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On the Exponential Diophantine Equation $10^x – 17^y = z^2$

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Abstract. In this study, our aim is to prove all the solutions of the exponential Diophantine equation $10^x – 17^y = z^2$, where *x, y* and *z* are non-negative integers. We applied the modular arithmetic and Catalan's conjecture to obtain all solutions. The result indicates that there are only two solutions to the equation.

Keywords: exponential Diophantine equation; factoring method; modular arithmetic method

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1. Introduction

The exponential Diophantine equations are classic problems in Number Theory. The most famous equation is $x^n + y^n = z^n$, where x, y and z are non-negative integers and $n \geq 3$. This equation was presented by Pierre de Fermat in 1637, and Andrew Wiles proved that the equation had no solution in 1993. Over a decade, many researchers computed and proved solutions to many equations. A major reason for the study is its wealth of application to cryptography, geometry, trigonometry and applied algebra. In 2004, Mihailescu [4] proved that the exponential Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min(a,b,x,y) > 1$ has a unique solution, $(a,b,x,y) = (3,2,2,3)$. In 2018, Rabago [5] studied and computed all solutions of $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$. In 2019, Thongnak et al. [7] proved that the exponential Diophantine equation $2^x - 3^y = z^2$ has only two solutions. Later, Burshtein [2] examined the exponential Diophantine equation $6^x - 11^y = z^2$ where $3 \leq x \leq 16$. He found one solution, $(x,y,z) = (2,1,5)$. In 2020, Buosi et al. [1] suggested the exponential Diophantine equation $p^x - 2^y = z^2$, where $p = k^2 + 2$ is a prime number and $k \geq 0$. They used Catalan's conjecture to compute the integer solutions, $(x,y,z) = (0,0,0)$ and $(1,1,k)$ with $k \geq 3$. Recently, many exponential Diophantine equations have been studied, for example, [6, 8, 9, 10]. These research articles motivated us to prove all solutions to other equations.

In this article, we aim to prove all solutions of the exponential Diophantine equation $10^x - 17^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

In this section, we introduce basic knowledge applied in this proof.

Theorem 2.1. (Euler's criterion [3]) Let p be an odd prime and $\gcd(a, p) = 1$. Then a is a quadratic residue of p if and only if $a^{(p-1)/2} \equiv 1 (\mathrm{mod}\ p)$.

Lemma 2.2. (Catalan's conjecture [4]) Let a, b, x and y be integers. The Diophantine equation $a^x - b^y = z^2$ with $\min\{a, b, x, y\} > 1$ has the unique solution $(a, b, x, y) = (3, 2, 2, 3)$.

3. Main result

Theorem 3.1. The exponential Diophantine equation $10^x - 17^y = z^2$ has exactly two non-negative integer solutions, $(x, y, z) = (0, 0, 0)$ and $(1, 0, 3)$.

Proof: Let x, y and z be non-negative integers such that

$$10^x - 17^y = z^2. \tag{1}$$

We separate into four cases, including case 1: $x = 0$ and $y = 0$, case 2: $x = 0$ and $y > 0$, case 3: $x > 0$ and $y = 0$, and case 4: $x > 0$ and $y > 0$.

Case 1: $x = 0$ and $y = 0$. From (1), we get $z^2 = 0$, implying that $z = 0$. Hence one solution to the equation is $(0, 0, 0)$.

Case 2: $x = 0$ and $y > 0$. In this case, (1) becomes $1 - 17^y = z^2 < 0$, which is impossible.

Case 3: $x > 0$ and $y = 0$. (1) becomes

$$10^x - z^2 = 1. \tag{2}$$

There are two cases to be considered: $x = 1$ and $x > 1$.

If $x = 1$, then (2) becomes $z^2 = 9$, which implies $z = 3$. Hence one solution is $(x, y, z) = (1, 0, 3)$.

If $x > 1$, then (2) implies $z > 1$. Lemma 2.2 (Catalan's conjecture) yields that (2) has no solution.

Case 4: $x > 0$ and $y > 0$, (1) implies that $z^2 \equiv 10^x (\mathrm{mod}\ 17)$, and it follows that 10^x is a quadratic residue of 17. By theorem 2.1, it follows that

$\left(10^x\right)^{(17-1)/2} = 10^{8x} \equiv 1\left(\bmod 17\right)$, and this yields $\left(-1\right)^x \equiv 1\left(\bmod 17\right)$. Thus x is even. We let $x = 2k$, $\exists k \in \square^{+}$. Clearly, (1) is equivalent to

$$17^y = 10^{2k} - z^2 = \left(10^k - z\right)\left(10^k + z\right).$$

There exists $\alpha \in \left\{0,1,2,3,...,y\right\}$ such that $10^k - z = 17^{\alpha}$ and $10^k + z = 17^{y-\alpha}$, where $\alpha < y - \alpha$. It follows that $2 \cdot 10^k = 17^{\alpha} + 17^{y-\alpha}$ or $2^{k+1} \cdot 5^k = 17^{\alpha}\left(1 + 17^{y-2\alpha}\right)$. Since $17 \nmid 2^{k+1} \cdot 5^k$, we have $\alpha = 0$ and $2^{k+1} \cdot 5^k = 1 + 17^y$. It yields $0 \equiv 2\left(\bmod 4\right)$, which is impossible. From all cases, $\left(0,0,0\right)$ and $\left(1,0,3\right)$ are the solutions to the equation $10^x - 17^y = z^2$.

4. Conclusion

In this work, we determined all solutions of the exponential Diophantine equation $10^x - 17^y = z^2$ where x, y and z are non-negative integers. The solutions, $\left(x, y, z\right)$, to the equation are $\left(0,0,0\right)$ and $\left(1,0,3\right)$.

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Conflicts of Interest: The authors declare that there is no conflict of interest.

Author's Contributions: All authors contributed equally.

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