Some Special Properties of I-rough Topological Spaces

Boby P. Mathew¹ and Sunil Jacob John²

¹Department of Mathematics, St. Thomas College, Pala
Kottayam – 686574, India. Email: bobynitc@gmail.com

²Department of Mathematics, National Institute of Technology, Calicut
Calicut – 673601, India. Email: sunil@nitc.ac.in

Abstract. This paper extends some essential topological properties in general topological spaces into the I-rough topological spaces, the topology of the rough universe. I-rough compactness and I-rough Hausdorffness are introduced and several properties are investigated. I-rough compactness of subsets of I-rough Hausdorff spaces are studied. Also the paper establishes I-rough connectedness in an I-rough topological space.

Keywords: I-rough topological spaces; I-rough continuous functions; I-rough compactness; I-rough Hausdorffness; I-rough connectedness

Mathematics Subject Classification (2010): 54A05, 54D05, 54D30, 54D70, 03Exx

1. Introduction
Rough set theory was introduced by Pawlak [6]. There are many approaches in rough sets. Some reviews can be seen in [12, 13]. Several applications of rough set models can be seen in [7, 8]. Iwinski [1] introduced the set oriented view of rough set in an algebraic method. A topology on a non-empty set is a collection of subsets of it, satisfying certain axioms [5, 11]. Mathew & Sunil [3] introduced I-rough topological spaces. The notion of I-rough continuous functions can be seen in [4]. Some related works in generalized topological spaces and fuzzy topological spaces can be seen in [2, 9, 10]. This paper is an attempt to strengthen the topology of the rough universe by extending the concepts of compactness, Hausdorffness and connectedness of general topological spaces into I-rough topological spaces.

2. Preliminaries
Some of the basic definitions for our further study need to be quoted before introducing the new concepts. Let $U$ be any non-empty set and let $\beta$ be a complete sub-algebra of the Boolean algebra $2^U$ of subsets of $U$. Then the pair $(U, \beta)$ is called a rough universe [1]. Let $(U, \beta)$ be a given fixed rough universe. Let $R$ be a relation on $\beta$ defined by $A = (A_1, A_2) \in R$ iff $A_1, A_2 \in \beta$ and $A_1 \subseteq A_2$. The elements of $R$ are called rough sets and the elements of $\beta$ are called exact sets [1]. In order to distinguish this definition of rough sets from Pawlak’s definition, this rough set is named as an I-rough set [12]. The
Boby P. Mathew and Sunil Jacob John

element \((X, X) \in R\) is identified with the element \(X \in \beta\) and hence an exact set is a rough set in the sense of the above definition. But a rough set need not be exact. Set theoretic operators on the rough sets are defined component wise using ordinary set operations as follows [1]. Let \(X = (X_1, X_2)\) and \(Y = (Y_1, Y_2)\) be any two I-rough sets in the rough universe \((U, \beta)\). Then,

\[
X \cup Y = (X_1 \cup Y_1, X_2 \cup Y_2)
\]

\[
X \cap Y = (X_1 \cap Y_1, X_2 \cap Y_2)
\]

\(X \subseteq Y\) if \(X \cap Y = X\). That is \(X \subseteq Y\) if \(X_1 \subseteq Y_1\) and \(X_2 \subseteq Y_2\)

\[
X - Y = (X_1 - Y_2, X_2 - Y_1).
\]

Hence \(X^C = (U, U) - (X_1, X_2) = (U - X_2, U - X_1) = (X_2^C, X_1^C)\).

The above set operations on I-rough sets are named as I-rough union, I-rough intersection, I-rough inclusion, I-rough difference and I-rough complement respectively [3]. Let \((U, \beta)\) be a fixed rough universe. Then a sub collection \(\tau\) of \(R\) is an I-rough topology on \((U, \beta)\) if the following 1, 2 and 3 hold [3].

1. \((\emptyset, \emptyset) \in \tau\) and \((U, U) \in \tau\)
2. \(\tau\) is closed under finite I-Rough intersection
3. \(\tau\) is closed under arbitrary I-Rough union.

If \(\tau\) be an I-rough topology on the rough universe \((U, \beta)\). Then the triple \((U, \beta, \tau)\) is called an I-rough topological space [3]. An I-rough set \((A_1, A_2)\) is an I-rough open set in an I-rough topological space \((U, \beta, \tau)\) if \((A_1, A_2) \in \tau\) and an I-rough set \((A_1, A_2)\) is an I-rough closed set if its I-rough complement \(A^C = (U - A_2, U - A_1)\) is I-rough open [3]. Let \(A\) be any subset of \(U\). Then \(\tau / A = \{G \cap A = (G_1 \cap A, G_2 \cap A) / G = (G_1, G_2) \in \tau\}\) is an I-rough topology on the rough universe \((A, \beta / A)\) induced by \(\tau\), where \(\beta / A = \{X \cap A / X \in \beta\}\) is the complete sub-algebra \(\beta\) restricted to \(A\). Then \(\tau / A\) is called the relative I-rough topology on \(A\) or the subspace I-rough topology on \(A\) and \((A, \beta / A, \tau / A)\) is called an I-rough subspace of the I-rough topological space \((U, \beta, \tau)\)[3].

