Split Feasibility Problem and Fixed Point Problem for Asymptotically Strictly Pseudo Nonspreading Mapping

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Abstract. The purpose of this paper is to introduce an iterative algorithm for finding a common element of the solution set of split feasibility problem and the fixed point set of an asymptotically strictly pseudo nonspreading mapping in the Hilbert Space.

Keywords: Split feasibility, Asymptotically strictly pseudo nonspreading mapping, Hilbert space

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1. Introduction

The Split feasibility problem was originally introduced by Censor and Elfving [7] for modeling phase retrieval problems and it later was studied extensively as an extremely powerful tool for the treatment of a wide range of inverse problem, such as medical image reconstruction and intensity modulated radiation therapy problems. For example we may refer to [8-10]. Let \( C \) and \( Q \) be two nonempty closed convex subset of real Hilbert spaces \( H_1 \) and \( H_2 \) respectively and \( A : H_1 \to H_2 \) be a bounded linear operator. The Split Feasibility Problem (SFP) is to find a point \( x \) such that

\[
x \in C, Ax \in Q
\]  

(1.1)

Throughout the paper, we denote by \( \Gamma \), the solution set of the split feasibility problem that is

\[
\Gamma = \{x \in C : Ax \in Q = C \cap (A^{-1}Q)\}
\]

Finding the common solution of a Split Feasibility Problem and fixed point problem is one of the core interest of many researchers. Recently Ceng et. al. [5] introduced a relaxed extragradient method with regularization for finding a common solution set of SFP and the set of \( \text{Fix}(T) \) of the fixed point of nonexpansive mapping \( T \). Recently Deepho [6] introduced and analyzed a relaxed extragradient method with regularization for finding a common element of the solution set \( \Gamma \) of the Split Feasibility Problem and fixed point set \( \text{Fix} \) of an uniformly Lipschitz continuous and asymptotically quasi nonexpansive mapping.
in the setting of real Hilbert space. Very recently Ansari et al. [2] deals with the weak convergence of the relaxed extragradient method with regularization for computing a common element of the solution set of Split Feasibility Problem and fixed point set of asymptotically \( k \) strict pseudo contractive mapping in intermediate sense.

### 2. Some definitions

**Definition 2.1.** [4] Let \( H \) be real Hilbert Space and \( C \) be a non empty closed convex subset of \( H \), a mapping \( T : C \to C \) is said to be nonspreading if

\[
2||x - y||^2 \leq ||Tx - x||^2 + ||Ty - x||^2 \quad \forall \ x, y \in C
\]

the above inequality is equivalent to

\[
||x - y||^2 \leq ||x - y||^2 + 2 < x - Tx, y - Ty > \quad \forall \ x, y \in C
\]

**Definition 2.2.** [4] Let \( H \) be real Hilbert Space. A mapping \( T : D(T) \subset H \to H \) is said to be \( k \) strict pseudo nonspreading mapping if there exist \( a, b \in (0, 1) \) such that

\[
||Tx - Ty||^2 \leq ||x - y||^2 + k||x - Tx - (y - Ty)||^2 + 2 < x - Tx, y - Ty > \quad \forall \ x, y \in D(T)
\]

Every nonspreading mapping is \( k \) strict pseudo nonspreading mapping.

**Definition 2.3.** [4] Let \( H \) be real Hilbert Space. A mapping \( T : D(T) \subset H \to H \) is said to be \( k \) asymptotically strictly pseudo nonspreading mapping if there exists \( r \in (0, 1) \) and a sequence \( \{r_n\} \subset [1, \infty) \) with \( r_n \to 1 \) (\( n \to \infty \)) such that

\[
||T^n x - T^n y||^2 \leq k_n ||x - y||^2 + k ||x - T^n x - (y - T^n y)||^2 + 2 < x - T^n x, y - T^n y > \quad \forall \ x, y \in D(T)
\]

It is easy to see that the class of \( k \) asymptotically strictly pseudo nonspreading mapping is more general than the class of \( k \) strictly pseudo nonspreading mapping and \( k \) asymptotically strictly pseudo contraction mapping.

