{547}{1735554833389122148620000000000000}\hbar^2t^{83}+,...$$

(78)

If we put $\hbar=-1,$ in equation (78), we have

$$x(t)=t^2+0+\left(\frac{61}{28755405000}-\frac{89}{3129840000}\right)t^{29}$$
$$+\left(-\frac{143}{72463620600000}+\frac{5588531}{1445407768556800000}\right)t^{38}$$
$$-\left(\frac{1}{3260862927000000}t+\frac{4575059}{2666777179872960000000}\right)t^{47}$$
$$+\frac{18643430059}{2109831432965211795840000000}t^{56}$$
$$-\frac{176249640017}{776585794831547446352880000000000}t^{65}$$
$$+\frac{230762317}{596645183833993769758920000000000}t^{74}$$
$$-\frac{547}{1735554833389122148620000000000000}t^{83}+,...$$

(79)

The result in (79) was obtained earlier by El-Sayed *et al.* [16] using the methods of ADM and RTM. It can be easily observed that for $\hbar = -1$, the series solution converges to the exact solution $x(t) = t^2$ as $m \to \infty$.

In **Table 4**, we compare the approximate solution obtained by using HAM with the exact solution. **Figure 7** and **Figure 8** show the comparison graph of the approximate solution by the HAM with the exact solution and the absolute error graph between them, respectively. It is worth noting that the approximate solutions in all the figures and tables are evaluated using only three terms from the homotopy analysis solution.

Table 4: Numerical comparison of solution of the equation (63) by HAM with the exact solution at different values of t

t	**Exact Solution**	**HAM Solution**	$\lvert x_{\text{Exact}} - x_{\text{HAM}} \rvert$
0.1	1.000000000000000e-002	1.000000000000000e-002	0
0.2	4.000000000000001e-002	4.000000000000001e-002	0
0.3	9.000000000000002e-002	9.000000000000002e-002	0
0.4	1.600000000000000e-001	1.600000000000000e-001	0
0.5	2.500000000000000e-001	2.499999999999999e-001	5.551115123125783e-017
0.6	3.600000000000000e-001	3.599999999999903e-001	9.714451465470120e-015
0.7	4.899999999999999e-001	4.899999999991527e-001	8.472667012426882e-013
0.8	6.400000000000001e-001	6.399999999592879e-001	4.071221137991188e-011
0.9	8.100000000000001e-001	8.099999987612172e-001	1.238782854073861e-009
1.0	1.000000000000000e+000	9.999999737220710e-001	2.627792905496307e-008

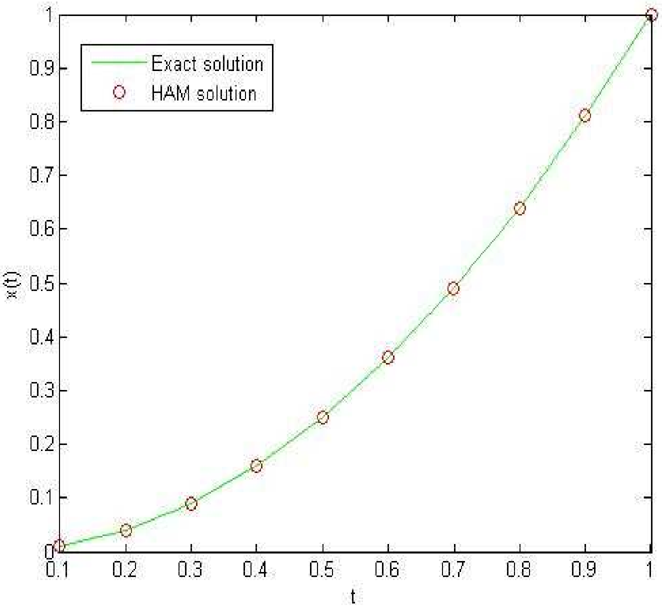

Figure 7: Comparison Graph of HAM solution with exact solution of equation (63), when $\hbar = -1$