Let \((X, \beta_1)\) and \((Y, \beta_2)\) be any two rough universes and let \(f : X \rightarrow Y\) be any function. Then the function \(f\) is an I- rough function if both \(f\) and \(f^{-1}\) maps exact sets on to exact sets. That is \(f\) is an I-rough function if \(f(A) \in \beta_2, \forall A \in \beta_1\) and \(f^{-1}(A) \in \beta_1, \forall A \in \beta_2\)[4]. Let \((X, \beta_1, \tau_1)\) and \((Y, \beta_2, \tau_2)\) are any two I-rough topological spaces and let \(f : (X, \beta_1) \rightarrow (Y, \beta_2)\) be any I-rough function. Then \(f\) is an
Some Special Properties of I-rough Topological Spaces

I-rough continuous function if \( f^{-1}(A) = \left( f^{-1}(A_1), f^{-1}(A_2) \right) \in \tau \), for every \( A = (A_1, A_2) \in \tau_2 \) [4]. An I-rough function \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) is an I-rough embedding if \( f \) is one-one and both \( f \) and \( f^{-1} \) are I-rough continuous functions [4]. Also \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) is an I-rough homeomorphism if \( f \) is one-one, onto and both \( f \) and \( f^{-1} \) are I-rough continuous functions [4].

3. I-rough compactness

In this section, the paper studies the compactness of the I-rough topological spaces. This is an attempt to generalize the concept of compactness in general topological space in to I-rough topological spaces.

**Definition 3.1.** A family \( V = \{ (A_i, A_{i'}) ; i \in I \} \) of I-rough subsets of a rough universe \((X, \beta)\) is an I-rough covering of an I-rough set \((E_1, E_2)\) if \((E_1, E_2) \subseteq \bigcup_{i \in I} (A_i, A_{i'}) = \left( \bigcup_{i \in I} A_i, \bigcup_{i \in I} A_{i'} \right) \). The I-rough covering \( V \) is finite I-rough covering if \( V \) contains only finitely many I-rough sets. Also the family \( V \) of I-rough subsets of a rough universe \((X, \beta)\) is an I-rough covering of \((X, \beta)\) if \( X \) is the I-rough union of all members of \( V \).

**Definition 3.2.** Let \((X, \beta, \tau)\) be an I-rough topological space. Then an I-rough covering \( V \) of I-rough subsets of a rough universe \((X, \beta)\) is an I-rough open covering if all the members of \( V \) are I-rough open sets.

**Definition 3.3.** Let \((X, \beta)\) be a rough universe and let \( U = \{ (A_i, A_{i'}) ; i \in I \} \) and \( V = \{ (B_i, B_{i'}) ; i \in J \} \) are any two I-rough covering of \((X, \beta)\). If for each \( i \in I, (A_i, A_{i'}) = (B_{i_j}, B_{i'}) \) for some \( j \in J \), then the I-rough covering \( U = \{ (A_i, A_{i'}) ; i \in I \} \) is an I-rough sub covering of the I-rough cover \( V = \{ (B_i, B_{i'}) ; i \in J \} \).

**Definition 3.4.** An I-rough topological space \((X, \beta, \tau)\) is I-rough compact if for every I-rough open covering of \((X, \beta)\) has a finite I-rough sub covering.

**Definition 3.5.** Let \((X, \beta, \tau)\) be an I-rough topological space and let \( Y \) be any subset of \( X \). Then \( Y \) is I-rough compact if the relative I-rough subspace \((Y, \beta/Y, \tau/Y)\) is I-rough compact.
Remark 3.1. Next theorem relates the I-rough compactness of an I-rough subspace \((Y, \beta/Y, \tau/Y)\) of an I-rough topological space \((X, \beta, \tau)\) to the I-rough topology \(\tau\) of \((X, \beta, \tau)\).

Theorem 3.1. Let \((X, \beta, \tau)\) be an I-rough topological space and let \(Y\) be any subset of \(X\). Then \(Y\) is I-rough compact iff for each I-rough open covering \(\{A_i, A_j\}: i \in I\) of \(Y\) such that for every \(i \in I\), \((A_i, A_j) \in \tau\), there is a finite I-rough sub covering of \(Y\).

Proof: Let \((X, \beta, \tau)\) be an I-rough topological spaces and let \(Y\) be any subset of \(X\). Suppose \(Y\) is I-rough compact. Let \(\{A_i, A_j\}: i \in I\) be an I-rough open covering of \(Y\) using I-rough open sets of \((X, \beta, \tau)\). Which implies that for every \(i \in I\), \((A_i, A_j) \in \tau\). Then \((A_i, A_j) \cap (Y, \tau) = \tau/Y\) for every \(i \in I\). Hence \(\{A_i, A_j\} \cap (Y, \tau) : i \in I\) is an I-rough open covering of \(Y\) using I-rough open sets in the relative I-rough topology of \((Y, \beta/Y, \tau/Y)\). Since \(Y\) is I-rough compact, this I-rough covering has a finite I-rough open covering of \(Y\). Let this finite I-rough sub covering be \(\{A_i, A_j\} \cap (Y, \tau) : j = 1, 2, 3, \cdots, n\}. Then clearly \(\{A_i, A_j\} : j = 1, 2, 3, \cdots, n\} covers \(Y\). Hence for each I-rough open covering \(\{A_i, A_j\} : i \in I\) of \(Y\) such that for every \(i \in I\), \((A_i, A_j) \in \tau\), there is a finite I-rough sub covering of \(Y\).

