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On the Exponential Diophantine Equation $3^x - 5^y = z^2$

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Abstract. In this study, we prove all solutions of the exponential Diophantine equation $3^x - 5^y = z^2$ where x, y and z are non-negative integers. The result indicates that the solutions (x, y, z) are (0, 0, 0) and (2, 1, 2).

Keywords: exponential Diophantine equation; divisibility; modular arithmetic method

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The exponential Diophantine equation is a classic topic in Number Theory. It has two or more unknown variables in an equation, and its solution must be integer. Because there is no general method to find a solution, it challenged mathematicians to determine how many solutions are. In 2004, Mihailescu [4] proved Catalan's conjecture $a^x - b^y = 1$ that it has exactly one solution when $\min(a,b,x,y) > 1$. Over five years ago, the exponential Diophantine equation was studied in the form $a^x - b^y = z^2$, where a, b, x, y and z are non-negative integers. In 2018, two exponential Diophantine equations, $4^x - 7^y = z^2$ and $4^x - 11^y = z^2$, were proved by Rabago [5]. He showed that the solutions to $4^x - 7^y = z^2$ are $(x, y, z) = (0,0,0)$ and $(2,1,3)$, and $4^x - 11^y = z^2$ has a unique solution, $(x, y, z) = (0,0,0)$. In 2019, Burshtein [2] suggested that $6^x - 11^y = z^2$ has a positive integer solution, $(x, y, z) = (2,1,5)$, and presumed that $6^x - 11^y = z^2$ has no solution when $x \geq 3$. After that, Thongnak et al. [9] proved that the equation $2^x - 3^y = z^2$ has three solutions, $(x, y, z) \in \{(0,0,0),(1,0,1),(2,1,1)\}$. From 2020 to 2022, many articles studied several equation in the form $a^x - b^y = z^2$ appearing in [1, 6, 10, 11, 12]. Recently, Tadee [8] showed that the non-negative integer solutions to the Diophantine equations, $9^x - 3^y = z^2$ and $13^x - 7^y = z^2$, are $(x, y, x) \in \{(r, 2r, 0)\}$

where $r \in \mathbb{N} \cup \{0\}$ and $(x, y, x) \in \{(0,0,0)\}$, respectively. In the same year, he proved the equation $3^x - p^y = z^2$ where p is prime. He showed the solutions with some conditions [7]. These previous works motivated us to study the remaining equations.

In this paper, we compute all solutions of the exponential Diophantine equation $3^x - 5^y = z^2$ where x, y and z are non-negative integers.

2. Preliminaries

In this section, we introduce the principle of Number Theory, which plays a critical role in the proof here.

Definition 2.1. [3] An integer b is said to be divisible by an integer $a \neq 0$, in symbols $a \mid b$, if there exists some integer c such that $b = ac$. We write $a \nmid b$ to indicate that b is not divisible by a.

Definition 2.2. [3] If n is a positive integer and $\gcd(a, n) = 1$, the least positive integer k such that $a^k \equiv 1 \pmod{n}$ is called the order of a modulo n and is denoted by $\operatorname{ord}_n a$.

Theorem 2.3. [3] If the integer a has order k modulo n, then $a^i \equiv a^j \pmod{n}$ if and only if $i \equiv j \pmod{k}$.

3. Main result

Theorem 3.1. Let x, y and z be non-negative integers. The exponential Diophantine equation $3^x - 5^y = z^2$ has only two solutions: $(x, y, z) = (0,0,0)$ and $(2,1,2)$.

Proof: Let x, y and z be non-negative integer

$$3^x - 5^y = z^2. \tag{1}$$

We consider four cases as follows:

Case 1: $x = y = 0$. (1) becomes $z^2 = 0$, and the solution is $(x, y, z) = (0,0,0)$.

Case 2: $x = 0$ and $y > 0$. By (1), we have $z^2 = 1 - 5^y < 0$, which is impossible.

Case 3: $x > 0$ and $y = 0$. By (1), we obtain $z^2 = 3^x - 1$, so $z^2 \equiv 2 \pmod{3}$. This is impossible because $z^2 \equiv 0,1 \pmod{3}$.

Case 4: $x > 0$ and $y > 0$, we separate into two subcases as follows:

Subcase 4.1: x is odd. It implies that $3^x \equiv 3 \pmod{4}$. We get from (1) that $z^2 \equiv 3^x - 1 \pmod{4}$ or $z^2 \equiv 2 \pmod{4}$. It is impossible because $z^2 \equiv 0,1 \pmod{4}$.

Subcase 4.2: x is even. We let $x = 2k, \exists\, k \in \mathbb{Z}^+$. We obtain $5^y = 3^{2k} - z^2$ or $5^y = (3^k - z)(3^k + z)$. Then, there exists $\alpha \in \{0,1,2,3,\ldots, y\}$ such that $3^k - z = 5^\alpha$ and $3^k + z = 5^{y-\alpha}$ with $\alpha < y - \alpha$. We obtain

$$2\cdot 3^k = 5^{\alpha}\left(1+5^{y-2\alpha}\right). \qquad (2)$$

Since $5 \nmid 2\cdot 3^k$ and (2), it yields that $\alpha = 0$. Thus, we have

$$2\cdot 3^k = 1+5^y. \qquad (3)$$

By (3), if $k=1$, then we obtain $5^y = 5$. It is easy to obtain that $y=1$, $x=2$, and $z = 3^1 - 1 = 2$. Hence, another solution is $(x,y,z)=(2,1,2)$.

If $k \geq 2$, we obtain from (3) that $5^y \equiv -1 \pmod 9$ which implies that $5^y \equiv 5^3 \pmod 9$. By theorem 2.3 and $\operatorname{ord}_9 5 = 6$, we get $y \equiv 3 \pmod 6$ yielding $y = 3+6l = 3(1+2l)$, where l is a non-negative integer. It is convenient to let $m = 1+2l$, so $y = 3m$. By (3), we have $2\cdot 3^k = 1+\left(5^3\right)^m$ or

$$2\cdot 3^k = 126\left(\left(5^3\right)^{m-1} - \left(5^3\right)^{m-2} + \cdots + \left(5^3\right)^2 - 5^3 + 1\right). \qquad (4)$$

Since $7 \mid 126$ and (4), we have $7 \mid 2\cdot 3^k$ which is impossible. In all cases, the solutions (x,y,z) of $3^x - 5^y = z^2$ are $(x,y,z)=(0,0,0)$ and $(2,1,2)$. □

4. Conclusion

We have proved and shown all solutions of the exponential Diophantine equation $3^x - 5^y = z^2$ where x, y and z are non-negative integers. In the proof, the modular arithmetic principle was applied to obtain all solutions. We have found that the solutions are $(x,y,z)=(0,0,0)$ and $(2,1,2)$.

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