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On the Diophantine Equation $55^x - 53^y = z^2$

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Abstract. This article aims to prove all solutions to the Diophantine equation $55^x - 53^y = z^2$ where *x*, *y* and *z* are non-negative integers. The results indicated that the solution is (*x*, *y*, *z*) = (0, 0, 0).

Keywords: Diophantine equation; factoring method, modular arithmetic method, order of a modulo

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1. Introduction

The diophantine equation is a topic in Number Theory. It concerns finding only an integer solution to an equation. The exponential diophantine equation is the diophantine equation where the unknown variables are exponents. Many researchers studied these equations. For example, in 2007, the exponential diophantine $2^x + 5^y = z^2$ was proved by Acu [1]. He found all non-negative integer solutions (x, y, z) including $(3,0,3)$ and $(2,1,3)$. In 2011, the diophantine equation $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ were studied and proved that both equations have no solution [10]. Then Sroysang [9] found that the unique solution to the equation $3^x + 5^y = z^2$ is $(x, y, z) = (1,0,2)$. During 2012-2017, many diophantine equations was studied, for example [2, 4, 5, 7, 8]. In 2018, Rabago [6] presented a diophantine equation in form $a^x - b^y = z^2$, where a, b are constant and non-negative integers and x, y, z are non-negative integers. He studied for $a = 4$ and b is prime. The solutions are $(x, y, z) = (q-1, 1, 2^{q-1} - 1)$, where q is prime. Next year, the diophantine equation $6^x - 11^y = z^2$ was studied by Brushtein [3]. He showed that there is no solution for $2 \le x \le 16$. After that, Thongnak et al. [12] proved that the equation $2^x - 3^y = z^2$ has three solutions (x, y, z): $(0,0,0)$, $(1,0,1)$ and $(2,1,1)$. After that, they investigated three equations which are $7^x - 5^y = z^2$, $7^x - 2^y = z^2$ and $5^x - 2 \cdot 3^y = z^2$ [13, 14, 15]. The first two equations had a trivial solution, and the last equation had no solution.

From previous works, many equations in the form $a^x - b^y = z^2$ were proved using various methods, but there was no general method to prove the solution to the equations. The study of individual equations has been necessary. In this paper, we aim to prove all solutions to $55^x - 53^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

In this section, we introduce basic knowledge applied in the proof.

Definition 2.1. [11] If n is a positive integer and $\gcd(a,n)=1$, the least positive integer k such that $a^k \equiv 1 \pmod n$ is called the order of a modulo n and is denoted by $ord_n(a)$.

Lemma 2.2. [11] If $ord_n(a)=k$, then $a^r \equiv a^s \pmod n$ if and only if $r \equiv s \pmod k$.

3. Main results

Theorem 3.1. For all $x, y, z \in \mathbb{N}^+ \cup \{0\}$, the Diophantine equation $55^x - 53^y = z^2$ where x, y, z are non-negative integers has only one solution that is $(x,y,z)=(0,0,0)$.

Proof: Let $x, y, z \in \mathbb{N}^+ \cup \{0\}$ such that

$$55^x - 53^y = z^2. \tag{1}$$

The proof is divided into four cases.

Case 1: $x = y = 0$. (1) becomes $z^2 = 0$. Thus $z = 0$, the solution (x,y,z) is $(0,0,0)$.

Case 2: $x = 0$ and $y > 0$. From (1), we have $1 - 53^y = z^2$. This means that $z^2 < 0$, impossible.

Case 3: $x > 0$ and $y = 0$. From (1), we obtain $z^2 = 55^x - 1$. We can see that $55 \equiv 0 \pmod{11}$, so $z^2 \equiv -1 \pmod{11}$. This means that $z^2 \equiv 10 \pmod{11}$. It is impossible because $z^2 \equiv 0,1,3,4,5,9 \pmod{11}$.

Case 4: $x > 0$ and $y > 0$, we separate into two subcases.

Subcase 4.1, x is odd. It follows that $3^x \equiv 3 \pmod 4$. Since $55 \equiv 3 \pmod 4$, we obtain $z^2 \equiv 3^x - 1 \pmod 4$. This yields $z^2 \equiv 2 \pmod 4$. It is impossible because $z^2 \equiv 0,1 \pmod 4$.