Conversely suppose for each I-rough open covering \(\{A_i, A_j\}: i \in I\) of \(Y\) such that for every \(i \in I\), \((A_i, A_j) \in \tau\), there is a finite I-rough sub covering of \(Y\). Let \(\{A_i, A_j\}: i \in I\) be an I-rough open covering of \(Y\) using I-rough open sets in the relative I-rough topology on \(Y\). That is for every \(i \in I\), \((A_i, A_j) \in \tau/Y\). Hence \((A_i, A_j) = (G_i, G_j) \cap (Y, \tau)\) where \((G_i, G_j) \in \tau\) for every \(i \in I\). Thus \(\{G_i, G_j\}: i \in I\) is an I-rough open covering of \(Y\) using I-rough open sets in \((X, \beta, \tau)\).

Then by our supposition, there is a finite I-rough sub covering \(\{G_j, G_i\} : j = 1, 2, 3, \cdots, n\} of \(Y\). Then \(\{G_j, G_i\} \cap (Y, \tau) : j = 1, 2, 3, \cdots, n\} is a finite I-rough sub covering of \(Y\). Hence \(Y\) is I-rough compact.

Definition 3.6. Let \((X, \beta, \tau)\) be an I-rough topological space and let \((Y_1, Y_2)\) be any I-rough set of the rough universe \((X, \beta)\). Then \((Y_1, Y_2)\) is I-rough compact set if for every I-rough open covering of \((Y_1, Y_2)\) using I-rough open sets of \((X, \beta, \tau)\) has a finite I-rough sub covering.

Theorem 3.2. Let \((X, \beta_1, \tau_1)\) and \((Y, \beta_2, \tau_2)\) are any two I-rough topological spaces and let \(f : (X, \beta_1) \rightarrow (Y, \beta_2)\) be any I-rough continuous function. Then if an I-rough set
Some Special Properties of I-rough Topological Spaces

$(A_1, A_2)$ of the rough universe $(X, \beta_1)$ is I-rough compact subset of $(X, \beta_1, \tau_1)$ then $f(A_1, A_2)$ is an I-rough compact subset of $(Y, \beta_2, \tau_2)$.

**Proof:** Let $f : (X, \beta_1) \to (Y, \beta_2)$ be any I-rough continuous function and let $(A_1, A_2)$ be an I-rough set of the rough universe $(X, \beta_1)$ is I-rough compact subset of $(X, \beta_1, \tau_1)$. Let $\{(G_i, G_j) : i \in I\}$ be an I-rough open covering of $f(A_1, A_2)$ using I-rough open sets in $(Y, \beta_2, \tau_2)$. This implies that

$$f(A_1, A_2) = f(A_1) \cup f(A_2) \subseteq \bigcup_{i \in I} (G_i, G_j) = \left( \bigcup_{i \in I} G_i, \bigcup_{i \in I} G_j \right).$$

Hence, $(A_1, A_2) \subseteq \bigcup_{i \in I} f^{-1}(G_i, G_j)$. Thus $\{f^{-1}(G_i, G_j) : i \in I\}$ be an I-rough covering of $(A_1, A_2)$. Since $f : (X, \beta_1) \to (Y, \beta_2)$ be any I-rough continuous function and $(G_i, G_j) \in \tau_2$ for every $i \in I$, clearly $f^{-1}(G_i, G_j) \in \tau_1$ for every $i \in I$. Hence $\{f^{-1}(G_i, G_j) : i \in I\}$ be an I-rough open covering of $(A_1, A_2)$. Since $(A_1, A_2)$ is I-rough compact set of $(X, \beta_1, \tau_1)$, it has a finite I-rough sub covering $\{f^{-1}(G_i, G_j) : i = 1, 2, 3, \ldots, n\}$ of $(A_1, A_2)$. That is $(A_1, A_2) \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i, G_j)$. Which implies that $f(A_1, A_2) = (f(A_1), f(A_2)) \subseteq \bigcup_{i=1}^{n} f^{-1}(G_i, G_j)$. Since $\{(G_i, G_j) : i \in I\}$ be an arbitrary I-rough open covering of $f(A_1, A_2)$ the theorem follows by theorem 3.1.

**Theorem 3.3.** Let $(X, \beta_1, \tau_1)$ and $(Y, \beta_2, \tau_2)$ are any two I-rough topological spaces, where $(X, \beta_1, \tau_1)$ is I-rough compact and $f : (X, \beta_1) \to (Y, \beta_2)$ be any I-rough continuous and onto function. Then $(Y, \beta_2, \tau_2)$ is I-rough compact.

**Proof:** Since $f : (X, \beta_1) \to (Y, \beta_2)$ is an onto function, clearly $f(X, X) = Y, Y)$. Then the proof follows from theorem 3.2.

**Theorem 3.4.** Let $(X, \beta_1, \tau_1)$ and $(Y, \beta_2, \tau_2)$ are I-rough homeomorphic I-rough topological spaces. Then $(X, \beta_1, \tau_1)$ is I-rough compact if $(Y, \beta_2, \tau_2)$ is I-rough compact.

**Proof:** Proof follows directly from theorem 3.3 and the definition of I-rough homeomorphism.

**Theorem 3.5.** Let $(X, \beta, \tau)$ be an I-rough compact I-rough topological space then every I-rough closed subsets of $(X, \beta, \tau)$ are also I-rough compact.

