Annals of Pure and Applied Mathematics
Vol. 31, No. 1, 2025, 53-58
ISSN: 2279-087X (P), 2279-0888(online)
Published on 31 March 2025
www.researchmathsci.org
DOI: http://dx.doi.org/10.22457/apam.v31n1a05962

Annals of Pure and Applied Mathematics

On the Diophantine Equation $p^x + (p+1)^y + (2p+1)^z = w^2$ where p is a prime number with $p \equiv 3, 5 \pmod 8$

Suton Tadee

Department of Mathematics
Faculty of Science and Technology
Thepsatri Rajabhat University, Lopburi 15000, Thailand
E-mail: suton.t@lawasri.tru.ac.th

Received 11 February 2025; accepted 30 March 2025

Abstract. In this article, for prime p with $p \equiv 3, 5 \pmod 8$, we consider the Diophantine equation $P^x + (p+1)^y + (2p+1)^z = w^2$, where x, y, z and w are non-negative integers. The result indicates that if $p \equiv 3, 5 \pmod 8$ and the equation has a solution, then $x = 0$ and z is odd. If $p \equiv 5 \pmod 8$ and the equation has a solution, then $x = 0$ and $y \geq 1$ according to the following conditions: (i) if $y = 1$ then z is even, (ii) if $y \geq 2$, then z is odd. Moreover, if $p \equiv 5, 19 \pmod{24}$, then the equation has no solution.

Keywords: Diophantine equation; Congruence; Quadratic residue; Legendre symbol

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2014, Bacani and Rabago [1] proved that $(0,0,1,3)$, $(1,1,0,3)$ and $(3,1,2,9)$ are the only solutions (x, y, z, w) to the Diophantine equation $3^x + 5^y + 7^z = w^2$ in non-negative integers. After that, in 2019, Burshtein [2] found some non-negative integer solutions of the Diophantine equation $p^x + (p+1)^y = z^2$, where p is a prime number. Burshtein [3, 4] presented all solutions of the Diophantine equations $p^x + (p+1)^y + (p+2)^z = M^n$,when p is a prime number, $1 \leq x, y, z \leq 2$ and $n = 1, 2$. In 2022, the non-negative integer solutions of the Diophantine equation $p_1^x + p_2^y + p_3^z = M^2$,when (p_1, p_2, p_3) is a prime triplet of the forms $(p, p+2, p+6)$ and $(p, p+4, p+6)$ for $1 \leq x, y, z \leq 2$ is investigated [7]. In 2023, Laipaporn, Kaewchay and Karnbanjong [6] found some conditions for non-existence of non-negative integer solutions of the Diophantine equation $a^x + b^y + c^z = w^2$. Recently, in 2024, Siraworakun and Tadee [8] also showed some conditions for non-existence of non-negative integer solutions (x, y, z, w) of the Diophantine equation $9^x + 9^y + n^z = w^2$, where n is a positive integer.

From the above research studies, we are interested in solving the Diophantine equation

$$p^x + (p+1)^y + (2p+1)^z = w^2, \tag{1}$$

where p is a prime number with $p \equiv 3,5 \pmod 8$ and x, y, z, w are non-negative integers.

2. Preliminaries

In the beginning of this section, we review the definition and properties of the quadratic residue and the Legendre symbol.

Definition 2.1. [5, p. 171] Let p be an odd prime number and a be an integer such that $\gcd(a,p)=1$. If the quadratic congruence $x^2 \equiv a \pmod p$ has an integer solution, then a is said to be a *quadratic residue* of p. Otherwise, a is called a *quadratic non-residue* of p.

