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On the Diophantine Equation $15^x - 13^y = z^2$

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Abstract. In this article, we prove that the Diophantine equation $15^x - 13^y = z^2$ has non-negative integer solution. The result reveals that the solution $(x, y, z) = (0, 0, 0)$.

Keywords: Diophantine equation; factoring method; modular arithmetic method

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

A popular topic in Mathematics is the Diophantine equation. This topic concerns finding a solution to an equation over an integer number. In [6], Mihailescu proved Catalan's conjecture. This theorem is very important because it has been applied to prove many Diophantine equations. In [1], the Diophantine equation $2^x + 5^y = z^2$ was presented by Acu. He applied congruent and modular arithmetic theories to prove that the two solutions (x, y, z) include $(3, 0, 3)$ and $(2, 1, 3)$. In [8], Suvarnamani et al. proved that $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no integer solution. Next, Chotchaisthit [5] demonstrated that $4^x + p^y = z^2$ where p is any positive prime number have no solution. In 2018, Rabago [7] proved that $4^x - p^y = 3z^2$ where p is prime has the set of all solutions (x, y, z) including $(0, 0, 0)$ and $(q-1, 1, 2^{q-1} - 1)$ where $p = 2^q - 1$ and q are prime. In 2019, Nechemia [3, 4] showed no solution to the Diophantine equation $7^x + 10^y = z^2$ when x, y, z are positive integers, and presented the equation $6^x - 11^y = z^2$ when x, y, z are positive integers. He suggested that the equation has one solution when $x = 2$, and no solution for $2 < x \le 16$. Next, the Diophantine equation $2^x - 3^y = z^2$ was presented [9]. The authors proved that there are three solutions to the equation. Then, a group of researchers suggested that $p^x - 2^y = z^2$ where $p = k^2 + 2$ is a prime number has two solutions including $(x, y, z) = (0, 0, 0)$ or $(1, 1, k)$ [2]. After that, the Diophantine equation $7^x - 5^y = z^2$ was proved that the solution (x, y, z) is $(0, 0, 0)$ [10]. Recently, the Diophantine equation $7^x - 2^y = z^2$ has been proved to have only one trivial solution

$(x, y, z) = (0,0,0)$ [11]. From the previous works, there is no general method to prove all sets of the Diophantine equation in the form $a^x - b^y = z^2$. We still need to prove individual equations.

In this work, we study the Diophantine equation $15^x - 13^y = z^2$. We use Modular Arithmetic to show all solutions to the equation.

2. Preliminaries

In this section, we introduce basic knowledge applying in the proof.

Lemma 2.1. For all $n \in \mathbb{N}^+$. Then $2^{2n-1} \equiv 2,5,6,7,8$ or $11 \pmod{13}$.

Proof: Let $P(n)$: $2^{2n-1} \equiv 2,5,6,7,8$ or $11 \pmod{13}$ **.**

For $n = 1$, we get $2^{2(1)-1} = 2 \equiv 2 \pmod{13}$. So $P(1)$ is true.

We assume that $P(k)$ is true for $k \in \mathbb{N}^+$ that is

$$2^{2k-1} \equiv 2,5,6,7,8 \text{ or } 11 \pmod{13} .$$

Now, to prove that $P(k+1)$ is true, we consider $2^{2(k+1)-1} = 2^{2k+1} = 4 \cdot 2^{2k-1}$. Then we have

$$\begin{aligned} 2^{2(k+1)-1} &\equiv 4(2),\ 4(5), 4(6), 4(7), 4(8) \text{ or } 4(11) \pmod{13} \\ &\equiv 8, 20, 24, 28, 32 \text{ or } 44 \pmod{13} \\ &\equiv 8, 7, 11, 2, 6 \text{ or } 5 \pmod{13}. \end{aligned}$$

Hence

$$2^{2(k+1)-1} \equiv 2,5,6,7,8 \text{ or } 11 \pmod{13} .$$

Thus, $P(k+1)$ is true. Therefore, by the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}^+$. □

Lemma 2.2. For all $x \in \mathbb{N}$. Then $x^2 \equiv 0,1,3,4,9,10$ or $12 \pmod{13}$.

Proof: Let $x \in \mathbb{N}$. There are $q, r \in \mathbb{N}$ such that $x = 13q + r$ for $0 \le r < 13$ **.** It follows that $x \equiv r \pmod{13}$ and $x^2 \equiv r^2 \pmod{13}$.

