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On the infinitude of solutions to the Diophantine Equation $p^x + q^y = z^2$ when $p = 2$ and $p = 3$

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Abstract. When $p = 2$ and also when $p = 3$, it is established and demonstrated that the title equation has a solution for each and every integer $x \geq 1$ with $y = 1$ or $y = 2$. The equation has infinitely many solutions.

Keywords: Diophantine equations

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1. Introduction

The history of Diophantine Equations dates back to antiquity. There are endless varieties of Diophantine Equations, and there is no general method of solution.

In this article, we consider the equation

$$p^x + q^y = z^2, \tag{1}$$

when $p = 2$, $p = 3$, $1 < q$ an odd integer, and x, y, z are positive integers. The literature contains a very large number of articles with various equations involving primes and powers of all kinds. Among them are for example [1, 2, 3, 4, 5, 6] which relate to (1).

In Section 2 for $p = 2$ with $y = 1$, and in Section 3 for $p = 3$ with $y = 1, 2$, it is established that equation (1) has a solution for each integer $x \geq 1$.

Thus, equation (1) has infinitely many solutions when $p = 2$ and also when $p = 3$.

2. The case $p = 2$

In the following Theorem 2.1, we prove that the equation $p^x + q^y = z^2$ has a solution for each integer $x \geq 1$ when $p = 2$ and $y = 1$ are fixed values.

Theorem 2.1. If in (1) $p = 2$, $1 < q$ is an odd integer and $y = 1$, then the equation

$$2^x + q^1 = z^2 \tag{2}$$

has a solution for each and every integer $x \geq 1$, i.e., the equation has infinitely many solutions.

Proof: In (2), we shall distinguish two cases, namely: $x = 2n$ and $x = 2n + 1$ for each integer $n \geq 1$. Afterwards, a solution for $x = 1$ will be demonstrated.

Suppose that $x = 2n$. From (2) we have $2^{2n} + q^1 = z^2$ or

$$(2^n)^2 + q = z^2. \tag{3}$$

The right-hand side of (3) is a square. Set the odd value q as $q = 2 \cdot 2^n + 1$ which yields $z^2 = (2^n + 1)^2$. The values $q = 2 \cdot 2^n + 1$ and $z^2 = (2^n + 1)^2$ substituted in (3) imply that

(3) is an identity valid for every integer $n \ge 1$. Thus, for all even values $x \ge 2$, the solution of equation (1) is

$$(p,\ q,\ x,\ y,\ z) = (2,\ 2\cdot 2^n + 1,\ 2n,\ 1,\ 2^n + 1).$$

Suppose that $x = 2n + 1$. Then from (2) we obtain

$$2^{2n+1} + q^1 = z^2. \tag{4}$$

Since q is odd, therefore z is also odd. It then follows from (4) that
$2^{2n+1} + (q - 1) = z^2 - 1$
$= (z - 1)(z + 1)$. The values $z - 1$ and $z + 1$ are both even. Denote $z - 1 = 2U$, hence $z = 2U + 1$ and $z + 1 = 2(U + 1)$. We then have

$$2^{2n+1} + (q - 1) = 2U \cdot 2(U + 1) = 4U(U + 1) \tag{5}$$

implying that $4|(q - 1)$ or $4M = q - 1$ and $q = 4M + 1$. Substituting $4M = q - 1$ into (5) yields

$$2^{2n-1} + M = U(U + 1).$$

The value 2^{2n-1} is even, as is $U(U + 1)$ the product of two consecutive integers. Therefore, M is even, and denote $M = 2T$. This in-turn implies that $q = 8T + 1$, and hence

$$2^{2n-1} + 2T = U(U + 1). \tag{6}$$

Any fixed value n in (6) yields various values T and U which satisfy (6). Evidently, to prove our case, it suffices to choose the smallest possible value U, such that $U(U + 1)$ exceeds 2^{2n-1} for the first time. The difference of the two even integers $U(U + 1)$ and 2^{2n-1} then equals the smallest possible value $2T$ in (6). Therefore, for any given value $n \ge 1$, the values U and T are determined, and thus the respective values q and z are established. Equation (4) is now satisfied for each value $n \ge 1$, and equation (1) has a solution for all odd values $x \ge 3$.

Finally, for $x = 1$, we have the following solution of (1)

$$(p,\ q,\ x,\ y,\ z) = (2,\ 7,\ 1,\ 1,\ 3).$$

This concludes the proof of the Theorem 2.1. □

The following Table 1 is in accordance with the above argument on equality (6). It demonstrates (6) when $n = 1, 2, 3, 4$, and the respective minimal values U, T, as well as the values q and z.