### 3. Preliminaries

Let \( H \) be a real Hilbert Space whose inner product and norm are denoted by \(<.,.>\) and \( ||.|| \) respectively. We denote the strong convergence and weak convergence of a sequence \( \{x_n\} \) to a point \( x \in X \) by \( x_n \to x \) and \( x_n \rightharpoonup x \) respectively. Let \( K \) be a nonempty closed convex subset of real Hilbert Space \( H \) for every point \( x \in H \), there exist a unique nearest point of \( K \), denoted by \( P_K x \), such that \( ||x - P_K x|| = ||x - y|| \quad \forall \ x, y \in K \). Such a \( P_K \) is called the metric projection from \( H \) onto \( K \). It is well known that \( P_K \) is firmly nonexpansive mapping from \( H \) onto \( C \), i.e.,

\[
||P_K x - P_K y||^2 \leq < P_K x - P_K y, x - y > \quad \forall \ x, y \in H
\]

**Proposition 3.1** [2]. For a given \( x \in H \) and \( y \in K \), we have

1. \( z = P_K x \) if and only if \( < x - z, y - z > \geq 0 \quad \forall \ y \in K \);
2. \( z = P_K x \) if and only if \( ||x - z||^2 \leq ||x - y||^2 - ||y - z||^2 \quad \forall \ y \in K \);
3. \( < P_K x - P_K y > \geq ||P_K x - P_K y||^2 \quad \forall \ y \in K \).
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Let $K$ be a non empty closed and convex subset of $H$ and $F : K \rightarrow H$ be a mapping. The Variational inequality problem (VIP) is to find $x \in K$ such that

$$<Fx, y - x> \geq 0 \, \forall \, y \in K$$

(3.1)

the solution of VIP is denoted by $VI(K, F)$. It is well known that $x^* \in VI(K, F) \iff x = P_K(x - \lambda Fx) \forall \, \lambda > 0$

A set valued mapping $T : H \rightarrow 2^H$ called monotone if $<x - y, f - g> \geq 0$ whenever $f \in Tx, g \in Ty$. It is said to be maximal monotone if, in addition, the graph $G(T) = \{(x, f) \in H \times H : f \in Tx\}$ of $T$ is not properly contained in the graph of any other monotone operator. It is well known that a monotone mapping $T$ is maximal if and only if,

for $(x, f) \in H \times H, <x - y, f - g> \geq 0$ for every $(y, g) \in G(T) \Rightarrow f \in Tx$

Let $F : K \rightarrow H$ be a monotone that is $<Fx - Fy, x - y> \geq 0$ for all $x, y \in K$ and $k$ lipschitz continuous mapping, let $N_K$ be the normal cone to $K$ at $v \in K$, that is

$$N_K v = \{w \in H : <v - u, w> \geq 0 \, \forall u \in K\}$$

Define

$$Tv = \begin{cases} Fv + N_K v, & \text{if } v \in K \\ \emptyset, & \text{otherwise} \end{cases}$$

Then $T$ is maximal monotone set valued mapping. It is well known that if $0 \in Tv$ then $-Fv \in N_K v$, which is further equivalent to the variational inequality.

**Proposition 3.2.** [5] Let $C$ and $Q$ be nonempty closed subsets of Hilbert space $H_1$ and $H_2$ respectively and $A : H_1 \rightarrow H_2$ be a bounded linear operator. For given $x^* \in H_1$, the following statement are equivalent

1. $x^*$ solves the SFP;
2. $x^*$ solves the Fixed point equation $P_C(I - \lambda A^*(I - P_Q)A)x^* = x^*$;
3. $x^*$ solves the VIP of finding $x^* \in C$ such that $<\nabla f(x^*), x - x^*> \geq 0$ for all $x \in C$ where $\nabla f = A^*(I - P_Q)A$ and $A^*$ is the adjoint of $A$.

