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Feebly p-clean Properties in Amalgamated Rings

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Abstract. In 1977, Nicholson initially proposed the idea of a "clean ring," where the ring R is called a clean ring if for each $x \in R$ there exist $e \in Id(R)$ and $u \in U(R)$ such that $x = e + u$. This paper proposes the idea of a feebly p-clean ring, where a ring R is said to be feebly p-clean if each member r can be expressed as $r = p + f - e$, where p is a pure element and f and e are orthogonal idempotents. In this paper, the transfer of the notion of feebly p-clean rings to the amalgamation of rings along an ideal is studied. In particular, the necessary and sufficient conditions for amalgamation to be a feebly p-clean ring are studied.

Keywords: clean ring, p-clean ring, feebly clean ring, feebly p-clean ring.

AMS Mathematics Subject Classification (2010): 13A15, 13D05

1. Introduction

As defined by Nicholson [10], an element r in a ring R is clean if it can be written as $r = u+e$, where $u \in U(R)$, the group of units of R, and $e \in Id(R)$, the set of idempotents of R. A ring R is clean if every element is clean. Arora and Kundu [3] introduced the concept of a feebly clean ring. An element r of a ring with identity is called feebly clean if $r = u+e_1 - e_2$, where $u \in U(R)$, $e_1,e_2 \in Id(R)$, and e_1,e_2 are orthogonal, that is, $e_1 e_2 = e_2 e_1 = 0$. A ring R is called a feebly clean ring if every element of R is feebly clean. They also studied S-feebly clean rings. For a non-empty $S \subseteq Id(R)$, R is a S-feebly clean ring if each $r \in R$ can be written as $r = u + e_1 - e_2$, where u is a unit and e_1, e_2 are orthogonal idempotents from S. An element p in a ring R is called a pure element if there exists q in R such that $p = pq$ [8], and the set of pure elements in R is written $Pu(R) = \{p \in R: p = pq, \text{ for some } q \in R\}$. In [9], Mohammed et al. detailed the concept of a p-clean ring: an element $c \in R$ is called p-clean if there exists $e \in Id(R)$ and $p \in Pu(R)$ such that $c = e + p$. The ring R is called a p-clean ring if each element in R expresses itself as the sum of an idempotent element and a pure element.

Let R and S be two rings with unity; let J be an ideal of S; and let $\phi: R \to S$ be a ring homomorphism. In [2], Anna et al. introduced and studied the new ring structure of the following subring of $R \times S$: $R \bowtie^{\phi} J := \{(r, f(r) + j) \mid r \in R, j \in J\}$ called the amalgamation of R with S along J with respect to ϕ. This new ring structure construction is a generalization of the amalgamated duplication of a ring along an ideal. Aruldoss et al. [4], Aruldoss and

Selvaraj [5, 6], Selvaganesh and Selvaraj [13], and Vijayanand and Selvaraj [14, 15] studied some ring properties and modules characterized via amalgamation construction.

The notion of a regular element was first introduced by von Neumann [16], where an element $r \in R$ is called a regular if there exists $s \in R$ such that $r = rsr$. A ring R is called a regular ring if each element in R is regular. Wardayani et al. [17] also studied regular rings and their properties. Ashrafi and Nasibi [7] introduced the concept of the r-clean ring, where the ring R is called r-clean ring if for each $a \in R$ there exists $e \in Id(R)$ and $r \in Reg(R)$ such that $a = e + r$. Anderson and Badawi [1] studied the idea of a von Neumann local ring, where a ring R is called a von Neumann local ring if for each $r \in R$ we have either $r \in Reg(R)$ or $1 - r \in Reg(R)$. Saravanan [11, 12] studied feebly r-clean rings, feebly r-clean ideals, and their properties. An element x in a ring R is called feebly r-clean if there exists a regular element $r \in Reg(R)$ and orthogonal idempotents $e, f \in Id(R)$ such that $x = r + e - f$. A ring R is called a feebly r-clean ring if every element of R is feebly r-clean.

This paper proposes the idea of a feebly p-clean ring and studies the transfer of the notion of feebly p-clean rings to the amalgamation of rings along an ideal. In particular, the necessary and sufficient conditions for amalgamation to be a feebly p-clean ring are studied. U(R), Id(R), Nilp(R), and Pu(R) denote the set of unit elements, the set of idempotents, the set of nilpotent elements, and the set of all pure elements of R, respectively.