**Proof:** Let $(D_1, D_2)$ be any I-rough closed subset of the I-rough topological space $(X, \beta, \tau)$ where $(X, \beta, \tau)$ is I-rough compact. Let $\{(A_i, A_j) : i \in I\}$ of $(D_1, D_2)$ such that for every $i \in I$, $(A_i, A_j) \in \tau$. Since $(D_1, D_2)$ is an I-rough closed subset of the I-
Boby P. Mathew and Sunil Jacob John

rough topological space \((X, \beta, \tau)\) implies \((X - D_{2}, X - D_{1}) \in \tau\). Now adjoin \((X - D_{2}, X - D_{1})\) to \(\{A_{i}, A_{i}^{j}: i \in I\}\) to get an I-rough open covering of \((X, \beta, \tau)\).

But since \((X, \beta, \tau)\) is I-rough compact it has a finite I-rough sub covering \(\{A_{i}, A_{i}^{j}: j = 1, 2, 3, \ldots, n\}\), which may or may not contains \((X - D_{2}, X - D_{1})\). If it contains \((X - D_{2}, X - D_{1})\), delete it from the finite I-rough sub covering to get a finite I-rough sub covering of \((X - D_{2}, X - D_{1}) = (D_{1}, D_{2})\). Since \(\{A_{i}, A_{i}^{j}: i \in I\}\) is arbitrary, the theorem follows by theorem 3.1.

**Remark 3.2.** Next theorem characterizes the I-rough compactness of an I-rough topological space in terms of its I-rough closed sets instead of I-rough open sets.

**Theorem 3.6.** An I-rough topological space \((X, \beta, \tau)\) is I-rough compact iff whenever a family \(\{F_{\alpha}, F_{\alpha}^{\epsilon}: \alpha \in I\}\) of I-rough closed sets of \((X, \beta, \tau)\) such that \(\bigcap_{\alpha \in I} F_{\alpha} = (\phi, \phi)\) then there is a finite subset of indices \(\alpha_{1}, \alpha_{2}, \alpha_{3}, \ldots, \alpha_{n}\) such that \(\bigcap_{i=1}^{n} F_{\alpha_{1}} \bigcap_{i=1}^{n} F_{\alpha_{2}} = (\phi, \phi)\).

**Proof:** Suppose that \((X, \beta, \tau)\) be an I-rough topological space which is I-rough compact. Let \(\{F_{\alpha}, F_{\alpha}^{\epsilon}: \alpha \in I\}\) be a family of I-rough closed sets of \((X, \beta, \tau)\) such that \(\bigcap_{\alpha \in I} F_{\alpha} = (\phi, \phi)\). Then \(\bigcup_{\alpha \in I} (F_{\alpha}^{\epsilon}, F_{\alpha}^{\epsilon}) = \bigcap_{\alpha \in I} F_{\alpha}^{\epsilon} = (\phi, \phi)^{\epsilon} = (X, X)\). Since each \((F_{\alpha}^{\epsilon}, F_{\alpha}^{\epsilon})\) is I-rough closed, \((F_{\alpha}^{\epsilon}, F_{\alpha}^{\epsilon})^{\epsilon}\) is I-rough open in \((X, \beta, \tau)\). Hence \(\bigcup_{\alpha \in I} (F_{\alpha}^{\epsilon}, F_{\alpha}^{\epsilon})^{\epsilon}\) is an I-rough open covering of \(X\), which is I-rough compact. Hence there is a finite I-rough sub covering \((F_{\alpha_{1}}^{\epsilon}, F_{\alpha_{1}}^{\epsilon})^{\epsilon}, (F_{\alpha_{1}}^{\epsilon}, F_{\alpha_{1}}^{\epsilon})^{\epsilon}, \ldots, (F_{\alpha_{n}}^{\epsilon}, F_{\alpha_{n}}^{\epsilon})^{\epsilon}\), such that \(\bigcup_{i=1}^{n} (F_{\alpha_{1}}^{\epsilon}, F_{\alpha_{1}}^{\epsilon})^{\epsilon} = (X, X)\). Then \(\bigcup_{i=1}^{n} (F_{\alpha_{1}}^{\epsilon}, F_{\alpha_{1}}^{\epsilon})^{\epsilon} = (X, X) = (\phi, \phi)\). But \(\bigcup_{i=1}^{n} (F_{\alpha_{1}}^{\epsilon}, F_{\alpha_{1}}^{\epsilon})^{\epsilon} = \bigcap_{i=1}^{n} F_{\alpha_{1}} \bigcap_{i=1}^{n} F_{\alpha_{2}} = (\phi, \phi)\).

Conversely suppose that for each family \(\{F_{\alpha}, F_{\alpha}^{\epsilon}: \alpha \in I\}\) of I-rough closed sets of \((X, \beta, \tau)\) such that \(\bigcap_{\alpha \in I} F_{\alpha} \bigcap_{\alpha \in I} F_{\alpha}^{\epsilon} = (\phi, \phi)\), there is a finite subset of indices
Some Special Properties of I-rough Topological Spaces

\{\alpha_{i}, \alpha_{2}, \alpha_{3},...,\alpha_{n}\} \text{ such that } \left(\bigcap_{i=1}^{n} F_{\alpha_{1}}, \bigcap_{i=1}^{n} F_{\alpha_{2}}\right) = (\phi, \phi). \text{ Let } \{(A_{\beta_{1}}, A_{\beta_{2}}), \beta \in J\} \text{ be any I-rough open covering of } (X, \beta, \tau). \text{ Then } \bigcup_{\beta \in J}(A_{\beta_{1}}, A_{\beta_{2}}) = (X, X). \text{ Then } \bigcup_{\beta \in J}(A_{\beta_{1}}, A_{\beta_{2}})^{C} = (X, X)^{C} = (\phi, \phi).