Definition 2.2. [5, p. 175] Let p be an odd prime number and a be an integer such that $\gcd(a,p)=1$. The *Legendre symbol*, $\left(\frac{a}{p}\right)$, is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 2.1. [5, p. 180] If p is an odd prime number, then

$$\left(\frac{2}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1,7 \pmod 8 \\ -1 & \text{if } p \equiv 3,5 \pmod 8. \end{cases}$$

Theorem 2.2. [5, p. 189] If $p \neq 3$ is an odd prime number, then

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } p \equiv 1,11 \pmod{12} \\ -1 & \text{if } p \equiv 5,7 \pmod{12}. \end{cases}$$

Moreover, there is an important theorem that can be used to find the non-negative integer solutions of the Diophantine equation (1), which was proved by Zhang and Li [9] in 2024.

Theorem 2.3. [9] The Diophantine equation $2 + 7^y = z^2$ has a unique non-negative integer solution $(y,z) = (1,3)$.

3. Main results

In this section, we present our results.

Lemma 3.1. Let p be a prime number with $p \equiv 3,5 \pmod 8$. If the Diophantine equation (1) has a non-negative integer solution (x, y, z, w), then $x = 0$.

Proof: Let x, y, z and w be non-negative integers such that the equation (1) is true. Assume that $x \neq 0$. Then $x \geq 1$ and so $p^x + (p+1)^y + (2p+1)^z \equiv 0+1+1 \equiv 2 \pmod p$. From the equation (1), it follows that $w^2 \equiv 2 \pmod p$. Therefore $\left(\frac{2}{p}\right) = 1$. By Theorem 2.1, we get $p \equiv 1,7 \pmod 8$. This is impossible since $p \equiv 3,5 \pmod 8$. Thus $x = 0$.

Theorem 3.2. Let p be a prime number with $p \equiv 3 \pmod 8$. If the Diophantine equation (1) has a non-negative integer solution (x, y, z, w), then $x = 0$ and z is odd.

Proof: Let x, y, z and w be non-negative integers such that the equation (1) is true. By Lemma 3.1, we get $x = 0$. Next, we consider the following cases:

Case 1. $y = 0$. From the equation (1) and $p \equiv 3 \pmod 8$, we have $w^2 \equiv 2 + (-1)^z \pmod 8$. Assume that z is even. Then $w^2 \equiv 3 \pmod 8$. This is impossible since $w^2 \equiv 0, 1, 4 \pmod 8$. Therefore, z is odd.

Case 2. $y = 1$. From the equation (1) and $p \equiv 3 \pmod 8$, we have $w^2 \equiv 5 + (-1)^z \pmod 8$. Assume that z is even. Then $w^2 \equiv 6 \pmod 8$. This is impossible since $w^2 \equiv 0, 1, 4 \pmod 8$. Therefore, z is odd.

Case 3. $y \geq 2$. From the equation (1) and $p \equiv 3 \pmod 8$, we have $w^2 \equiv 1 + (-1)^z \pmod 8$. Assume that z is even. Then $w^2 \equiv 2 \pmod 8$. This is impossible since $w^2 \equiv 0, 1, 4 \pmod 8$. Therefore, z is odd.

Corollary 3.3. If p is a prime number with $p \equiv 3 \pmod 8$, then the Diophantine equation

$$p^x + (p+1)^y + (2p+1)^{2z} = w^2 \qquad (2)$$

has no non-negative integer solution.

Proof: Assume that there exist non-negative integers x, y, z and w such that the equation (2) is true. It implies that $(x, y, 2z, w)$ is a non-negative integer solution of the equation (1). By Theorem 3.2, we obtain that $2z$ is odd, which is a contradiction. Hence, the equation (2) has no non-negative integer solution.

Corollary 3.4. If $p = 3$, then the Diophantine equation (1) has a unique non-negative integer solution $(x, y, z, w) = (0, 0, 1, 3)$.

