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On the Diophantine Equation $7^x + 2.p^y = z^2$

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Abstract. This paper investigates the Diophantine equation $7^x + 2.p^y = z^2$, where p is a prime number and x, y, z are non-negative integers. We first observe that if $p = 2$, then the equation has only three solutions (x, y, z), namely (1, 0, 3), (0, 2, 3), and (2, 4, 9). For $p = 7$, all solutions of the equation are of the following form $(x, y, z) = (2t+1, 2t, 3.7^t)$, where t is a non-negative integer. Finally, using the properties of the Legendre symbol, we show that if $p = \pm 5, \pm 11, \pm 13 \pmod{28}$, then the equation has a unique non-negative integer solution $(x, y, z) = (1, 0, 3)$.

Keywords: Diophantine equation; Legendre symbol; Mihailescu's theorem

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

In 2019, Laipaporn, Wananiyakul and Khachorncharoenkul [1] studied the Diophantine equation $3^x + p5^y = z^2$, where p is a prime number and x, y, z are non-negative integers. In 2022, Thongnak, Chuayjan and Kaewong [2] found that $(x, y, z) = (2, 0, 10)$ is the unique non-negative integer solution of the Diophantine equation $11 \cdot 3^x + 11^y = z^2$. Moreover, they also showed that the Diophantine equation $5^x - 2 \cdot 3^y = z^2$ has no solution in non-negative integers [3]. In the same year, Siraworakun and Tadee [4] proved that two Diophantine equations $16^x + qp^y = z^4$ and $16^x - qp^y = z^4$, where p and q are prime numbers with $p \equiv 3 \pmod 4$ and $q \equiv 3 \pmod 4$, have no positive integer solution. In 2023, Tadee [5] investigated the Diophantine equations $(n+2)^x - 2 \cdot n^y = z^2$ and $(n+2)^x + 2 \cdot n^y = z^2$, where n is a positive integer with $n = 2$ or $n \equiv 3 \pmod 4$. Porto, Bousi and Ferreira [6] found non-negative integer solutions of the Diophantine equation $p \cdot 3^x + p^y = z^2$, when p is a prime number. Tadee [7] presented all non-negative integer solutions of the Diophantine equation $p^x + pq^y = z^2$, where p and q are distinct prime numbers. In 2024, Tadee [8] studied the Diophantine equation $(p+2)^x + 4 \cdot p^y = z^2$, where p and $p+2$ are prime numbers with $\operatorname{ord}_p 2 = p - 1$. In the same year, he found all non-negative integer solutions of the Diophantine equations $(p-1)^x + 2\cdot p^y = z^2$ and $(p-1)^x - 2\cdot p^y = z^2$, where p is a prime number [9]. Phosri and Tadee [10] presented some conditions of the non-existence of non-negative integer solutions for the Diophantine equations $q^x + p(2q+1)^y = z^2$ and $q^x + p(4q+1)^y = z^2$, when p and q are prime numbers. In 2025, Neres [11] studied the Diophantine equation $5^x + 3\cdot p^y = z^2$, where p is a prime number with $p \equiv 1, 2, 3, 4 \pmod 5$.

Motivated by previous studies on Diophantine equations, we will consider the Diophantine equation $7^x + 2\cdot p^y = z^2$, where p is a prime number and x, y, z are non-negative integers.

2. Preliminaries

In this section, we present some helpful Theorems and Definitions.

Theorem 2.1. (Mihailescu's theorem) [12] The Diophantine equation $a^x - b^y = 1$, where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$, has the unique integer solution $(a, b, x, y) = (3, 2, 2, 3)$.

Theorem 2.2. [13] The Diophantine equation $2 + 7^y = z^2$ has the unique non-negative solution $(y, z) = (1, 3)$.

Definition 2.1. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. If the congruence $z^2 \equiv a \pmod p$ has an integer solution, then a is said to be a quadratic residue of p. Otherwise, a is called a quadratic non-residue of p.

Definition 2.2. Let p be an odd prime and a be an integer such that $\gcd(a, p) = 1$. The Legendre symbol, $\left(\frac{a}{p}\right)$, is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue of } p \\ -1 & \text{if } a \text{ is a quadratic non-residue of } p. \end{cases}$$

Theorem 2.3. [14] Let p be an odd prime and a, b be integers with $\gcd(a, p) = 1$ and $\gcd(b, p) = 1$.

$$\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right).$$

Theorem 2.4. [15] Let p be an odd prime with $p \neq 7$.

$$\left(\frac{7}{p}\right)=\begin{cases} 1 & \text{if } p\equiv \pm1,\pm3,\pm9 \pmod{28} \\ -1 & \text{if } p\equiv \pm5,\pm11,\pm13 \pmod{28}. \end{cases}$$

3. Main results

In this section, we present our research results, starting with Case $p=2$, Case $p=7$, and Case $p\equiv \pm5,\pm11,\pm13 \pmod{28}$, respectively.