**Lemma 3.3.** [2] Let $H$ be real Hilbert space. Then for all $x, y \in H$ we have

- $||x - y||^2 \leq ||x||^2 + ||y||^2$
- $||\lambda x - (1 - \lambda)y||^2 = \lambda ||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)||x - y||^2$, for all $\lambda \in [0, 1]$

**Lemma 3.4.** [4] Let $K$ be a non empty closed convex subset of a real Hilbert space $H$ and let $T : K \rightarrow K$ be a continuous $k$ asymptotically strictly pseudo nonspreading mapping if $F(T) \neq \phi$, then it is a closed and convex subset.

**Lemma 3.5.** [4] Let $K$ be a non empty closed convex subset of a real Hilbert space $H$ and let $T : K \rightarrow K$ be a continuous $k$ asymptotically strictly pseudo nonspreading mapping
then \((I - T)\) is demiclosed at 0 that is, if \(x_n \to x^*\) and \(\limsup_{m \to \infty} \limsup_{n \to \infty} \|((I - T^m)x_n)\| = 0\) then \(\|(I - T)x^*\| = 0\).

**Lemma 3.6.** [3] Let \(K\) be a non empty closed convex subset of a real Hilbert space \(H\) and let \(T : K \to K\) be a continuous \(k\) asymptotically strictly pseudo nonspreading mapping and uniformly \(L\) Lipschitzian mapping then for any sequence \(x_n\) in \(K\) converging weakly to a point \(p\) and \(\{\|x_n - T x_n\|\}\) converging strongly to 0, we have \(p = Tp\).

Motivated by the above, the purpose of this paper is to introduce an iterative algorithm for finding a common element of the solution set of split feasibility problem and the fixed point set of a asymptotically strictly pseudo nonspreading mapping in the Hilbert Space which improve and extends the results of [1].

**4. Main result**

**Theorem 4.1.** Let \(C\) be a nonempty closed convex subset of a real Hilbert space \(H\) and \(T : C \to C\) be an uniformly \(L\) lipschitzian and \(k\) asymptotically strictly pseudo nonspreading mapping such that \(\text{Fix}(T) \cap \Gamma \neq \emptyset\).

\[ x_0 \in H \]
\[ y_n = P_C(I - \lambda_n \nabla f(\alpha_n)(x_n)) \]
\[ x_{n+1} = (1 - \beta_n) x_n + \beta_n T^n(y_n) \]

Assume that the sequence \(\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\}, \{k_n\}\) satisfy the following conditions,

1. \(\sum_{n=1}^{\infty} \alpha_n < \infty\);
2. \(\{\lambda_n\} \subset [a, b] \) for some \(a, b \in (0, 1]\), \(\sum_{n=1}^{\infty} \lambda_n < \infty\);
3. \(\{\nabla f(\alpha_n)(x_n)\}_{n=1}^{\infty}\) is bounded sequence;
4. \(\{\beta_n\} \subset [d, e] \) for some \(d, e \in (0, 1)\);
5. \(\{k_n\} \subset [1, \infty) \), \(k \in [0, 1)\).

Then both the sequence \(\{x_n\}\) and \(\{y_n\}\) converges weakly to an element \(x^* \in \text{Fix}(T) \cap \Gamma\).

**Proof.** Let \(p \in \text{Fix}(T) \cap \Gamma\) be arbitrary chosen, then we have \(T(p) = p \in C\) and \(Ap \in Q\). Therefore,
\[ P_C(p) = p \text{ and } P_Q(Ap) = Ap \]

Since \(P_C\) is nonexpansive, we have
\[ \|y_n - p\|^2 = \|P_C(I - \lambda_n \nabla f(\alpha_n)(x_n)) - P_C(p)\|^2 \]
\[ \leq \|(x_n - p) - \lambda_n \nabla f(\alpha_n)(x_n)\|^2 \]
\[ \|y_n - p\|^2 \leq \|x_n - p\|^2 + \lambda_n^2 \left\| \nabla f(\alpha_n)(x_n) \right\|^2 \] (4.1)