That is \left(\bigcup_{\beta \in J}(A_{\beta_{1}}, A_{\beta_{2}})^{C}\right) = \bigcap_{\beta \in J}(A_{\beta_{1}}, A_{\beta_{2}})^{C} = \left(\bigcap_{\beta \in J}(A_{\beta_{2}}, A_{\beta_{1}})^{C}\right) = \left(\bigcup_{\beta \in J}(A_{\beta_{1}}, A_{\beta_{2}})^{C}\right) = (X, X).

Hence by our assumption there is a finite subset of indices \{\beta_{1}, \beta_{2}, \beta_{3},...,\beta_{n}\} \text{ such that } \left(\bigcap_{i=1}^{n} A_{\beta_{1}}, \bigcap_{i=1}^{n} A_{\beta_{2}}\right) = (\phi, \phi). \text{ Then } \left(\bigcap_{i=1}^{n} A_{\beta_{1}}, \bigcap_{i=1}^{n} A_{\beta_{2}}\right)^{C} = (\phi, \phi)^{C} = (X, X). \text{ That is } \left(\bigcap_{i=1}^{n} A_{\beta_{1}}, \bigcap_{i=1}^{n} A_{\beta_{2}}\right)^{C} = \left(\bigcap_{i=1}^{n} A_{\beta_{2}}, \bigcap_{i=1}^{n} A_{\beta_{1}}\right)^{C} = \left(\bigcup_{i=1}^{n} A_{\beta_{1}}, \bigcup_{i=1}^{n} A_{\beta_{2}}\right) = (X, X).

That is \left(\bigcup_{i=1}^{n} A_{\beta_{1}}, \bigcup_{i=1}^{n} A_{\beta_{2}}\right) = (X, X). \text{ Hence the I-rough open covering } \{(A_{\beta_{1}}, A_{\beta_{2}}), \beta \in J\}

has a finite I-rough sub covering. Since the I-rough open covering is arbitrary \(X, \beta, \tau\) is I-rough compact.

4. I-rough Hausdorff spaces

This section introduces the Hausdorff property in I-rough topological spaces and studies the I-rough compactness of subsets of I-rough Hausdorff spaces.

**Definition 4.1.** An I-rough topological space \((X, \beta, \tau)\) is I-rough Hausdorff space if for any two distinct points \(x, y\) in \(X\), there exist \((U_{1}, U_{2}), (V_{1}, V_{2}) \in \tau\), such that \(x \in (U_{1}, U_{2})\) and \((U_{1}, U_{2}) \cap (V_{1}, V_{2}) = (U_{1} \cap V_{1}, U_{2} \cap V_{2}) = (\phi, \phi)\).

**Theorem 4.1.** Let \((X, \beta, \tau)\) be an I-rough Hausdorff space and \(x \in X\). Let \((F_{1}, F_{2})\) be an I-rough set of the I-rough universe \((X, \beta)\), which is an I-rough compact subset not containing \(x\). Then there exist I-rough open sets \((U_{1}, U_{2})\) and \((V_{1}, V_{2})\) such that \(x \in (U_{1}, U_{2})\) and \((U_{1}, U_{2}) \cap (V_{1}, V_{2}) = (U_{1} \cap V_{1}, U_{2} \cap V_{2}) = (\phi, \phi)\).

**Proof:** Since \((F_{1}, F_{2})\) is I-rough compact subset not containing \(x\), for each \(y \in (F_{1}, F_{2})\), there exist I-rough open sets \((U_{y_{1}}, U_{y_{2}})\) and \((V_{y_{1}}, V_{y_{2}})\) such that \(x \in (U_{y_{1}}, U_{y_{2}})\) and \((U_{y_{1}}, U_{y_{2}}) \cap (V_{y_{1}}, V_{y_{2}}) = (U_{y_{1}} \cap V_{y_{1}}, U_{y_{2}} \cap V_{y_{2}}) = (\phi, \phi)\). Then the
Boby P. Mathew and Sunil Jacob John

family \( \{V_{y_1, V_{y_2}}: y \in (F_1, F_2)\} \) is an I-rough open covering of \((F_1, F_2)\). Since \((F_1, F_2)\) is I-rough compact, there is a finite I-rough sub cover for \((F_1, F_2)\) say \(\{V_{y_1, V_{y_2}}, V_{y_2, V_{y_2}}\}, \ldots, (V_{y_{n-1}, V_{y_n}})\}. \ Let \((U_1, U_2) = \bigcap_i (U_{y_1, U_{y_2}})\) and \((F_1, F_2) = \bigcup_i (V_{y_1, V_{y_2}})\). Then clearly \((U_1, U_2)\) and \((F_1, F_2)\) are I-rough open sets and \(x \in (U_1, U_2)\), \((F_1, F_2) \subseteq (V_1, V_2)\) and \((U_1, U_2) \cap (V_1, V_2) = (U_1 \cap V_1, U_2 \cap V_2) = (\emptyset, \emptyset)\).