Theorem 3.1. The Diophantine equation $7^x+2\cdot 2^y=z^2$ has exactly three non-negative integer solutions (x,y,z), namely, $(1,0,3)$, $(0,2,3)$ and $(2,4,9)$.

Proof: Let x, y and z be non-negative integers such that $7^x+2^{y+1}=z^2$.

Case 1. $y=0$. Therefore $7^x+2=z^2$. By Theorem 2.2, it follows that $x=1$ and $z=3$. Then $(x,y,z)=(1,0,3)$.

Case 2. $y\geq 1$. Therefore $7^x+2^{y+1}\equiv(-1)^x \pmod 4$. It implies that $z^2\equiv(-1)^x \pmod 4$. Assume that x is odd. Then $z^2\equiv 3 \pmod 4$. This is impossible since $z^2\equiv 0,1 \pmod 4$. Therefore, x is even. There exists a non-negative integer k such that $x=2k$. Then $(z-7^k)(z+7^k)=2^{y+1}$. Thus $z-7^k=2^u$ and $z+7^k=2^{y+1-u}$ for some non-negative integer u with $u\leq y+1$. It follows that $2\cdot 7^k=2^u\left(2^{y+1-2u}-1\right)$. Then $u=1$ and so $7^k=2^{y-1}-1$ or $2^{y-1}-7^k=1$. It easy to see that $y\neq 1$. If $y=2$, then $7^k=1$ and so $k=0$. It implies that $x=0$ and $z=3$. Therefore $(x,y,z)=(0,2,3)$. Suppose that $y>2$. By Theorem 2.1, we get $k\leq 1$. Next, we consider the following subcases:

Subcase 2.1. $k=0$. Then $2^{y-1}=2$ and so $y=2$. This is impossible.

Subcase 2.2. $k=1$. Then $2^{y-1}=8$ and so $y=4$. It implies that $x=2$ and $z=9$. Then $(x,y,z)=(2,4,9)$.

From Case 1 and 2, we conclude that the Diophantine equation $7^x+2\cdot 2^y=z^2$ has exactly three non-negative integer solutions $(x,y,z)\in\{(1,0,3),(0,2,3),(2,4,9)\}$.

Theorem 3.2. All non-negative integer solutions of the Diophantine equation $7^x+2\cdot 7^y=z^2$ are of the following form $(x,y,z)=\left(2t+1, 2t, 3\cdot 7^t\right)$, where t is a non-negative integer.

Proof: Let x, y and z be non-negative integers such that $7^x+2\cdot 7^y=z^2$. Assume that $x<y$. Then $7^x\left(1+2\cdot 7^{y-x}\right)=z^2$. Since $\gcd\left(7^x, 1+2\cdot 7^{y-x}\right)=1$, there exists a non-negative integer k such that $1+2\cdot 7^{y-x}=k^2$. Then $(k-1)(k+1)=2\cdot 7^{y-x}$. We consider the following cases:

Case 1. $k-1=1$ and $k+1=2\cdot 7^{y-x}$. Then $2=2\cdot 7^{y-x}-1$ and so $3=2\cdot 7^{y-x}$. This is impossible.

Case 2. $k-1=2$ and $k+1=7^{y-x}$. Then $2=7^{y-x}-2$ and so $4=7^{y-x}$. This is impossible.

Case 3. $k-1=7^{y-x}$ and $k+1=2$. Then $2=2-7^{y-x}$ and so $0=7^{y-x}$. This is impossible.
Case 4. $k-1=2\cdot 7^{y-x}$ and $k+1=1$. Then $2=1-2\cdot 7^{y-x}$ and so $-1=2\cdot 7^{y-x}$. This is impossible.
Thus $x \ge y$. Then $7^y\left(7^{x-y}+2\right)=z^2$. Since $\gcd\left(7^y,7^{x-y}+2\right)=1$, there exists a non-negative integer t such that $y=2t$. Then $7^{x-2t}+2=\left(\frac{z}{7^t}\right)^2$. By Theorem 2.2, we obtain that $x=2t+1$ and $z=3\cdot 7^t$. Hence, all non-negative integer solutions of the Diophantine equation $7^x+2\cdot 7^y=z^2$ are of the following form $(x,y,z)=\left(2t+1,2t,3\cdot 7^t\right)$, where t is a non-negative integer.