The paper is organized as follows: In Section 2, the concept of a feebly p-clean ring is introduced and many properties of feebly p-clean rings are studied. In Section 3, the transfer of the notion of feebly p-clean rings to the amalgamation of rings along an ideal is studied. In particular, the necessary and sufficient conditions for amalgamation to be a feebly p-clean ring are studied. Section 4 contains conclusions.

2. Feebly p-clean ring

In this section, the concept of a feebly p-clean ring is introduced and many properties of feebly p-clean rings are studied.

Definition 2.1. An element $r \in R$ is called feebly p-clean if $r = p + f - e$, where $p \in Pu(R)$, $f, e \in Id(R)$, and f, e are orthogonal, that is, $fe = ef = 0$.

Definition 2.2. Let R be a ring. Then R is called a feebly p-clean ring if each element of R is feebly p-clean.

Example 2.3. The matrix ring $M_2(Z_3)$ is a feebly p-clean ring.

Proof: For any element $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z_3)$, there is an element $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in M_2(Z_3)$ such that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then all the elements in $M_2(Z_3)$ are pure elements. Clearly,

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are orthogonal idempotents in $M_2(Z_3)$. Now one can write $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z_3)$ as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-1 & b \\ c & d+1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Therefore, each element of $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(Z_3)$ is feebly p-clean. Hence, $M_2(Z_3)$ is a feebly p-clean ring.

Definition 2.4. Let R be a ring and a non-empty set S ⊆ Id(R). Then R is called a S-feebly p-clean ring if each element r of R can be written as r = p + f – e, where p is a pure element and f and e are orthogonal idempotents from S.

Proposition 2.5. Every feebly clean ring is a feebly p-clean ring.
Proof: Let R be a feebly clean ring, and r ∈ R. Then r = u + f – e, where u ∈ U(R) and f, e ∈ Id(R), with fe = ef = 0. To prove that r is feebly p-clean, it remains only to prove that u is a pure element. Since u ∈ U(R), then u = u.1, and so u is a pure element. Thus, r is feebly p-clean. Therefore, R is a feebly p-clean ring.

The converse of the above proposition is not true.

Example 2.6. The matrix ring $M_2(Z_3)$ is a feebly p-clean ring but not a feebly clean ring because not each element in $M_2(Z_3)$ is a unit.

We can choose Id(R) = {0,1} to show the converse of the previous proposition.

Theorem 2.7. Let R be a ring and Id(R) = {0,1}. Then R is a feebly p-clean ring if and only if it is a feebly clean ring.
Proof: By Proposition 2.5, every feebly clean ring is a feebly p-clean ring. Conversely, suppose that R is a feebly p-clean ring. Let r ∈ R. Since r is a feebly p-clean, there exist p ∈ Pu(R), f, e ∈ Id(R), and f, e are orthogonal such that r = p + f – e. Since p ∈Pu (R), then there is a non-zero element d ∈ R such that p = pd. Consider d = qp, then p = pqp. Now, $(pq)^2$ = (pq) (pq) = (pqp) q = pq, which implies that pq ∈ Id (R), and hence, by hypothesis, either pq = 0 or pq = 1. If pq = 0, then p = 0 or q = 0, which is a contradiction, thus pq = 1. On the other hand, $(qp)^2$ = (qp) (qp) = q(pqp) = qp, which implies that qp ∈ Id (R), and hence by hypothesis either qp = 0 or qp = 1. If qp = 0, then p = 0 or q = 0, which is a contradiction, thus qp = 1. This implies that p ∈ U(R). Then r is the sum of a unit and orthogonal idempotent elements, and hence r is a feebly clean element. Therefore, R is a feebly clean ring.

Proposition 2.8. Every feebly r-clean ring is a feebly p-clean ring.
Proof: Let R be a feebly r−clean ring, and let x ∈ R. Then x = r + e – f, where r ∈ Reg (R) and orthogonal idempotents e and f ∈ Id(R). To prove x is a feebly p−clean element in R, it is enough to prove that r is a pure element. Since r ∈ Reg (R), then there is s ∈ R such

that r = r s r. Let q = s r, then q ∈ R. Hence, r = r q. Thus, r is a pure element, which implies x is a feebly p-clean element. Therefore, R is a feebly p-clean ring.

