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On the Diophantine Equations $(n + 2)^x - 2.n^y = z^2$ and $(n + 2)^x + 2.n^y = z^2$

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Abstract. In this article, we solve the Diophantine equations $(n + 2)^x - 2.n^y = z^2$ and $(n + 2)^x + 2.n^y = z^2$, where x, y, z are non-negative integers and n is a positive integer with $n \equiv 2$ or $n \equiv 3 \pmod 4$.

Keywords: Diophantine equation; integer solution; congruence

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In the past few years, many researchers have studied the Diophantine equations of the form $a^x - b^y = z^2$, where a, b are positive integers and x, y, z are non-negative integers (see for instance [1-11]).

Recently, Thongnak, Chuayjan and Kaewong [12] studied the Diophantine equation $5^x - 2\cdot 3^y = z^2$ and found that the equation has no non-negative integer solution. In this paper, we will generalize their results by considering the Diophantine equation $(n+2)^x - 2\cdot n^y = z^2$, where n is a positive integer with some conditions. Moreover, we will solve the Diophantine equation $(n+2)^x + 2\cdot n^y = z^2$.

2. The Diophantine equation $(n + 2)^x - 2.n^y = z^2$

We begin this section by considering case $n = 2$.

Theorem 2.1. The Diophantine equation $4^x - 2\cdot 2^y = z^2$ has the non-negative integer solutions $(x, y, z) \in \{(r, 2r-1, 0) : r \in \mathbb{N}\}$.

Proof: Let x, y and z be non-negative integers such that $4^x - 2\cdot 2^y = z^2$. It implies that $(2^x - z)(2^x + z) = 2^{y+1}$. There exists a non-negative integer r such that $2^x - z = 2^r$ and $2^x + z = 2^{y+1-r}$. Consequently, $y + 1 \geq 2r$ and $2^{x+1} = 2^r(2^{y+1-2r} + 1)$. Thus $y + 1 - 2r = 0$ and so $x + 1 = r + 1$. Then $y = 2r - 1$ and $x = r$, respectively. Since y is a non-negative integer and $y = 2r - 1$, we get $r \neq 0$. Since $x = r$ and $2^x - z = 2^r$, we have $z = 0$. Hence,

$(x, y, z) \in \{(r, 2r-1, 0) : r \in \mathbb{N}\}$ are all non-negative integer solutions of the Diophantine equation $4^x - 2 \cdot 2^y = z^2$.
Now, we consider case $n \equiv 3 \pmod 4$.

Theorem 2.2. Let n be a positive integer with $n \equiv 3 \pmod 4$. Then the Diophantine equation $(n+2)^x - 2 \cdot n^y = z^2$ has no non-negative integer solution.

Proof: Assume that x, y and z are non-negative integers such that $(n+2)^x - 2 \cdot n^y = z^2$. Since $n \equiv 3 \pmod 4$, it implies that $z^2 = (n+2)^x - 2 \cdot n^y \equiv 1 - 2(-1)^y \pmod 4$.

Case 1: y is even. Then $z^2 \equiv 1 - 2(1) \equiv -1 \equiv 3 \pmod 4$.

Case 2: y is odd. Then $z^2 \equiv 1 - 2(-1) \equiv 3 \pmod 4$.

Both cases are impossible since $z^2 \equiv 0, 1 \pmod 4$.

By Theorem 2.2, if $n = 3$, then we have the result of Thongnak, Chuayjan and Kaewong [12]:

Corollary 2.1. [12] The Diophantine equation $5^x - 2 \cdot 3^y = z^2$ has no non-negative integer solution.

Corollary 2.2. Let m and n be positive integers with $n \equiv 3 \pmod 4$. Then the Diophantine equation $(n+2)^x - 2 \cdot n^y = z^{2m}$ has no non-negative integer solution.