Subcase 4.2, x is even. Let $x = 2k$, $\exists k \in \mathbb{N}^+$, (1) becomes $53^y = 55^{2k} - z^2$ or $53^y = (55^k - z)(55^k + z)$. There exist $A, B \in \mathbb{N}^+ \cup \{0\}$ for which $55^k - z = 53^A$ and $55^k + z = 53^B$ where $A < B$ and $A + B = y$. Then, we have $2 \cdot 55^k = 53^A + 53^B$ or $2 \cdot 5^k \cdot 11^k = 53^A (1 + 53^{B-A})$. Since, $53 \nmid 2 \cdot 5^k \cdot 11^k$, we obtain $A = 0$ and $B = y$. This yields

$$2 \cdot 5^k \cdot 11^k = 1 + 53^y. \tag{2}$$

So $1+9^y \equiv 0 \pmod{11}$. This implies that $9^y \equiv 10 \pmod{11}$. Since $2^6 \equiv 9 \pmod{11}$ and $2^5 \equiv 10 \pmod{11}$, these yield $2^{6y} \equiv 2^5 \pmod{11}$. Because $ord_{11} 2 = 10$, we have $6y \equiv 5 \pmod{10}$. There exist $m \in \mathbb{N}$ such that $6y = 5+10m$. This implies that $2(3y-5m) = 5$ which is impossible. Hence the solution is $(x,y,z) = (0,0,0)$. □

4. Conclusion

In this article, we have proved the solution to the equation $55^x - 53^y = z^2$ where x, y and z are non-negative integers. The knowledge of number theory such as factoring method and modular arithmetic was applied to obtain the solution. We found that the equation has only one solution which is $(x,y,z) = (0,0,0)$.

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Conflict of interest. The paper is written by single author so there is no conflict of interest.

Authors' Contributions. It is a single author paper. So, full credit goes to the author.

REFERENCES

1. D.Acu, On a Diophantine equation, *General Mathematics*, 15 (2007) 145-148.
2. S.Asthana, On the Diophantine equation $8^x + 113^y = z^2$, *International Journal of Algebra*, 11 (2017) 225-230.
3. N.Burshtein, A Short Note on Solutions of the Diophantine Equation $6^x + 11^y = z^2$ and $6^x - 11^y = z^2$ in Positive Integers x, y, z, *Annals of Pure and Applied Mathematics*, 19 (2) (2019) 55-56.
4. A.Hoque and H.Kalita, On the Diophantine equation $(p^q - 1)^x + p^{qy} = z^2$, *Journal of Analysis & Number Theory*, 3(2) (2015) 117-119.
5. J.F.T.Rabago, On the Diophantine equation $2^x + 17^y = z^2$, *Journal of the Indonesian Mathematical Society*, 22(2) (2016) 85-88.
6. J.F.T.Rabago, On the Diophantine equation $4^x - p^y = 3z^2$ where p is a Prime, *Thai Journal of Mathematics*, 16 (2018) 643-650.
7. B.Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *International Journal of Pure and Applied Mathematics*, 89(1) (2013) 111-114.
8. B.Sroysang, On two Diophantine equations $7^x + 19^y = z^2$ and $7^x + 91^y = z^2$, *International Journal of Pure and Applied Mathematics*, 92(1) (2014) 113-116.
9. B.Sroysang, On the Diophantine equation $3^x + 5^y = z^2$, *International Journal of Pure and Applied Mathematics*, 81 (2012), 605-608.
10. A.Suvarnamani, On two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$, *Science and Technology RMUTT Journal*, 1(1) (2011) 25-28.
11. J.J.Tattersall, *Elementary Number Theory in Nine Chapters*, Cambridge University Press, (1999).

12. S.Thongnak, W.Chuayjan and T.Kaewong, On the exponential diophantine equation $2^x - 3^y = z^2$, *Southeast-Asian Journal of Sciences*, 7 (1) (2019) 1-4.
13. S.Thongnak, W.Chuayjan and T.Kaewong, The solution of the exponential diophantine equation $7^x - 5^y = z^2$, *Mathematical Journal*, 66 (703) (2021) 62-67.
14. S.Thongnak, W.Chuayjan and T.Kaewong, On the diophantine equation $7^x - 2^y = z^2$ where x, y and z are non-negative integers, *Annals of Pure and Applied Mathematics*, 25 (2) (2022) 63-66.
15. S.Thongnak, W.Chuayjan and T.Kaewong, On the exponential diophantine equation $5^x - 2 \cdot 3^y = z^2$, *Annals of Pure and Applied Mathematics*, 25 (2) (2022) 109-112.