The converse of the above proposition is not true.

Example 2.9. The ring (Z, +, ·) is a feebly p-clean ring but not a feebly r-clean ring because each element in Z is not a regular element.

Proposition 2.10. Let R be a ring and r ∈ R. Then r is feebly p-clean if and only if – r is feebly p-clean.

Proof: Let R be a ring, and r ∈ R. Assume that r is feebly p − clean, then r = p + f – e with p ∈ Pu(R) and f, e ∈ Id(R) and f, e are orthogonal. Now that − r = − (p + f – e) = − p + e – f, we have to prove that (−p) ∈ Pu (R). Since p ∈ Pu (R), then there is q ∈ R such that p = pq. Hence – p = −pq = (−p) q, thus (−p) ∈ Pu (R). Therefore, – r is feebly p − clean. Conversely, let – r be feebly p−clean, then – r = p + f – e with p ∈ Pu(R) and f, e ∈ Id(R) and f, e are orthogonal. Now, r = − (p + f – e) = − p + e – f; as a prior proof, (−p) ∈ Pu (R), and so r is a feebly p–clean.

Proposition 2.11. Let R be a feebly p-clean ring and R′ be a ring. If *f*: R → R′ is an epimorphism, then R′ is a feebly p-clean ring.

Proof: Let r′ ∈ R′. Since *f*: R → R′ is an epimorphism, there is an r ∈ R such that r′ = *f*(r). Since R is a feebly p-clean ring, r = p + f – e with p ∈ Pu(R) and f, e ∈ Id(R) and f, e are orthogonal. Now r′ = *f*(r) = *f*(p + f – e) = *f* (p) + *f* (f) – *f* (e). Now, it needs to be proven that *f*(p) ∈ Pu (R′), *f*(f) ∈ Id(R′), and *f*(e) ∈ Id(R′). Since p ∈ Pu (R), then there is q ∈ R such that p = pq. Hence, *f*(p) = *f*(pq) = *f* (p).*f*(q), but q ∈ R, then *f* (q) ∈ R′, which implies that *f*(p) ∈ Pu (R′), Since f, e ∈ Id (R), then f^2= f and e^2= e. Hence *f*(f) =*f*(f^2) = $[f(f)]^2$ and *f*(e) =*f*(e^2) = $[f(e)]^2$. Also, *f*(f) *f*(e) = *f*(f.e) = f(0) = 0, and *f*(e) *f*(f) = *f*(e.f) = f(0) = 0. Thus, *f*(f) and *f*(e) ∈ Id (R′) and are orthogonal. Hence r′ is a feebly p-clean. Therefore, R′ is a feebly p-clean ring.

Proposition 2.12. Let I be an ideal of a feebly p-clean ring R. Then R/I is a feebly p-clean ring.

Proof: Let r + I ∈ R/I. Then r ∈ R, since R is a feebly p−clean ring, r = p + f – e with p ∈ Pu(R) and f, e ∈ Id(R) and f, e are orthogonal. Hence, r + I = p + f – e + I = p + I + f + I – e + I. To prove r + I is a feebly p-clean element in R/I, we have to prove that p + I is a pure element in R/I and f + I and e + I are orthogonal idempotent elements in R/I. Since p ∈ Pu (R), there is q ∈ R such that p = pq. Now, p+I = pq+I = (p+I) (q+I), and so p+I is a pure element in R/I. Since f and e are in Id(R), f^2 = f and e^2 = e. Hence, f + I = f^2 + I = f. f + I = (f + I). (f + I) = $(f + I)^2$, and e + I = e^2 + I = e. e + I = (e + I). (e + I) = $(e + I)^2$. Also (f + I). (e + I) = f. e + I = I. Similarly, (e + I). (f + I) = I. Thus, f + I and e + I are orthogonal idempotent elements in R/I. Thus, r+I is a feebly p-clean element in R/I. Hence, R/I is a feebly p − clean ring.

Theorem 2.13. For every ring R, there are the following statements:

(i) If e is a central idempotent element of R and eRe and (1−e)R(1−e) are both feebly p-clean, then so is R;

(ii) Let $e_1, e_2, \cdots, e_n$ be orthogonal central idempotents with $e_1+e_2+\cdots+e_n = 1$. Then e_iRe_i is feebly p-clean for each i, if and only if so is R.
(iii) If R is feebly p-clean, then so is the matrix ring $M_n(R)$ for any $n > 1$.