Proof: Let x, y, z and w be non-negative integers such that the equation (1) is true. Since $p = 3$ and Theorem 3.2, we obtain that $x = 0$ and z is odd. Next, we consider the following cases:

Case 1. $y=0$. From the equation (1), it implies that $2+7^z=w^2$. By Theorem 2.3, we have $(z,w)=(1,3)$. Thus, $(x,y,z,w)=(0,0,1,3)$.
Case 2. $y=1$. From the equation (1), we have $5+7^z=w^2$. It easy to check that $z\geq 1$. Therefore, $w^2\equiv 5(\text{mod}\,7)$. This is impossible since $w^2\equiv 0,1,2,4(\text{mod}\,7)$.
Case 3. $y\geq 2$. From the equation (1), it follows that $w^2=1+4^y+7^z\equiv 1+7^z(\text{mod}\,16)$. Since z is odd, we get $w^2\equiv 8(\text{mod}\,16)$. This is impossible since $w^2\equiv 0,1,4,9(\text{mod}\,16)$.
From the three cases above, $(x,y,z,w)=(0,0,1,3)$ is the unique non-negative integer solution of the equation (1) for $p=3$.

Theorem 3.5. Let p be a prime number with $p\equiv 5(\text{mod}\,8)$. If the Diophantine equation (1) has a non-negative integer solution (x,y,z,w), then $x=0$ and $y\geq 1$ according to the following conditions:

(i) if $y=1$, then z is even,
(ii) if $y\geq 2$, then z is odd.

Proof: Let x,y,z and w be non-negative integers such that the equation (1) is true. By Lemma 3.1, we get $x=0$. Next, we consider the following cases:
Case 1. $y=0$. From the equation (1) and $p\equiv 5(\text{mod}\,8)$, we get $w^2\equiv 2+3^z\equiv 3,5(\text{mod}\,8)$. This is impossible since $w^2\equiv 0,1,4(\text{mod}\,8)$.
Case 2. $y=1$. From the equation (1) and $p\equiv 5(\text{mod}\,8)$, we have $w^2\equiv 7+3^z(\text{mod}\,8)$. Assume that z is odd. Then $w^2\equiv 2(\text{mod}\,8)$. This is impossible since $w^2\equiv 0,1,4(\text{mod}\,8)$. Thus, z is even.
Case 3. $y=2$. From the equation (1) and $p\equiv 5(\text{mod}\,8)$, we have $w^2\equiv 5+3^z(\text{mod}\,8)$. Assume that z is even. Then $w^2\equiv 6(\text{mod}\,8)$. This is impossible since $w^2\equiv 0,1,4(\text{mod}\,8)$. Thus, z is odd.
Case 4. $y\geq 3$. From the equation (1) and $p\equiv 5(\text{mod}\,8)$, we have $w^2\equiv 1+3^z(\text{mod}\,8)$. Assume that z is even. Then $w^2\equiv 2(\text{mod}\,8)$. This is impossible since $w^2\equiv 0,1,4(\text{mod}\,8)$. Thus, z is odd.

Corollary 3.6. Let p be a prime number with $p\equiv 5(\text{mod}\,8)$. If the Diophantine equation (2) has a non-negative integer solution (x,y,z,w), then $x=0$ and $y=1$.
Proof: Let x,y,z and w be non-negative integers such that the equation (2) is true. Then $(x,y,2z,w)$ is a non-negative integer solution of the equation (1). Since $2z$ is even and Theorem 3.5, we obtain that $x=0$ and $y=1$.

Corollary 3.7. If $p=13$, then the Diophantine equation (2) has a unique non-negative integer solution $(x,y,z,w)=(0,1,0,4)$.

Proof: Let x, y, z and w be non-negative integers such that the equation (2) is true. Since $p = 13$ and Corollary 3.6, we obtain that $x = 0$ and $y = 1$. From the equation (2), we get $15 + 27^{2z} = w^2$. It follows that $(w - 27^z)(w + 27^z) = 15$. Since $w - 27^z \leq w + 27^z$, we consider the following cases:

Case 1. $w - 27^z = 1$ and $w + 27^z = 15$. Then $2 \cdot 27^z = 14$ or $27^z = 7$. This is impossible.