Since \(y_n \in C\) and \(T^n y_n \in C\), we have
\[ \|y_n - T^n y_n\|^2 = \|P_C(I - \lambda_n \nabla f(\alpha_n)(x_n)) - P_C(T^n y_n)\|^2 \]
\[ \leq \|(x_n - T^n y_n) - \lambda_n \nabla f(\alpha_n)(x_n)\|^2 \]
\[ \|y_n - T^n y_n\|^2 \leq \|(x_n - T^n y_n) + \lambda_n^2 \left\| \nabla f(\alpha_n)(x_n) \right\|^2 \] (4.2)
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By $k$ asymptotically strictly pseudo nonspreading mapping of $T$, by Lemma (3.3), (4.1) and (4.2)

\[
\|x_{n+1} - p\|^2 = \| (1 - \beta_n) x_n + \beta_n T^n (y_n) - p \|^2 \\
= \| x_n - \beta_n x_n + \beta_n T^n (y_n) - \beta_n p + \beta_n p - p \|^2 \\
= \| x_n - \beta_n x_n + \beta_n (T^n (y_n) - p) + (\beta_n p - p) \|^2 \\
= \| (1 - \beta_n) x_n + \beta_n (T^n (y_n) - p) + (\beta_n - 1) p \|^2 \\
= \| (1 - \beta_n) (x_n - p) + \beta_n (T^n (y_n) - p) \|^2 \\
= (1 - \beta_n) \| x_n - p \|^2 + \beta_n \| T^n (y_n) - p \|^2 - \beta_n (1 - \beta_n) \| x_n - p - T^n y - p \|^2 \\
= (1 - \beta_n) \| x_n - p \|^2 + \beta_n \| T^n (y_n) - p \|^2 - \beta_n (1 - \beta_n) \| x_{n+1} - p \|^2 \| T^n y - x_n \|^2 \\
= (1 - \beta_n) \| x_n - p \|^2 + \beta_n k_n \| x_n - p \|^2 + \beta_n k_n \lambda_n^2 \| \nabla f (x_n) \|^2 + \beta_n k_n \| y_n - T^n y_n \|^2 \\
- \beta_n (1 - \beta_n) \| T^n y - x_n \|^2 \\
= (1 - \beta_n + \beta_n k_n) \| x_n - p \|^2 + \beta_n k_n \lambda_n^2 \| \nabla f (x_n) \|^2 + \beta_n k_n \| x_n - T^n y_n \|^2 \\
+ \lambda_n \| \nabla f (x_n) \|^2 \| \nabla f (x_n) \|^2 - \beta_n (1 - \beta_n) \| T^n y - x_n \|^2 \\
= (1 - \beta_n + \beta_n k_n) \| x_n - p \|^2 + \beta_n k_n \lambda_n^2 \| \nabla f (x_n) \|^2 + \beta_n k_n \| x_n - T^n y_n \|^2 \\
+ \lambda_n \| \nabla f (x_n) \|^2 \| \nabla f (x_n) \|^2 - \beta_n (1 - \beta_n) \| T^n y - x_n \|^2 \\
= (1 - \beta_n + \beta_n k_n) \| x_n - p \|^2 + (\lambda_n \| \nabla f (x_n) \|^2 + \beta_n k_n (k_n + \lambda_n)^2) \| \nabla f (x_n) \|^2 \\
+ \beta_n k_n \| x_n - T^n y_n \|^2 - \beta_n (1 - \beta_n) \| T^n y - x_n \|^2 \\
= (1 + \beta_n (k_n - 1) \| x_n - p \|^2 + (\lambda_n \| \nabla f (x_n) \|^2 + \beta_n k_n (k_n + \lambda_n)) M \\
+ \| x_n - T^n y_n \|^2 (\beta_n k_n (k_n - 1) - \beta_n) \\
\| x_{n+1} - p \|^2 \leq (1 + \beta_n (k_n - 1)) \| x_n - p \|^2 + (\lambda_n \| \nabla f (x_n) \|^2 + \beta_n k_n (k_n + \lambda_n)) M \\
+ \| x_n - T^n y_n \|^2 (\beta_n k_n (k_n - 1) - \beta_n) \\
\| x_{n+1} - p \|^2 \leq (1 + \beta_n (k_n - 1)) \| x_n - p \|^2 + b_n (4.3)
\]
By (4.4),
Thus from (4.3), we obtain