Proof: (i) For convenience, write $\bar{e} = 1 - e$ for each $e \in Id(R)$. We use the Pierce decomposition of A: we have

$$R = eRe \oplus eR\bar{e} \oplus \bar{e}Re \oplus \bar{e}R\bar{e}.$$

Since e, $\bar{e}$ are central, we have $R = eRe \oplus \bar{e}R\bar{e} \cong \begin{bmatrix} eRe & 0 \\ 0 & \bar{e}R\bar{e} \end{bmatrix}$.

Then each matrix $M \in R$ is of the form $\begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix}$, where $m \in eRe$ and $n \in \bar{e}R\bar{e}$.

By hypothesis, m and n are feebly p-clean. Then $m = p_1 + f_1 - e_1$ and $n = p_2 + f_2 - e_2$, where $p_1 \in Pu(eRe) \subseteq Pu(R)$, $p_2 \in Pu(\bar{e}R\bar{e}) \subseteq Pu(R)$, $f_1, e_1 \in Id(eRe) \subseteq Id(R)$, $f_2, e_2 \in Id(\bar{e}R\bar{e}) \subseteq Id(R)$ and f_i and e_i are orthogonal for i = 1, 2. Then

$$M = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} = \begin{bmatrix} p_1 + f_1 - e_1 & 0 \\ 0 & p_2 + f_2 - e_2 \end{bmatrix} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} + \begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix} - \begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}.$$

Since $p_1, p_2 \in Pu(R)$, there exist q_1, q_2 in R such that $p_1 = p_1q_1$ and $p_2 = p_2q_2$. Hence, we have $\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}\begin{bmatrix} q_1 & 0 \\ 0 & q_2 \end{bmatrix} = \begin{bmatrix} p_1q_1 & 0 \\ 0 & p_2q_2 \end{bmatrix} = \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ and so $\begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix}$ is a pure element.

Clearly, $\begin{bmatrix} f_1 & 0 \\ 0 & f_2 \end{bmatrix}$ and $\begin{bmatrix} e_1 & 0 \\ 0 & e_2 \end{bmatrix}$ are orthogonal idempotents, since f_i and e_i are orthogonal idempotents for i = 1, 2. Hence R, is a feebly p-clean ring.
(ii) By induction, one direction of (ii) comes from (i). Proposition 2.12 provides an alternative direction.
(iii) follows from (ii).

Theorem 2.14. Let R_k (k = 1, 2, $\cdots$, n) be a feebly p-clean ring. Then $\prod_{k=1}^{n} R_k$ is a feebly p-clean ring.

Proof: Let R_k (k = 1, 2, $\cdots$, n) be a feebly p-clean ring. Let $r = (r_k) \in \prod_{k=1}^{n} R_k$. For each k, there exist $p_k \in Pu(R_k)$ and orthogonal idempotents $f_k, e_k \in Id(R_k)$ such that $r_k = p_k + f_k - e_k$. Then $r = p + f - e$, where $p = (p_k) \in Pu\left(\prod_{k=1}^{n} R_k\right)$ and $f = (f_k)$ and $e = (e_k)$ are orthogonal idempotents of $\prod_{k=1}^{n} R_k$. Hence, $\prod_{k=1}^{n} R_k$ is a feebly p-clean ring.

Proposition 2.15. Every homomorphic image of a feebly p-clean ring is a feebly p-clean ring.

Proof: Since every homomorphic image of a pure element and an idempotent element is a pure and an idempotent element, respectively, every homomorphic image of a feebly p-clean ring is a feebly p-clean ring.

3. Feebly p-clean properties in amalgamated rings

In this section, the transfer of the notion of feebly p-clean rings to the amalgamation of rings along an ideal is studied.

Proposition 3.1 Let ϕ: R → S be a ring homomorphism and J an ideal of S. If $R \bowtie^{\phi} J$ is a feebly p-clean ring, then R and f(R) + J are feebly p-clean rings.

