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On the Diophantine Equation $p^x + q^y = z^2$

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Dedicated to the memory of my late parents

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Abstract. It has been shown in [2] that the title equation has infinitely many solutions when $p = 2$ and also when $p = 3$. In this article, it is established and demonstrated for each prime $p > 3$, that the equation has a solution for each and every integer $x \geq 1$. We also discuss separately two distinct particular cases of the equation. One is related to the Sophie Germain conjecture, and the other to the Goldbach conjecture.

Keywords: Diophantine equations, Sophie Germain conjecture, Goldbach conjecture

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction

The history of Diophantine equations dates back to antiquity. There are endless varieties of Diophantine Equations, and there is no general method of solution.

Consider the equation

$$p^x + q^y = z^2 \tag{1}$$

where p is prime, and $1 < q$ is an odd integer. The values x, y, z, and all other values occurring in our discussion represent positive integers.

We now introduce the relation between equation (1) and Sophie Germain primes.

A Sophie Germain (1776 – 1831) prime is a prime p such that $2p + 1$ is also prime. From [5] we also cite: As of 29.2.2016, the largest known proven Sophie Germain prime p is

$$p = 2618163402417 \cdot 2^{1290000} - 1$$

having 388342 decimal digits.

The well-known Sophie Germain conjecture, i.e., there exist infinitely many Sophie Germain primes is an extremely difficult problem which is still unsolved. Under the assumption that the Sophie Germain conjecture is indeed true, i.e., there exist infinitely many Sophie Germain primes, the author [1] established that each Sophie Germain prime with $x = 2$ and $y = 1$ determines a solution of equation (1). This is discussed in Section 2.

The author [2] has proved for $p = 2$ and also for $p = 3$ that equation (1) has infinitely many solutions for each integer $x \geq 1$ when $y = 1$ or $y = 2$. Therefore, the main objective of this article is to show in particular: First (Section 2) that equation (1) has infinitely many solutions for each prime $p > 3$, and secondly (Section 3) that a relation exists between a certain case of equation (1) and the Goldbach conjecture.

2. The main result

In this section, in Theorem 2.1 it is established for every prime $p > 3$ that equation (1) has a solution for each integer $x \geq 1$.

Theorem 2.1. For each and every prime $p > 3$, the equation

$$p^x + q^y = z^2 \qquad q \text{ odd}, \qquad y = 1 \tag{2}$$

has a solution for every integer $x \geq 1$, i.e., the equation has infinitely many solutions.

Proof: We shall distinguish two cases in (2), namely: $x = 2n$ and $x = 2n+1$ for every integer $n \geq 1$. The case $x = 1$ will then be demonstrated.

Suppose that $x = 2n$. From (2) we have $p^{2n} + q^1 = z^2$ or

$$(p^n)^2 + q = z^2. \tag{3}$$

Set the odd value q as $q = 2{\cdot}p^n + 1$. Then $z^2 = (p^n + 1)^2$. Thus, equation (3) has the form

$$(p^n)^2 + 2{\cdot}p^n + 1 = (p^n + 1)^2$$

which is an identity valid for each prime p, and every integer $n \geq 1$. Hence, the solution of equation (1) for each prime $p > 3$ and all even values $x \geq 2$ is given by

$$(p,\ q,\ x,\ y,\ z) = (p,\ 2{\cdot}p^n + 1,\ 2n,\ 1,\ p^n + 1).$$

The above solution and the Sophie Germain primes are connected as follows. When $x = 2$ ($n = 1$), the solution yields

$$(p,\ q,\ x,\ y,\ z) = (p,\ 2p+1,\ 2,\ 1,\ p + 1).$$

If p is a Sophie Germain prime, then by definition $q = 2p+1$ is also prime. Moreover, the primes $p = 2$ and $p = 3$ are also Sophie Germain primes. Evidently, each Sophie Germain prime $p \geq 2$ satisfies the above solution of equation (1). Under the assumption that there exist infinitely many Sophie Germain primes, it then follows that the above solution is valid for each and every such prime. Hence, when $x = 2$ equation (1) has infinitely many solutions.

Suppose that $x = 2n + 1$. From (2) we obtain

$$p^{2n+1} + q^1 = z^2. \tag{4}$$

Each prime $p > 3$ is either of the form $4N + 1$ or of the form $4N + 3$, where $N \geq 1$ is an integer. We shall consider two cases as follows:

(a) $p = 4N + 1$,
(b) $p = 4N + 3$.

(a) Suppose that $p = 4N + 1$. Let p, hence N be fixed. For every fixed value n in (4), there exists a respective fixed integer V satisfying $p^{2n+1} = (4N + 1)^{2n+1} = 4V + 1$, where V is odd or even. Let T be an integer. If $q = 4T + 1$, the left-hand side of (4) is clearly not a square. Therefore $q = 4T + 3$. Then (4) yields

$$(4V + 1) + (4T + 3) = 4(V + T + 1) = z^2. \tag{5}$$

Hence z is even. Denote $z = 2R$ where R is an integer. Then (5) implies

$$V + T + 1 = R^2 \tag{6}$$

For any fixed value V, evidently there exists a value T odd or even, which satisfies (6), and therefore the values q, R, z are determined. Moreover, since q is odd, q may also be prime. The conditions for q being prime are not pursued here. The above argument is illustrated in Table 1 at the end of the theorem.