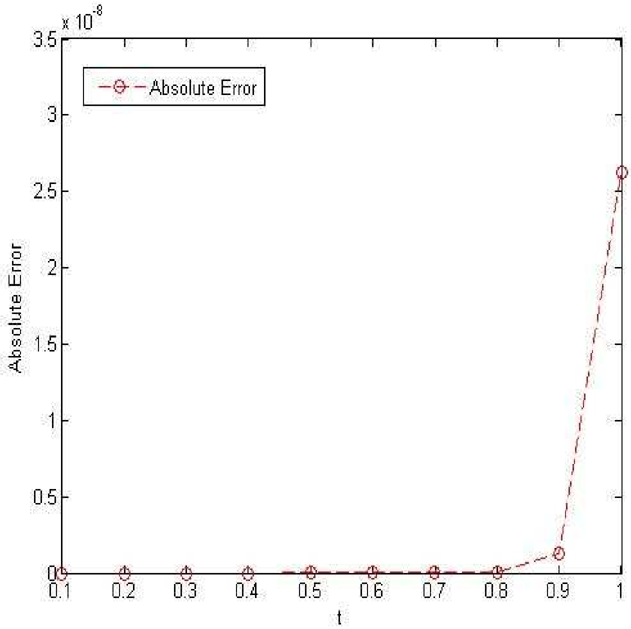

Figure 8: Graph of absolute error between HAM solution and exact solution of equation(63), when $\hbar = -1$

5. Concluding remark

In this work, we have determined the approximate solutions of the quadratic integral equations by the use of the homotopy analysis method (HAM).We compared the approximate and exact solutions graphically. The obtained results demonstrate that the proposed approach is highly efficient and straightforward, and it may be implemented to investigate solutions to other nonlinear problems.

Acknowledgment. The authors would like to thank the editor and anonymous reviewers for their careful reading and constructive suggestions and comments, which helped a lot in the improvement of the manuscript.

Conflict of interest. The authors declare that they have no conflict of interest.

Authors' Contributions. All the authors contributed equally to this work.

REFERENCES

1. Z.Abbas, S.Vahdati, F.Ismail and A.K.Dizicheh, Application of homotopy analysis method for linear integro-differential equations. *In International Mathematical Forum,* 5(5) (2010) 237-249.

A. Adawi, F.Awawdeh and H.Jaradat, A numerical method for solving linear integral equations, *Int. J. Contemp. Math. Sciences*, 4(10) (2009) 485-496.

2. H.Al-badrani, F.A.Hendi, and W.Shammakh, Variational homotopy perturbation method of quadratic integral equations. *Applied Mathematical Sciences*, 11(13) (2017) 623-635.
3. M.F.Araghi and S.S.Behzadi, Solving nonlinear volterra-fredholm integro-differential equations using the modified adomian decomposition method, *Computational Methods in Applied Mathematics,* 9(4) (2009) 321-331.
4. M.F.Araghi and S.S.Behzadi, Numerical solution of nonlinear Volterra-Fredholm integro-differential equations using Homotopy analysis method, *Journal of Applied Mathematics and Computing,* 37(1) (2011) 1-12.
5. I.K.Argyros, Quadratic equations and applications to Chandrasekhar's and related equations, *Bulletin of the Australian Mathematical Society*, 32(2) (1985) 275-292.
6. J.Banaś and A.Martinon Monotonic solutions of a quadratic integral equation of Volterra type, *Computers & Mathematics with Applications*, 47 (2004) 271-279.
7. J.Banaś, M.Lecko and W.G.El-Sayed, Existence theorems for some quadratic integral equations, *Journal of Mathematical Analysis and Applications,* 222(1) (1998) 276-285.
8. J.Biazar and H.Ghazvini, Numerical solution for special non-linear Fredholm integral equation by HPM, *Applied Mathematics and Computation,* 195(2) (2008) 681-687.
9. N.Bildik and M.Inc, Modified decomposition method for nonlinear Volterra-Fredholm integral equations, *Chaos, Solitons & Fractals*, 33(1) (2007) 308-313.
10. I.W.Busbridge, The mathematics of radiative transfer, University Press, Cambridge, England, (1960).
11. S.Chandrasekhar, Radiative transfer, *Dover Publications, Inc., New York*, (1960).