Case 2. $w - 27^z = 3$ and $w + 27^z = 5$. Then $2 \cdot 27^z = 2$ or $27^z = 1$. It implies that $z = 0$ and so $w = 4$. Hence, $(x, y, z, w) = (0, 1, 0, 4)$ is the unique solution of the equation (2) for $p = 13$.

Theorem 3.8. Let p be a prime number with $p \equiv 5, 19 (\text{mod } 24)$.Then, the Diophantine equation (1) has no non-negative integer solution.

Proof: Assume that there exist non-negative integers x, y, z and w such that the equation (1) is true. Since $p \equiv 5, 19 (\text{mod } 24)$, we get $p \equiv 3, 5 (\text{mod } 8)$ and $p \equiv 5, 7 (\text{mod } 12)$. By Lemma 3.1, we obtain that $x = 0$. Then $p^x + (p+1)^y + (2p+1)^z \equiv 1 + 1 + 1 \equiv 3 (\text{mod } p)$. From the equation (1), we have $w^2 \equiv 3 (\text{mod } p)$. Thus $\left(\frac{3}{p}\right) = 1$. By Theorem 2.2, we get $p \equiv 1, 11 (\text{mod } 12)$. This is impossible since $p \equiv 5, 7 (\text{mod } 12)$.

Corollary 3.9. If n is a positive integer and p is a prime number with $p \equiv 5, 19 (\text{mod } 24)$, then the Diophantine equation

$$p^x + (p+1)^y + (2p+1)^z = w^{2n} \quad (3)$$

has no non-negative integer solution.

Proof: Assume that there exist non-negative integers x, y, z and w such that the equation (3) is true. Then (x, y, z, w^n) is a non-negative integer solution of the equation (1). This is impossible by Theorem 3.8.

Acknowledgements. The author would like to thank reviewers for careful reading of this manuscript and the useful comments. This work was supported by the Research and Development Institute and Faculty of Science and Technology, Thepsatri Rajabhat University, Thailand.

Conflict of interest. The paper is written by a single author so there is no conflict of interest.

Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

REFERENCES

1. J.B. Bacani and J.F.T. Rabago, On the Diophantine equation $3^x + 5^y + 7^z = w^2$, *Konuralp Journal of Mathematics*, 2(2) (2014) 64-69.

2. N. Burshtein, A note on the Diophantine equation $p^x + (p+1)^y = z^2$, *Annals of Pure and Applied Mathematics*, 19(1) (2019) 19-20.
3. N. Burshtein, Solutions of the Diophantine equations $p^x + (p+1)^y + (p+2)^z = M^2$ for primes $p \geq 2$ when $1 \leq x, y, z \leq 2$, *Annals of Pure and Applied Mathematics*, 22(1) (2020) 41-49.
4. N. Burshtein, All the solutions of the Diophantine equations $p^x + (p+1)^y + (p+2)^z = M^3$ when p is prime and $1 \leq x, y, z \leq 2$, *Annals of Pure and Applied Mathematics*, 23(1) (2021) 7-15.
5. D.M. Burton, *Elementary number theory*. 7th ed., New York: McGraw-Hill, 2010.
6. K. Laipaporn, S. Kaewchay and A. Karnbanjong, On the Diophantine equations $a^x + b^y + c^z = w^2$, *European Journal of Pure and Applied Mathematics*, 16(4) (2023) 2066-2081.
7. V. Pandichelvi and P. Sandhya, Investigation of solutions to an exponential Diophantine equation $p_1^x + p_2^y + p_3^z = M^2$ for prime triplets (p_1, p_2, p_3), *International Journal of Scientific Research and Engineering Development*, 5(2) (2022) 22-31.
8. A. Siraworakun and S. Tadee, On the Diophantine equation $9^x + 9^y + n^z = w^2$, *Journal of Science and Technology Thonburi University*, 8(2) (2024) 12-17.
9. J. Zhang and Y. Li, On the equation $(-1)^\alpha p^x + (-1)^\beta (2^k(2p-1))^y = z^2$ for prime pairs $(p, 2p-1)$, *Integers*, 24 (2024) 1-16.