Since
Using limiting conditions of
=
thus
We have
exists. Also, \( \lim_{n \to \infty} ||y_n - p|| \) exists.
Thus from (4.3), we obtain
\[
||x_{n+1} - p||^2 \leq (1 + \beta_n(k_n - 1)||x_n - p||^2 + (\lambda_n^2 \beta_n(k + k_n)M + ||x_n - T^n y_n||^2 (\beta_n k - \beta_n(1 - \beta_n)))
\]
\[
||x_n - T^n y_n||^2 (\beta_n k - \beta_n(1 - \beta_n)) \leq (1 + \beta_n(k_n - 1)||x_n - p||^2 + (\lambda_n^2 \beta_n(k + k_n)M - ||x_{n+1} - p||^2)
\]
Using limiting conditions of \( M \) and \( k \), we get,
\[
\lim_{n \to \infty} ||T^n y_n - x_n|| = 0
\]
(4.4)
\[
\lim_{n \to \infty} ||T^n y_n - y_n|| = 0
\]
(4.5)
By (4.4),
\[
||x_{n+1} - x_n|| = ||(1 - \beta_n)x_n + \beta_n T^n (y_n) - x_n||
\]
\[
= ||(x_n - \beta_n x_n) + \beta_n T^n (y_n) - x_n||^2
\]
\[
= ||(x_n - \beta_n x_n) - \beta_n T^n (y_n) - x_n||^2
\]
\[
= \lim_{n \to \infty} \beta_n ||T^n y_n - x_n|| \to 0 .
\]
\[
\sum \left( \frac{\lambda_n}{||x_n - y_n||} \right) \leq (1 + \sum \lambda_n)^2 \beta_n(k + k_n)M - ||x_{n+1} - p||^2
\]
Since \( y_n = P_C \left( x_n - \lambda_n \nabla f(x_n) \right) \) and by proposition 3.1, we have
\[
||y_n - p||^2 \leq ||x_n - \lambda_n \nabla f(x_n) - p||^2 - ||x_n - \lambda_n \nabla f(x_n) - y_n||
\]
\[
= ||x_n - p||^2 - ||x_n - y_n||^2 + 2 \lambda_n < \nabla f(x_n), p - y_n >
\]
\[
= ||x_n - p||^2 - ||x_n - y_n||^2 + 2 \lambda_n < \nabla f(x_n), p - x_n > + 2 \lambda_n < \nabla f(x_n), x_n - y_n >
\]
\[
= ||x_n - p||^2 - ||x_n - y_n||^2 + 2 \lambda_n < \nabla f(x_n), x_n - y_n > + 2 \lambda_n < \nabla f(x_n), p - x_n > + 2 \lambda_n < \nabla f(x_n), p - x_n > + 2 \lambda_n < \nabla f(x_n), x_n - y_n >
\]
\[
= ||x_n - p||^2 - ||x_n - y_n||^2 + 2 \lambda_n < \nabla f(x_n), x_n - y_n > + 2 \lambda_n < \nabla f(x_n), x_n - y_n > + 2 \lambda_n < \nabla f(x_n), x_n - p > + ||x_n - p||^2 - ||x_n - y_n||^2
\]
\[
= ||x_n - p||^2 - ||x_n - y_n||^2 + 2 \lambda_n < \nabla f(x_n), x_n - y_n > + 2 \lambda_n < \nabla f(x_n), p - x_n > + 2 \lambda_n < \nabla f(x_n), x_n - p > + ||x_n - p||^2 - ||x_n - y_n||^2
\]
\[
= ||x_n - p||^2 - ||x_n - y_n||^2 + 2 \lambda_n < \nabla f(x_n), x_n - y_n > + 2 \lambda_n < \nabla f(x_n), p - x_n > + 2 \lambda_n < \nabla f(x_n), x_n - p > + ||x_n - p||^2 - ||x_n - y_n||^2
\]
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\begin{equation}
= \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n < \alpha_n p, p - x_n > + 2\lambda_n < \nabla f(\alpha_n)(x_n), x_n - p >
> + 2\lambda_n < \nabla f(\alpha_n)(x_n), y_n - p >
\end{equation}