12. P.Darania and K.Ivaz, Numerical solution of nonlinear Volterra–Fredholm integro-differential equations, *Computers & Mathematics with Applications,* 56(9) (2008) 2197-2209.

A. El-Sayed and H.H.G.Hashem, Integrable and continuous solutions of a nonlinear quadratic integral equation, *Electronic Journal of Qualitative Theory of Differential Equations*, (25) (2008) 1-10.

13. A.M.A.El-Sayed and H.H.G.Hashem, Solvability of nonlinear Hammerstein quadratic integral equations, *Journal of Nonlinear Sciences & Applications,* 2(3) (2009) 152-160.

14. A.M.A.El-Sayed, M.M.Saleh, and E.A.A.Ziada, Numerical and analytic solution for nonlinear quadratic integral equations. *Math. Sci. Res. J*, 12(8) (2008) 183-191.

15. W.G.El-Sayed and B.Rzepka, Non decreasing solutions of a quadratic integral equation of Urysohn type, *Computers & Mathematics with Applications*, 51(6-7) (2006) 1065-1074.

16. S.Z.Fu, Z.Wang and J.S.Duan, Solution of quadratic integral equations by the Adomian decomposition method, *CMES-Comput. Model. Eng. Sci,* 92(4) (2013) 369-385.

17. J.M.Heris, Solving the Integro-Differential Equations Using the Modified Laplace Adomian Decomposition Method.*Journal of Mathematical Extension*, 6, (2012)

18. S.Hu, M.Khavanin and W.A.N.Zhuang, Integral equations arising in the kinetic theory of gases, *Applicable Analysis*, 34(3-4) (1989) 261-266.

19. J.Izadian and S.Salahshour, Solution of nonlinear Volterra and Fredholm integral equations by HAM, *ICPACS*, 1 (2012) 1-6.

20. H.Jafari and S.Seifi, Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation, *Communications in Nonlinear Science and Numerical Simulation,* 14(5) (2009) 2006-2012.

21. J.Juang, K.Y.Lin and W.W.Lin, Spectral analysis of some iterations in the Chandrasekhar's H-functions, *Numerical Functional Analysis & Optimization*, 24(5-6) (2003) 575-586.

22. S.Liao, Beyond perturbation: introduction to the homotopy analysis method. *CRC press*, (2003).

23. S.Liao, On the homotopy analysis method for nonlinear problems, *Applied Mathematics and Computation,* 147(2) (2004) 499-513.

24. S.Liao, Notes on the homotopy analysis method: some definitions and theorems, *Communications in Nonlinear Science and Numerical Simulation,* 14(4) (2009) 983-997.

25. S.Liao and Y.Tan, A general approach to obtain series solutions of nonlinear differential equations, *Studies in Applied Mathematics,* 119(4) (2007) 297-354.

26. S.J.Liao, Beyond perturbation: a review on the basic ideas of the homotopy analysis method and its applications, *Adv. Mech.,* 38 (2008) 1–34.

27. Z.M.Odibat, Differential transform method for solving Volterra integral equation with separable kernels, *Mathematical and Computer Modelling,* 48(7-8) (2008) 1144-1149.

28. A.Shidfar, A.Molabahrami, A.Babaei and A.Yazdanian, A series solution of the nonlinear Volterra and Fredholm integro-differential equations. *Communications in Nonlinear Science and Numerical Simulation,* 15(2) (2010) 205-215.

29. L.Song, and H.Zhang, Application of homotopy analysis method to fractional KdV–Burgers–Kuramoto equation, *Physics Letters A*, 367(1-2) (2007) 88-94.
30. H.Xu, S.J.Liao and X.C.You,Analysis of nonlinear fractional partial differential equations with the homotopy analysis method, *Communications in Nonlinear Science and Numerical Simulation*, 14(4) (2009) 1152-1156.