**Theorem 4.2.** Let \((X, \beta, \tau)\) be an I-rough Hausdorff space and \((Y_1, Y_2)\) be an I-rough compact subset of the rough universe \((X, \beta)\). Then \((Y_1, Y_2)\) is I-rough closed.

**Proof:** Let \((Y_1, Y_2)\) be an I-rough compact subset of an I-rough Hausdorff space \((X, \beta, \tau)\). Then by theorem 4.1, for any \(x \in (X, X) - (Y_1, Y_2)\) there exist I-rough open sets \((U_1, U_2)\) and \((V_1, V_2)\) such that \(x \in (U_1, U_2)\), \((Y_1, Y_2) \subseteq (V_1, V_2)\) and \((U_1, U_2) \cap (V_1, V_2) = (U_1 \cap V_1, U_2 \cap V_2) = (\emptyset, \emptyset)\). In particular \((U_1, U_2) \cap (Y_1, Y_2) = (\emptyset, \emptyset)\), and \((U_1, U_2) \subseteq (X - Y_1, X - Y_1)\). Thus \((X, X) - (Y_1, Y_2)\) is an I-rough neighbourhood of each of its points. Hence \((X, X) - (Y_1, Y_2)\) is I-rough open and then \((Y_1, Y_2)\) is I-rough closed.

**Theorem 4.3.** Let \((X, \beta, \tau)\) be an I-rough compact Hausdorff space. Then an I-rough set \((Y_1, Y_2)\) of the rough universe \((X, \beta)\) is I-rough compact iff \((Y_1, Y_2)\) is I-rough closed.

**Proof:** Proof follows from theorem 3.5 and theorem 4.2.

**Theorem 4.4.** Let \(f : (X, \beta_1) \to (Y, \beta_2)\) be an I-rough continuous function from an I-rough compact topological space \((X, \beta_1, \tau_1)\) on to an I-rough Hausdorff topological space \((Y, \beta_2, \tau_2)\). Then an I-rough set \(A_1, A_2\) of the rough universe \((Y, \beta)\) is I-rough closed in \((Y, \beta_2, \tau_2)\) iff \(f^{-1}(A_1, A_2)\) is I-rough closed in \((X, \beta_1, \tau_1)\).

**Proof:** First suppose that the I-rough set \((A_1, A_2)\) of the rough universe \((Y, \beta)\) is I-rough closed in \((Y, \beta_2, \tau_2)\). Since \(f : (X, \beta_1) \to (Y, \beta_2)\) be an I-rough continuous function, clearly \(f^{-1}(A_1, A_2)\) is I-rough closed in \((X, \beta_1, \tau_1)\). Conversely suppose \(f^{-1}(A_1, A_2)\) is I-rough closed in the I-rough compact topological space \((X, \beta_1, \tau_1)\). Then by theorem 3.5, \(f^{-1}(A_1, A_2)\) is I-rough compact. Then by theorem 3.2, \(f(f^{-1}(A_1, A_2)) = (A_1, A_2)\) is I-rough compact subset of \((Y, \beta_2, \tau_2)\). Then being an I-rough compact I-rough set of an I-rough Hausdorff topological space \((Y, \beta_2, \tau_2)\), \((A_1, A_2)\) is I-rough closed in \((Y, \beta_2, \tau_2)\) by theorem 4.3.
Some Special Properties of I-rough Topological Spaces

**Theorem 4.5.** Let \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) be an I-rough continuous function from an I-rough compact topological space \((X, \beta, \tau_1)\) on to an I-rough Hausdorff topological space \((Y, \beta_2, \tau_2)\). Then \( f : (X, \beta) \rightarrow (Y, \beta_2) \) is I-rough closed mapping.

**Proof:** Let \((A_1, A_2)\) be any I-rough closed subset of \((X, \beta, \tau_1)\). Then by theorem 3.5, \((A_1, A_2)\) is I-rough compact subset of \((X, \beta, \tau_1)\). Then by theorem 3.2, \(f(A_1, A_2)\) is an I-rough compact subset of \((Y, \beta_2, \tau_2)\). Now since \((Y, \beta_2, \tau_2)\) is I-rough Hausdorff, \(f(A_1, A_2)\) is I-rough closed by theorem 4.3. Hence \( f : (X, \beta) \rightarrow (Y, \beta_2) \) maps I-rough closed sets in to I-rough closed sets and hence it is an I-rough closed mapping.

**Theorem 4.6.** Let \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) be an I-rough continuous bijective function from an I-rough compact topological space \((X, \beta, \tau_1)\) on to an I-rough Hausdorff topological space \((Y, \beta_2, \tau_2)\). Then \( f : (X, \beta) \rightarrow (Y, \beta_2) \) is an I-rough homeomorphism.

**Proof:** Let \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) be an I-rough continuous bijective function where \((X, \beta_1, \tau_1)\) is an I-rough compact topological space and \((Y, \beta_2, \tau_2)\) is an I-rough Hausdorff topological space. Let \((G_1, G_2)\) be an I-rough open subset of \((X, \beta_1, \tau_1)\). Then \((G_1, G_2)^C = (X, X) \setminus (G_1, G_2)\) is I-rough closed in \((X, \beta_1, \tau_1)\). Then since \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) is a bijective I-rough function, \( f((G_1, G_2)^C) = f((X, X) \setminus (G_1, G_2)) = (Y, Y) \setminus f(G_1, G_2) = (f(G_1, G_2))^C\) is I-rough closed in \((Y, \beta_2, \tau_2)\) by theorem 4.5. So \( f(G_1, G_2)\) is I-rough open in \((Y, \beta_2, \tau_2)\). Hence \( f : (X, \beta_1) \rightarrow (Y, \beta_2) \) is an I-rough open function. Thus \( f : (X, \beta) \rightarrow (Y, \beta_2) \) is I-rough continuous, I-rough open bijective function and hence an I-rough homeomorphism.