\begin{equation}
\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2 + 2\lambda_n \alpha_n ||p|| ||p - x_n||
+ 2\lambda_n ||\nabla f(\alpha_n)(x_n)|| ||x_n - p|| + 2\lambda_n ||\nabla f(\alpha_n)(x_n)|| ||y_n - p||
\end{equation}

\begin{equation}
\text{Taking limit both sides and using conditions, we obtain}
\lim_{n \to \infty} \|x_n - y_n\| = 0
\end{equation}

\begin{equation}
\|y_{n+1} - y_n\| = \|P_C (I - \lambda_{n+1} \nabla f(\alpha_{n+1})(x_{n+1})) - P_C (I - \lambda_n \nabla f(\alpha_n)(x_n))\|
\leq (\|x_{n+1} - x_n\| + \lambda_{n+1} ||\nabla f(\alpha_{n+1})(x_{n+1})||^2) + \lambda_n ||\nabla f(\alpha_n)(x_n)||^2
\end{equation}

Taking limit both the sides

\begin{equation}
\lim_{n \to \infty} \|y_{n+1} - y_n\| = 0
\end{equation}

Since \(\|y_{n+1} - y_n\| \to 0\), \(\|T^n y_n - y_n\| \to 0\) as \(n \to \infty\). \(T\) is uniformly Lipschitz by Lemma (3.6), \(\|T^n y_n - y_n\| \to 0\) as \(n \to \infty\). Since \(\{x_n\}\) is bounded sequence, there exist a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) that converges weakly to \(x^*\) i.e., \(x_{n_j} \to x^*\).