Proof: Assume that a, b and c are non-negative integers such that $(n+2)^a - 2 \cdot n^b = c^{2m}$. Then $(x, y, z) = (a, b, c^m)$ is a non-negative integer solution of the Diophantine equation $(n+2)^x - 2 \cdot n^y = z^2$. This contradicts to Theorem 2.2.

3. The Diophantine equation $(n + 2)^x + 2.n^y = z^2$

We begin this section by considering case $n = 2$.

Theorem 3.1. The Diophantine equation $4^x + 2 \cdot 2^y = z^2$ has the non-negative integer solutions $(x, y, z) \in \{(r-1, 2r, 3 \cdot 2^{r-1}) : r \in \mathbb{N}\}$.

Proof: Let x, y and z be non-negative integers such that $4^x + 2 \cdot 2^y = z^2$. It implies that $(z - 2^x)(z + 2^x) = 2^{y+1}$. There exists a non-negative integer r such that $z - 2^x = 2^r$ and $z + 2^x = 2^{y+1-r}$. Thus $y + 1 > 2r$ and $2^{x+1} = 2^r(2^{y+1-2r} - 1)$. Consequently, $2^{y+1-2r} - 1 = 1$ and $x + 1 = r$. Then $y = 2r$ and $x = r - 1$, respectively. Since x is a non-negative integer, we get $r \neq 0$. Since $z - 2^x = 2^r$ and $x = r - 1$, we obtain that $z = 2^r + 2^{r-1} = 3 \cdot 2^{r-1}$. Hence, $(x, y, z) \in \{(r-1, 2r, 3 \cdot 2^{r-1}) : r \in \mathbb{N}\}$ are all non-negative integer solutions of the Diophantine equation $4^x + 2 \cdot 2^y = z^2$.
Now, we consider case $n \equiv 3 \pmod 4$.

Theorem 3.2. Let n be a positive integer with $n \equiv 3 \pmod{4}$. Then the Diophantine equation $(n+2)^x + 2 \cdot n^y = z^2$ has no non-negative integer solution.

Proof: Assume that x, y and z are non-negative integers such that $(n+2)^x + 2 \cdot n^y = z^2$. Since $n \equiv 3 \pmod{4}$, it implies that $z^2 = (n+2)^x + 2 \cdot n^y \equiv 1 + 2(-1)^y \pmod{4}$.

Case 1: y is even. Then $z^2 \equiv 1 + 2(1) \equiv 3 \pmod{4}$.

Case 2: y is odd. Then $z^2 \equiv 1 + 2(-1) \equiv -1 \equiv 3 \pmod{4}$.

Both cases are impossible since $z^2 \equiv 0,1 \pmod{4}$.

By Theorem 3.2, we have the following corollaries:

Corollary 3.1. The Diophantine equation $5^x + 2 \cdot 3^y = z^2$ has no non-negative integer solution.

Corollary 3.2. Let m and n be positive integers with $n \equiv 3 \pmod{4}$. Then the Diophantine equation $(n+2)^x + 2 \cdot n^y = z^{2m}$ has no non-negative integer solution.

Proof: Assume that a, b and c are non-negative integers such that $(n+2)^a + 2 \cdot n^b = c^{2m}$. Then $(x, y, z) = (a, b, c^m)$ is a non-negative integer solution of the Diophantine equation $(n+2)^x + 2 \cdot n^y = z^2$. This contradicts to Theorem 3.2.

4. Conclusion

In this work, using elementary methods, we investigated non-negative integer solutions of the Diophantine equations $(n+2)^x - 2 \cdot n^y = z^2$ and $(n+2)^x + 2 \cdot n^y = z^2$, where n is a positive integer with $n = 2$ or $n \equiv 3 \pmod{4}$. Nevertheless, the Diophantine equations on the other case remain an open problem.

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Conflict of interest. The paper is written by single author so there is no conflict of interest.

Authors' Contributions. It is a single author paper. So, full credit goes to the author.

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