**Corollary 4.1.** Every I-rough continuous, one-one I-rough function from an I-rough compact topological space in to an I-rough Hausdorff topological space is an I-rough embedding.

**Proof:** Proof follows from theorem 4.6 and definition of I-rough embedding.

**Theorem 4.7.** Let \((U, \beta, \tau)\) be an I-rough topological space and let \((X_1, X_2)\) and \((Y_1, Y_2)\) are any two I-rough compact subsets of \((U, \beta, \tau)\). Then \((X_1, X_2) \cup (Y_1, Y_2)\) is again an I-rough compact subset of \((U, \beta, \tau)\).

**Proof:** Let \( \{A_i, A_j\} : i \in I \) be an arbitrary I-rough open covering of \((X_1, X_2) \cup (Y_1, Y_2)\) using I-rough open sets of \((U, \beta, \tau)\). Then clearly \( \{A_i, A_j\} : i \in I \) is an I-rough open covering of \((X_1, X_2)\) and \((Y_1, Y_2)\) using I-rough open sets of \((U, \beta, \tau)\). Since \((X_1, X_2)\) and \((Y_1, Y_2)\) are I-rough compact subspaces of \((U, \beta, \tau)\), there exist finite I-rough open sub covers \( \{A_{k_i}, A_{k_j}\} : k = 1, 2, 3, \ldots, n\) covers \((X_1, X_2)\) and
Boby P. Mathew and Sunil Jacob John

\[
\{A_{i_j}, A_{j_1}: j = 1, 2, 3, \ldots, m\}
\]

covers \((Y_1, Y_2)\), by theorem 3.1. Then clearly the union of these two sub collections

covers \((X_1, X_2) \cup (Y_1, Y_2)\). Hence the union of these two sub collections is a finite I-

rough open sub covering of \(\{A_{i_j}, A_{j_1}: i \in I\}\). Since \(\{A_{i_j}, A_{j_1}: i \in I\}\) is arbitrary I-

rough open covering, \((X_1, X_2) \cup (Y_1, Y_2)\) is I-rough compact.

**Theorem 4.8.** Finite I-rough union of I-rough compact subspaces of an I-rough
topological space is again I-rough compact.

**Proof:** The argument in theorem 4.7 is valid for a finite number of I-rough compact
subspaces of an I-rough topological space.

**Theorem 4.9.** Arbitrary I-rough intersection of I-rough compact subsets of an I-rough
Hausdorff topological space is again I-rough compact.

**Proof:** Let \((U, \beta, \tau)\) be an I-rough topological spaces and let \(Y_i = (Y_{i_1}, Y_{i_2})\) where \(i \in I\)
be any I-rough compact subsets of \((U, \beta, \tau)\). Then by theorem 4.3, each \(Y_i = (Y_{i_1}, Y_{i_2})\),
where \(i \in I\) is I-rough closed in \((U, \beta, \tau)\). Let \((Y_1, Y_2) = \bigcap_{i \in I} (Y_{i_1}, Y_{i_2})\). Being arbitrary I-

rough intersection of I-rough closed subsets of an I-rough topological spaces, \((Y_1, Y_2)\) is
again I-rough closed. That is \((Y_1, Y_2)\) is an I-rough closed subset of an I-rough Hausdorff
topological space \((U, \beta, \tau)\). Then by theorem 4.3, \((Y_1, Y_2)\) is I-rough compact.

**Remark 4.1.** The I-rough Hausdorff property in theorem 4.9 is very important. Since I-

rough intersection of I-rough compact subspaces of an I-rough topological space need not
be I-rough compact in general. Consider the following example.

**Example 4.1.** Let \(N\) be the set of natural numbers and let \(x, y \notin N\) be any two real
numbers. Let \(U = N \cup \{x, y\}\). Let \(\beta = P(U)\). Also let \(\tau\) be the collection of I-rough
sets of the rough universe \((U, \beta)\) obtained by adding the following four I-rough sets to
the discrete I-rough topology on \(N\) such as \((N, N \cup \{x\}), (N, N \cup \{y\}),
(N, N \cup \{x, y\}), (N \cup \{x, y\}, N \cup \{x, y\})\). Then clearly \(\tau\) is an I-rough topology on
the rough universe \((U, \beta)\). Let \(A = (N, N \cup \{x\})\). Note that the only I-rough open sets containing \(A = (N, N \cup \{x\})\) are \((N, N \cup \{x\}), (N, N \cup \{x, y\})\).