Proof: Define p_R: $R \bowtie^{\phi} J$ → R by $p_R(r, \phi(r) + k) = r$ and p_S: $R \bowtie^{\phi} J$ → S by $p_S(r, \phi(r) + k) = \phi(r) + k$. Then $R \bowtie^{\phi} J /(\{0\} \times J) \cong R$ and $R \bowtie^{\phi} J /(\phi^{(-1)}(J) \times \{0\}) \cong \phi(R) + J$. By proposition 2.15, R and $\phi(R)$ + J are feebly p-clean rings.

The converse of the above proposition is not true.

Proposition 3.2. Let ϕ: R → S be a ring homomorphism and J an ideal of S. Assume that $(\phi(R)+J)/J$ is uniquely feebly p-clean and S is an integral domain. Then $R \bowtie^{\phi} J$ is a feebly p-clean ring if and only if R and $\phi(R)$ + J are feebly p-clean rings.

Proof: By Proposition 3.1, $R \bowtie^{\phi} J$ is a feebly p-clean ring, which implies R and $\phi(R)$ + J are feebly p-clean rings. Conversely, if it is assumed that R and $\phi(R)$ + J are feebly p-clean rings. Since R is feebly p-clean, we can write r = p + f – e with p ∈ Pu(R) and f, e ∈ Id(R) and f, e are orthogonal. Similarly, since $\phi(R)$ + J is feebly p-clean, we can write $\phi(r) + j = \phi(p_1) + j_1 + \phi(f_1) + j_2 - (\phi(e_1) + j_3)$ with $\phi(p_1) + j_1$ is a pure element and $\phi(f_1) + j_2$ and $\phi(e_1) + j_3$ are orthogonal idempotent elements. Clearly, $\overline{\phi(p_1)} = \overline{\phi(p_1) + j_1}$ = (resp., $\overline{\phi(p)}$) is a pure element of $(\phi(R) + J)/J$ and $\overline{\phi(f_1)} = \overline{\phi(f_1) + j_2}$ (resp., $\overline{\phi(f)}$) and $\overline{\phi(e_1)} = \overline{\phi(e_1) + j_3}$ (resp., $\overline{\phi(e)}$) are orthogonal idempotent elements of $(\phi(R) + J)/J$. Then we have $\overline{\phi(r)} = \overline{\phi(p)} + \overline{\phi(f)} - \overline{\phi(e)} = \overline{\phi(p_1)} + \overline{\phi(f_1)} - \overline{\phi(e_1)}$. Since $(\phi(R) + J)/J$ is uniquely feebly p-clean, $\overline{\phi(p)} = \overline{\phi(p_1)}$, $\overline{\phi(f)} = \overline{\phi(f_1)}$ and $\overline{\phi(e)} = \overline{\phi(e_1)}$. Consider j_1', j_2', j_3' ∈ J such that $\phi(p_1) = \phi(p) + j_1'$, $\phi(f_1) = \phi(f) + j_2'$ and $\phi(e_1) = \phi(e) + j_3'$. Then $(r, \phi(r) + j) = (p + f - e, \phi(p_1) + j_1 + \phi(f_1) + j_2 - (\phi(e_1) + j_3)) = (p, \phi(p) + j_1' + j_1) + (f, \phi(f) + j_2' + j_2) - (e, \phi(e) + j_3' + j_3)$. Clearly, $(f, \phi(f) + j_2' + j_2)$ and $(e, \phi(e) + j_3' + j_3)$ are an orthogonal idempotent elements of $R \bowtie^{\phi} J$. Since $\phi(p) + j_1' + j_1$ is pure in $\phi(R)$ + J, there exists an element $\phi(a_0) + j_0$ such that $\phi(p) + j_1' + j_1 = (\phi(p) + j_1' + j_1)(\phi(a_0) + j_0)$. Since p = pq for some q ∈ R, we have $\overline{\phi(p)\phi(q)} = \overline{\phi(p)} = \overline{\phi(p)\phi(a_0)}$. Since S is an integral domain, $\overline{\phi(q)} = \overline{\phi(a_0)}$. This implies $\phi(a_0) = \phi(q) + j_0'$ and hence $\phi(a_0) + j_0 = \phi$

(q) + j_0' + j_0. Therefore, ϕ(p) + j_1' + j_1 = (ϕ(p) + j_1' + j_1)(ϕ(q) + j_0' + j_0). Hence, (p, ϕ(p) + j_1' + j_1) = (pq, (ϕ(p) + j_1' + j_1)(ϕ(q) + j_0' + j_0)) = (p, ϕ(p) + j_1' + j_1) (q, ϕ(q) + j_0' + j_0). Therefore, (p, ϕ(p) + j_1' + j_1) is a pure element in R ⋈$^\phi$ J and hence (r, ϕ(r) + j) is a feebly p-clean in R ⋈$^\phi$ J. Hence, R ⋈$^\phi$ J is a feebly p-clean ring.