Case 1: $r = 0$, then we get $x^2 \equiv 0 \pmod{13}$.

Case 2: $r = 1$ or 12, then we get $x^2 \equiv 1 \pmod{13}$.

Case 3: $r = 2$ or 11, then we get $x^2 \equiv 4 \pmod{13}$.

Case 4: $r = 3$ or 10, then we get $x^2 \equiv 9 \pmod{13}$.

Case 5: $r = 4$ or 9, then we get $x^2 \equiv 3 \pmod{13}$.

Case 6: $r = 5$ or 8, then we get $x^2 \equiv 12 \pmod{13}$.

Case 7: $r = 6$ or 7, then we get $x^2 \equiv 10 \pmod{13}$.

From all cases, we have $x^2 \equiv 0,1,3,4,9,10$ or $12 \pmod{13}$. □

3. Main result

Theorem 3.1. For all $x, y, z \in \mathbb{N}^+ \cup \{0\}$. The Diophantine equation $15^x - 13^y = z^2$ has a unique solution $(x, y, z) = (0,0,0)$.

Proof: Let $x, y, z \in \mathbb{N}^+ \cup \{0\}$ such that

$$15^x - 13^y = z^2. \quad (1)$$

The equation can be solved by considering the following four cases. 1) $x = 0$ and $y = 0$ 2) $x = 0$ and $y > 0$ 3) $x > 0$ and $y = 0$ 4) $x > 0$ and $y > 0$.

Case 1: $x = 0$ and $y = 0$. It is easy to see that $z = 0$. We get the solution $(x, y, z) = (0,0,0)$.

Case 2: $x = 0$ and $y > 0$. The equation (1) becomes $1 - 13^y = z^2$. Because $1 - 13^y < 0$, we obtain $z^2 < 0$, impossible.

Case 3: $x > 0$ and $y = 0$. So (1) becomes $z^2 = 15^x - 1$. Because $15 \equiv 0 \pmod 3$, this implies that $z^2 \equiv -1 \pmod 3$ or $z^2 \equiv 2 \pmod 3$, impossible.

Case 4: $x > 0$ and $y > 0$, we consider the following two subcases.

Subcase 4.1, x is odd. By Lemma 2.1, it is easy to see that $2^x \equiv 2,5,6,7,8,11 \pmod{13}$. By (1), we have $z^2 \equiv 2^x \pmod{13}$. This yields $z^2 \equiv 2,5,6,7,8,11 \pmod{13}$. By Lemma 2.2, this is impossible.

Subcase 4.2, x is even. Then $x = 2k$, $\exists k \in \mathbb{N}^+ \cup \{0\}$. It follows that $13^y = 15^{2k} - z^2$. This is equivalent to $13^y = (15^k - z)(15^k + z)$. There are α and $\beta \in \mathbb{N}^+ \cup \{0\}$ such that $15^k - z = 13^\alpha$ and $15^k + z = 13^\beta$ where $\alpha < \beta$ and $\alpha + \beta = y$. This implies that $2 \cdot 15^k = 13^\alpha + 13^\beta$ or $2 \cdot 3^k \cdot 5^k = 13^\alpha (1 + 13^{\beta - \alpha})$.

Since $13 \nmid 2 \cdot 3^k \cdot 5^k$, we easily get $\alpha = 0$. This implies that $\beta = y$ and we obtain

$$2 \cdot 3^k \cdot 5^k = 1 + 13^y. \quad (2)$$

Since $13 \equiv 1 \pmod 3$, (2) implies that $2 \equiv 0 \pmod 3$. This is impossible. In all cases, it can be concluded that $(0,0,0)$ is a solution to the equation. □

4. Conclusion

In this work, we have proved that the Diophantine equation $15^x - 13^y = z^2$ has a unique solution $(x, y, z) = (0,0,0)$. In the proof, we consider four cases, including cases 1: $x = 0$ and $y = 0$, case 2: $x = 0$ and $y > 0$, case 3: $x > 0$ and $y = 0$, and case 4: $x > 0$ and $y > 0$, and we use Modular Arithmetic. We obtain that the equation has only a trivial solution $(x, y, z) = (0,0,0)$.

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Conflict of interest. The paper is written by single author so there is no conflict of interest.

Authors' Contributions. It is a single author paper. So, full credit goes to the author.

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