\((N \cup \{x, y\}, N \cup \{x, y\})\). Hence any I-rough open covering of \(A\) has a finite I-rough
open sub covering. Hence \(A = (N, N \cup \{x\})\) is I-rough compact. Similarly
\(B = (N, N \cup \{y\})\) is also I-rough compact. But \(A \cap B = (N, N)\) is not I-rough compact, since
\(\{(n, N), n \in N\}\) is an I-rough open covering of \(A \cap B = (N, N)\) which
does not have any finite I-rough open sub covering. Also note that \(A^c = (N, N \cup \{x\})^c
= ((N \cup \{x\})^c, N^c) = (\phi, \{x\})\) and \(B^c = (N, N \cup \{y\})^c = ((N \cup \{y\})^c, N^c) = (\phi, \{y\})\)
Some Special Properties of I-rough Topological Spaces

are not I-rough open in \((U, \beta, \tau)\). Hence \(A\) and \(B\) are not I-rough closed in \((U, \beta, \tau)\). Hence by theorem 4.3, the I-rough topological space \((U, \beta, \tau)\) is not I-rough Hausdorff.

5. I-rough connectedness

This section generalizes the connectedness property in general topological spaces in to the I-rough connectedness in an I-rough topological spaces.

Definition 5.1. An I-rough topological spaces \((\tau_\beta, \beta, X)\) is I-rough connected if it is impossible to find two non-empty exact I-rough open sets \((A, A)\) and \((B, B)\) such that \((X, X) = (A, A) \cup (B, B)\) and \((A, A) \cap (B, B) = (\phi, \phi)\). If an I-rough topological space is not I-rough connected it is called I-rough disconnected.

Remark 5.1. It is clear that the I-rough open sets in the definition of I-rough connected spaces can be replaced by I-rough closed sets. Hence an I-rough topological spaces \((\tau_\beta, \beta, X)\) is I-rough connected if it is impossible to find two non-empty exact I-rough closed sets \((A, A)\) and \((B, B)\) such that \((X, X) = (A, A) \cup (B, B)\) and \((A, A) \cap (B, B) = (\phi, \phi)\).

Theorem 5.1. An I-rough topological spaces \((X, \beta, \tau)\) is I-rough connected space iff the only exact I-rough clopen sets of \((\tau_\beta, \beta, X)\) are \((\phi, \phi)\) and \((X, X)\).

Proof: First suppose that \((X, \beta, \tau)\) is an I-rough connected space. If possible there exist an I-rough clopen subset \((A, A)\) other than \((\phi, \phi)\) and \((X, X)\). Now \((A, A)\) is I-rough open implies \((A, A)^c = (X - A, X - A)\) is I-rough closed. Again \((A, A)\) is I-rough closed implies \((A, A)^c = (X - A, X - A)\) is I-rough open. That is \((A, A)\) and \((X - A, X - A)\) are two exact I-rough open sets of \((X, \beta, \tau)\) such that \((X, X) = (A, A) \cup (X - A, X - A)\) and \((A, A) \cap (X - A, X - A) = (\phi, \phi)\). Which is a contradiction since \((X, \beta, \tau)\) is an I-rough connected space.

Conversely suppose that the only exact I-rough clopen sets of \((X, \beta, \tau)\) are \((\phi, \phi)\) and \((X, X)\). If \((X, \beta, \tau)\) is I-rough connected then there are two non-empty exact I-rough open sets \((A, A)\) and \((B, B)\) such that \((X, X) = (A, A) \cup (B, B)\) and \((A, A) \cap (B, B) = (\phi, \phi)\). Then clearly \((A, A)\) and \((B, B) = (X - A, X - A)\) are exact I-rough clopen sets of \((X, \beta, \tau)\) other than \((\phi, \phi)\) and \((X, X)\). Which is a contradiction to our assumption.

Example 5.1. Let \((X, \beta)\) be any I-rough universe. Then \((X, \beta)\) disconnected in the discrete I-rough topology and \((X, \beta)\) is connected in the indiscrete I-rough topology.
Theorem 5.2. The image of an I-rough connected space under an I-rough continuous function is I-rough connected.

Proof: Let \((X, \beta_1, \tau_1)\) and \((Y, \beta_2, \tau_2)\) are two I-rough topological spaces, where \((X, \beta_1, \tau_1)\) is I-rough connected and let \(f: (X, \beta_1) \rightarrow (Y, \beta_2)\) be an I-rough continuous bijective function. If \((Y, \beta_2, \tau_2)\) is not I-rough connected then there are two non-empty exact I-rough open sets \((A, A)\) and \((B, B)\) of \((Y, \beta_2, \tau_2)\) such that \((Y, Y) = (A, A) \cup (B, B)\) and \((A, A) \cap (B, B) = (\phi, \phi)\). Since \(f: (X, \beta_1) \rightarrow (Y, \beta_2)\) is an I-rough continuous bijective function \(f^{-1}(A, A)\) and \(f^{-1}(B, B)\) are exact I-rough open subsets of \((X, \beta_1, \tau_1)\) and \(f^{-1}(A, A) \cup f^{-1}(B, B) = (f^{-1}(A \cup B), f^{-1}(A \cup B))\) \(= (f^{-1}(Y), f^{-1}(Y)) = (X, X)\). Also \(f^{-1}(A, A) \cap f^{-1}(B, B) = (f^{-1}(A \cap B), f^{-1}(A \cap B)) = (f^{-1}(\phi), f^{-1}(\phi)) = (\phi, \phi)\). Which is a contradiction to the fact that \((X, \beta_1, \tau_1)\) is I-rough connected. Hence our assumption is wrong and \((Y, \beta_2, \tau_2)\) is also I-rough connected.

REFERENCES