Next, we show that $(x,y,z)=(1,0,3)$ is the unique solution of the Diophantine equation $7^x+2\cdot p^y=z^2$, where p is a prime number with $p\equiv \pm 5,\pm 11,\pm 13 \pmod{28}$. First, we will prove two helpful lemmas.

Lemma 3.3. Let p be a prime number with $p\ne 2$ and $p\ne 7$. If x is even, then the Diophantine equation $7^x+2\cdot p^y=z^2$ has no non-negative integer solution.

Proof: Assume that there exist non-negative integers x,y and z such that $7^x+2\cdot p^y=z^2$. Since x is even, we get $x=2k$ for some non-negative integer k. It implies that $\left(z-7^k\right)\left(z+7^k\right)=2\cdot p^y$. Since p is a prime number with $p\ne 2$ and $p\ne 7$, we have the following cases:
Case 1. $z-7^k=1$ and $z+7^k=2\cdot p^y$. It follows that $2\cdot 7^k=2\cdot p^y-1$ and so $2\mid 1$. This is impossible.
Case 2. $z-7^k=2$ and $z+7^k=p^y$. It follows that $2\cdot 7^k=p^y-2$ and so $2\mid p^y$. This is impossible.
Case 3. $z-7^k=p^y$ and $z+7^k=2$. It follows that $2\cdot 7^k=2-p^y$ and so $2\mid p^y$. This is impossible.
Case 4. $z-7^k=2\cdot p^y$ and $z+7^k=1$. It follows that $2\cdot 7^k=1-2\cdot p^y$ and so $2\mid 1$. This is impossible.

Corollary 3.4. Let p be a prime number with $p\ne 2$ and $p\ne 7$. Then the Diophantine equation $49^x+2\cdot p^y=z^2$ has no non-negative integer solution.

Proof: Assume that there exist non-negative integers x,y and z such that $49^x+2\cdot p^y=z^2$ or $7^{2x}+2\cdot p^y=z^2$. This is impossible since $2x$ is even and Lemma 3.3.

Lemma 3.5. Let p be a prime number with $p\equiv \pm 5,\pm 11,\pm 13 \pmod{28}$. If x is odd, then the Diophantine equation $7^x+2\cdot p^y=z^2$ has the unique non-negative integer solution $(x,y,z)=(1,0,3)$.

Proof: Let x, y and z be non-negative integers such that $7^x + 2\cdot p^y = z^2$.

Case 1. $y = 0$. Then $7^x + 2 = z^2$. By Theorem 2.2, we obtain that $(x, y, z) = (1, 0, 3)$.

Case 2. $y \geq 1$. Then $z^2 = 7^x + 2\cdot p^y \equiv 7^x \pmod{p}$. It implies that $\left(\frac{7^x}{p}\right) = 1$. By Theorem 2.3, we have $\left(\frac{7^x}{p}\right) = \left(\frac{7}{p}\right)^x = 1$. Since x is odd, we get $\left(\frac{7}{p}\right) = 1$. This is impossible since $p \equiv \pm 5, \pm 11, \pm 13 \pmod{28}$ and Theorem 2.4.

From Lemma 3.3 and 3.5, we have the following theorem.

Theorem 3.6. Let p be a prime number with $p \equiv \pm 5, \pm 11, \pm 13 \pmod{28}$. Then the Diophantine equation $7^x + 2\cdot p^y = z^2$ has the unique non-negative integer solution $(x, y, z) = (1, 0, 3)$.

4. Conclusion

Based on Mihailescu's theorem, we investigate the Diophantine equation $7^x + 2\cdot p^y = z^2$, where p is a prime number and x, y, z are non-negative integers. For $p = 2$, we establish that there are only three non-negative integer solutions to the equation. The solutions are $(x, y, z) \in \{(1, 0, 3), (0, 2, 3), (2, 4, 9)\}$. For $p = 7$, all non-negative integer solutions of the equation are of the following form $(x, y, z) = (2t + 1, 2t, 3\cdot 7^t)$, where t is a non-negative integer. Moreover, by using the properties of Legendre symbol, we prove that if $p \equiv \pm 5, \pm 11, \pm 13 \pmod{28}$, then the equation has the unique non-negative integer solution $(x, y, z) = (1, 0, 3)$.

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Authors' Contributions. It is a single-author paper. So, full credit goes to the author.

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