Remark 3.3. Let ϕ: R → S be a ring homomorphism and J an ideal of S.

1. If S = J, then R ⋈$^\phi$ S is feebly p-clean if and only if R and S are feebly p-clean since R ⋈$^\phi$ J = R × S
2. If ϕ^{-1}(J) = 0, then by [Anna et al. [2], Proposition 5.1(3)], R ⋈$^\phi$ J is feebly p-clean if and only if ϕ(R) + J is feebly p-clean.

Corollary 3.4. Let R be a ring and I an ideal such that R/I is uniquely feebly p-clean. Then R ⋈$^\phi$ J is feebly p-clean if and only if R is feebly p-clean.

Theorem 3.5. Let ϕ: R → S be a ring homomorphism and J an ideal of S such that ϕ(p) + j ∈ Pu(S) for each p ∈ Pu(R) and j ∈ J. Then R ⋈$^\phi$ J is feebly p-clean if and only if R is feebly p-clean.

Proof: By Proposition 3.1, R ⋈$^\phi$ J is feebly p-clean, which implies R is feebly p-clean. Conversely, assume that R is feebly p-clean and ϕ(p) + j ∈ Pu(S) for each p ∈ Pu(R) and j ∈ J. Since R is feebly p-clean, we can write r = p + f – e with p ∈ Pu(R) and f, e ∈ Id(R) and f, e are orthogonal. Since p is pure in R, there exists q ∈ R such that p = pq. Therefore, (p, ϕ(p) + j) (q, ϕ(q)) = (pq, (ϕ(p) + j) (q)) = (pq, ϕ(p) ϕ(q)+j) = (p, ϕ(p)+j). Thus, (p, ϕ(p)+j) is pure in R ⋈$^\phi$ J. Hence, (r, ϕ(p)+j) = (p, ϕ(p)+j) + (f, ϕ(f)) – (e, ϕ(e)) in R ⋈$^\phi$ J such that (p, ϕ(p)+j) ∈ Pu(R) and (f, ϕ(f)), (e, ϕ(e)) ∈ Id(R) and (f, ϕ(f)) and (e, ϕ(e)) are orthogonal. Therefore, R ⋈$^\phi$ J is a feebly p-clean.

Theorem 3.6. Let ϕ: R → S be a ring homomorphism and J an ideal of S. Set $\overline{R}$ = R/Nilp(R), $\overline{S}$ = S/Nilp(S). $\pi : S \to \overline{S}$, the canonical projection, and $\overline{J} = \pi(J)$. Consider a ring homomorphism $\overline{\phi} : \overline{R} \to \overline{S}$ defined by $\overline{\phi}\left(\overline{r}\right) = \overline{\phi(r)}$. Then R ⋈$^\phi$ J is feebly p-clean (resp., uniquely feebly p-clean) if and only if $\overline{R}$ ⋈$^\phi$ $\overline{J}$ is feebly p-clean (resp., uniquely feebly p-clean).

Proof: Clearly, $\overline{\phi}$ is well defined and a ring homomorphism. Consider the map χ : (R ⋈$^\phi$ J)/Nilp(R ⋈$^\phi$ J) → $\overline{R}$ ⋈$^\phi$ $\overline{J}$ defined by $\chi\left(\overline{(r, \phi(r)+j)}\right) = \left(\overline{r}, \overline{\varphi}\left(\overline{r}\right) + \overline{j}\right)$. To prove R ⋈$^\phi$ J is feebly p-clean (resp., uniquely feebly p-clean) if and only if $\overline{R}$ ⋈$^\phi$ $\overline{J}$ is feebly p-clean (resp., uniquely feebly p-clean), it is enough to prove that the above defined function χ is an isomorphism. If $\overline{(r, \phi(r)+j)} = \overline{(s, \phi(s)+j)}$, then (r − s, ϕ(r − s) + j − j′) ∈ Nilp(R ⋈$^\phi$ J). Therefore, r − s ∈ Nilp(R) and j − j′ ∈ Nilp(S). Then $\overline{r} = \overline{s}$ and j = j′. Hence, χ is well defined. It can be easily checked χ is a ring homomorphism. Moreover, $\left(\overline{r}, \overline{\varphi}\left(\overline{r}\right) + \overline{j}\right)$