Let \(\{x_{n_j}\}\) of \(\{x_n\}\) such that converges weakly to \(x^*\) i.e., \(x_{n_j} \to x^*\).
Assume $x' \neq x^*$. By Opial’s condition
\[
\lim_{n \to \infty} ||x_n - x^*|| = 0
\]
\[
\lim_{n \to \infty} \inf ||x_{n_i} - x^*|| < \lim_{n \to \infty} ||x_{n_i} - x'||
\]
\[
\lim_{n \to \infty} \inf ||x_{n_i} - x'|| = \lim_{n \to \infty} ||x_{n_i} - x'|| < \lim_{n \to \infty} ||x_n - x^*||
\]
This contradict to our assumption $x' \neq x^*$.
Hence $x_{n_i} \to x^*$, $x_n \to x^*$. For all $f \in H$, $f(x_n) \to f(x^*)$
Next we show that $y_n \to x^*$
\[
||f(y_n) - f(x^*)|| = ||f(y_n) + f(x_n) - f(x_n) - f(x^*)||
\]
\[
\leq ||f(y_n) - f(x_n)|| + ||f(x_n) - f(x^*)||
\]
\[
\lim_{n \to \infty} ||f(y_n) - f(x^*)|| = 0 \text{ for all } f \in H, f(x_n) \to f(x^*), y_n \to x^*
\]
By lemma 3.5, $x^* \in \text{Fix}(T)$.
Now we show that $x^* \in \Gamma$
\[
S\omega_1 = \begin{cases} \lambda_n \nabla f_{\omega_1} + N_C \omega_1 & \text{if } \omega_1 \in C \\ \emptyset & \text{otherwise} \end{cases}
\]
\[
N_C \omega_1 = \{z \in H : < \omega_1 - u, z > \geq 0 \text{ for all } u \in C\}
\]
To show that $x^* \in \Gamma$ it is sufficient to show that $0 \in Sx^*$
Let $(\omega_1, z) \in G(C)$.
We have, $z \in S\omega_1 - \lambda_n \nabla f_{\omega_1} + N_C \omega_1$
And, $z - \lambda_n \nabla f_{\omega_1} \in N_C \omega_1$
So we have $< \omega_1 - u, z - \lambda_n \nabla f_{\omega_1} > \geq 0$ for all $u \in C$
Since
We have
\[
y_n = P_C(I - \lambda_n \nabla f(\alpha_n))(x_n)
\]
And by Proposition 3.1.
\[
< (x_n - \lambda_n \nabla f(\alpha_n)(x_n)) - y_n, y_n - \omega_1 > \geq 0
\]
\[
< \omega_1 - y_n, y_n - x_n + \lambda_n \nabla f(\alpha_n)(x_n) > \geq 0
\]
\[
z_n + \lambda_n \nabla f_{\omega_1} \in N_C \omega_1 \text{ and } y_{n_1} \in C. \text{ It follows that}
\]
\[
< \omega_1 - y_{n_1}, z > \geq < \omega_1 - y_{n_1}, \nabla f_{\omega_1} >
\]
\[
\leq < \omega_1 - y_{n_1}, \lambda_n \nabla f_{\omega_1} > = < \omega_1 - y_{n_1}, y_{n_1} - x_{n_1} + \lambda_n \nabla f(\alpha_n)(x_{n_1}) >
\]
\[
> (\omega_1 - y_{n_1}, \lambda_n \nabla f_{\omega_1} > - < \omega_1 - y_{n_1}, y_{n_1} - x_{n_1} + \lambda_n \nabla f(\alpha_n)(x_{n_1}) >
\]
\[
< \omega_1 - y_{n_1}, \lambda_n >
\]
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Pseudo Nonsnearing Mapping

\[ \langle \omega_1 - y_{n_i}, \lambda_n \nabla f_{\omega_1} - \lambda_n \nabla f_{y_{n_i}} \rangle \geq \langle \omega_1 - y_{n_i}, y_{n_i} - x_{n_i} \rangle 
- \lambda_n (y_{n_i} - x_{n_i}) \]
\[ \geq \langle \omega_1 - y_{n_i}, \lambda_n \nabla f_{\omega_1} - \lambda_n \nabla f_{x_{n_i}} \rangle 
- \langle \omega_1 - y_{n_i}, y_{n_i} - x_{n_i} \rangle > -\lambda_n (\alpha_{n_i}) \]
\[ \langle \omega_1 - y_{n_i}, x_{n_i} \rangle \]

Taking limit as \( i \to \infty \), we obtain \( \langle \omega_1 - x^*, z \rangle \geq 0 \) as \( i \to \infty \)

Since \( \langle \omega_1 - x^*, z \rangle \geq 0 \) for every \( (\omega_1, z) \in G(S) \).
Therefore the maximality of \( S \) implies that \( 0 \in Sx^* \). Thus we have, \( x^* \in VI(G, \nabla f) \) finally, proposition [5] implies that \( x^* \in \Gamma \). This completes the proof.

**Remark 2.6.** Theorem [10] improve and extends [1] in the following aspects:

1. The technique of proving weak convergence in [10] is different from that in [1] because of our technique to use \( k \) asymptotically strictly pseudo nonspreading mapping and the property of maximal monotone mappings.
2. The problem of finding a common element of \( \text{Fix}(T \cap \Gamma) \) for \( k \) asymptotically strictly pseudo nonspreading mappings which is more general than that for nonexpansive mappings and the problem of finding a solution of the SFP in [1].
3. The problem of finding a common element of \( \text{Fix}(T \cap \Gamma) \) for \( k \) asymptotically \( k \) strictly pseudo nonspreading mappings which is more general than that for asymptotically \( k \) strict pseudo contractive mappings and the problem of finding a solution of the SFP in [2].

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