Table 1.

n	$2^{2n-1} + 2T = U(U+1)$	U	$U+1$	T	$q = 8T + 1$	$z = 2U + 1$
1	$2^1 + 2T = U(U + 1)$	2	3	2	17	5
2	$2^3 + 2T = U(U + 1)$	3	4	2	17	7
3	$2^5 + 2T = U(U + 1)$	6	7	5	41	13
4	$2^7 + 2T = U(U + 1)$	11	12	2	17	23

3. The case $p = 3$

In the following Theorem 3.1 we prove that the equation $p^x + q^y = z^2$ has a solution for each integer $x \ge 2$ when $p = 3$, and $y = 1$ or $y = 2$ are fixed values. For $x = 1$, a solution of equation (1) is exhibited separately.

Theorem 3.1. Suppose that in (1) $p = 3$ and $1 < q$ is an odd integer. Let $n \geq 1$ be an integer. If $x = 2n$ and $y = 1$, or if $x = 2n + 1$ and $y = 2$, then each of the equations

$$3^{2n} + q^1 = z^2 \tag{7}$$

and

$$3^{2n+1} + q^2 = z^2 \tag{8}$$

has a solution for every integer $n \geq 1$, i.e., each equation has infinitely many solutions.

Proof: The two cases will be considered separately.

Suppose (7), i.e., $3^{2n} + q^1 = z^2$. Then $(3^n)^2 + q = z^2$. Set the odd value q as $q = 2\cdot3^n + 1$.

Hence $z^2 = (3^n + 1)^2$. The form of (7) is then

$$(3^n)^2 + (2\cdot3^n + 1)^1 = (3^n + 1)^2 ,$$

an identity valid for every integer $n \geq 1$. The solution of equation (1) for all even values $x \geq 2$ is

$$(p, q, x, y, z) = (3, 2\cdot3^n + 1, 2n, 1, 3^n + 1).$$

Suppose (8), i.e., $3^{2n+1} + q^2 = z^2$. Since $3^{2n+1} + q^2$ is equal to a square, the value 3^{2n+1} satisfies $3^{2n+1} = 2q + 1$ or

$$q = \frac{3^{2n+1}-1}{2}, \text{ and } z = q + 1 = \frac{3^{2n+1}-1}{2} + 1 = \frac{3^{2n+1}+1}{2}.$$

It is easily verified that q is always odd, and therefore z is even. Thus, equation (8) is the identity of the form

$$3^{2n+1} + \left(\frac{3^{2n+1}-1}{2}\right)^2 = \left(\frac{3^{2n+1}+1}{2}\right)^2$$

valid for each integer $n \geq 1$. For all odd values $x \geq 3$, equation (1) has the solution

$$(p, q, x, y, z) = \left(3, \frac{3^{2n+1}-1}{2}, 2n + 1, 2, \frac{3^{2n+1}+1}{2}\right).$$

The proof of Theorem 3.1 is complete. □

For $x = 1$ when $p = 3$, q prime and $y = 1$, the smallest possible solution of (1) is

$$(p, q, x, y, z) = (3, 13, 1, 1, 4).$$

Other solutions for $x = 1$, $p = 3$, q prime and $y = 1$ also exist.

It is easily seen from equation (1) that the values $x = 1$, $p = 3$ and $y = 2$ yield $q = 1$ contrary to $q > 1$.

Final remark. In [2], the author has exhibited five solutions of (1), in which $p = 3$, $x = 1, 2, 3, 4, 5$, q prime and $y = 1$. He raised the question, whether for $p = 3$ equation (1) has a solution for each integer $x > 5$, and also conjectured that the answer is affirmative. In this paper, it has been shown that the conjecture is indeed true.

REFERENCES

1. D. Acu, On a diophantine equation $2^x + 5^y = z^2$, *Gen. Math.*, 15 (2007) 145 – 148.
2. N. Burshtein, On solutions of the Diophantine equation $p^x + q^y = z^2$, *Annals of Pure and Applied Mathematics*, 13 (1) (2017) 143 – 149.

3. Md. A.- A. Khan, A. Rashid and Md. S. Uddin, Non-negative integer solutions of two Diophantine equations $2^x + 9^y = z^2$ and $5^x + 9^y = z^2$, *Journal of Applied Mathematics and Physics*, 4 (2016) 762 – 765.
4. J.F.T. Rabago, A note on two Diophantine equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$, *Math. J. Interdisciplinary Sci.*, 2 (2013) 19 – 24.
5. B. Sroysang, On the Diophantine equation $3^x + 17^y = z^2$, *Int. J. Pure Appl. Math.*, 89 (2013) 111 – 114.
6. A. Suvarnamani, Solution of the Diophantine equation $p^x + q^y = z^2$, *Int. J. Pure Appl. Math.*, 94 (2014) 457 – 460.