= (0, 0) implies that r ∈ Nilp(R) and j ∈ Nilp(S). Consequently, (r, ϕ(r)+j) ∈ Nilp(R $\bowtie^{\phi}$ J). Hence, $\overline{(r,\phi(r)+j)}$ = (0, 0) and so χ is injective. Clearly, by the construction, χ is surjective and so χ is an isomorphism. Hence proved.

Proposition 3.7. Let ϕ: R → S be a ring homomorphism and let (e) be an ideal of S generated by the idempotent element e of S. Then R$\bowtie^{\phi}$(e) is feebly p-clean if and only if R and ϕ(R) + (e) are feebly p-clean. In particular, if e is an element of R, then R$\bowtie^{\phi}$(e) is feebly p-clean if and only if R is feebly p-clean.

Proof: By Proposition 3.1, R$\bowtie^{\phi}$(e) is feebly p-clean, which implies R and ϕ(R) + (e) are feebly p-clean. Conversely, assume that R and ϕ(R) + (e) are feebly p-clean. Let (r, ϕ(r) + se) be an element of R$\bowtie^{\phi}$(e) with r ∈ R and s ∈ S. Since R is feebly p-clean, there exists a pure element p and orthogonal idempotent elements f and v such that r = p + f – v. Also, since ϕ(R)+(e) is feebly p-clean, there exists a pure element p′ and orthogonal idempotent elements f′ and v′ such that ϕ(r) + se = p′ + f′ - v′. We have (r, ϕ(r) + se) = (p, ϕ(p) + (p′ − ϕ(p))e) + (f, ϕ(f) + (p′ − ϕ(f))e) – (v, ϕ(v) + (v′− ϕ(v))e). On the other hand, $[\phi(f)+(f'-\phi(f))e]^2 = [\phi(f)(1-e)+f'e]^2 = \phi(f)(1-e)+f'e = \phi(f)+(f'-\phi(f))e$ and $[\phi(v)+(v'-\phi(v))e]^2 = [\phi(v)(1-e)+v'e]^2 = \phi(v)(1-e)+v'e = \phi(v)+(v'-\phi(v))e$. Now $[\phi(f)+(f'-\phi(f))e]\,[\phi(v)+(v'-\phi(v))e] = [\phi(f)(1-e)+f'e][\phi(v)(1-e)+v'e] = \phi(fv)(1-e)+f'v'e = 0 + 0 = 0$. Similarly, $[\phi(v)+(v'-\phi(v))e]\,[\phi(f)+(f'-\phi(f))e] = 0$. Also $[\phi(p)+(p'-\phi(p))e][\phi(q)+(q'-\phi(q))e] = [\phi(p)(1-e)+p'e][\phi(q)(1-e)+q'e] = \phi(pq)(1-e)+p'q'e = \phi(p)(1-e)+p'e = \phi(p)+(p'-\phi(p))e$. Then (f, ϕ(f) + (f′ − ϕ(f))e) and (v, ϕ(v) + (v′ − ϕ(v))e) are orthogonal idempotents, and (p, ϕ(p) + (p′− ϕ(p))e) is a pure element in R$\bowtie^{\phi}$(e). Hence, R$\bowtie^{\phi}$(e) is a feebly p-clean. Moreover, if R = S and $\phi = id_A$, then R$\bowtie^{\phi}$(e) = R$\bowtie$ (e) and ϕ(R) + (e) = R. Then R is feebly p-clean ring.

4. Conclusion

This paper proposes the idea of a feebly p-clean rings and studies the transfer of the notion of feebly p-clean rings to the amalgamation of rings along an ideal. In particular, the necessary and sufficient conditions for amalgamation to be a p-clean ring are studied. This study will further help in studying the properties of other ring structures, such as the A + XB[X] and A + XB[[X]] constructions. In the future, there is scope to study the generalization of amalgamated rings, namely bi-amalgamation rings with feebly p-clean properties.

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