This completes the proof of case **(a)**.

The proof of case **(b)** is in its entirety the same proof as that of case **(a)**. Nevertheless, in order to make each case self-contained, we shall present the complete proof of case **(b)**.

(b) Suppose that $p = 4N + 3$. Let p, hence N be fixed. For every fixed value n in (4), there exists therefore a respective fixed integer U satisfying $p^{2n+1} = (4N + 3)^{2n+1} = 4U + 3$, where U is odd or even. Let S be an integer. If $q = 4S + 3$, the left-hand side of (4) is never a square. Hence $q = 4S + 1$. Then (4) implies

$$(4U + 3) + (4S + 1) = 4(U + S + 1) = z^2. \qquad (7)$$

Thus, z is even. Denote $z = 2W$ where W is an integer. Then (7) yields

$$U + S + 1 = W^2. \qquad (8)$$

For any fixed value U, evidently there exists a value S odd or even so that (8) is satisfied, and thus the values q, W, z are determined. Furthermore, since q is odd, q may also be prime. This argument is presented in Table 2 at the end of the theorem.

This completes the proof of case **(b)**.

In (3) and (4), all values $x > 1$ were considered. We conclude our proof by showing that the assertion is also true when $x = 1$.

From (2) when $x = 1$, we have

$$p^1 + q^1 = z^2. \qquad (9)$$

For each prime $p > 3$, one can certainly obtain a value q, such that (9) is satisfied. If p, q are both of the form $4N + 1$ or both of the form $4M + 3$, then (9) is never equal to a square. Therefore, p and q must be of two different forms. Two examples, one for each form of p with q prime are given by

$$p = 5, \qquad q = 11, \qquad p + q = 16 = z^2,$$

and

$$p = 7, \qquad q = 29, \qquad p + q = 36 = z^2.$$

This completes the proof of Theorem 2.1. □

As mentioned earlier, we now present the two tables for cases **(a)** and **(b)**.

In Table 1 the first three values of $p = 4N + 1$ are provided. When $x = 3$ ($n = 1$), the respective values of V odd/even, T odd/even, z, and q prime are demonstrated.

Table 1.

$p = 4N+1$	n	V	T	$V+T+1=R^2$	$z^2 = 4R^2$	z	$q = 4T+3$
5	1	31	4	36	144	12	19 prime
5	1	31	17	49	196	14	71 prime
13	1	549	26	576	2304	48	107 prime
13	1	549	179	729	2916	54	719 prime
17	1	1228	67	1296	5184	72	271 prime
17	1	1228	140	1369	5476	74	563 prime

In Table 2, the first three values of $p = 4N + 3$ are given. When $x = 3$ ($n = 1$), the respective values of U odd/even, S odd/even, z, and q prime are presented.

Table 2.

$p = 4N+3$	n	U	S	$U+S+1=W^2$	$z^2 = 4W^2$	z	$q = 4S+1$
7	1	85	58	144	576	24	233 prime
11	1	332	28	361	1444	38	113 prime
19	1	1714	49	1764	7056	84	197 prime

3. On $p^x + q^y = z^2$ and the Goldbach conjecture

The Goldbach (1690 – 1764) Conjecture is one of the oldest, most famous and very difficult unsolved problem in Number Theory today. It states the every even integer greater than 2 can be expressed as the sum of two primes.

The relation between $p^x + q^y = z^2$ and the Goldbach Conjecture will now be shown for a particular case of the equation.

Suppose that $A < B$ are positive integers of the same parity. Then, for each and every even value $z \geq 4$ the equality

$$A + B = z^2$$

holds. If indeed the Goldbach Conjecture is true, i.e., every even number greater than 2 is a sum of two primes, then under this assumption a particular case of equation (1) is now derived from the above equality, namely

$$p^1 + q^1 = z^2.$$

The relation mentioned earlier has been shown. For each and every even value $z \geq 4$, this equation is satisfied with distinct primes p and q. The equation has infinitely many solutions. This is an immediate consequence, since the infinite set of all even squares $z^2 \geq 16$ is a subset of the infinite set of all even integers.

4. Conclusion

It is also observed, that for a given even value z^2, more than one solution of equation (1) exists when $p < q$ are odd primes and $x = y = 1$. For each of the values $z^2 = 4^2$, 6^2, 8^2, 10^2, we demonstrate all the possible pairs (p, q) of equation (1) as follows:

$3 + 13 = 5 + 11 = 4^2$ two pairs
$5 + 31 = 7 + 29 = 13 + 23 = 17 + 19 = 6^2$ four pairs
$3 + 61 = 5 + 59 = 11 + 53 = 17 + 47 = 23 + 41 = 8^2$ five pairs
$3 + 97 = 11 + 89 = 17 + 83 = 29 + 71 = 41 + 59 = 47 + 53 = 10^2$ six pairs

In view of the above, we may ask:

Question 1. Let $p < q$ be odd primes satisfying $p + q = z^2$. For each such value z^2, what is the maximal number of pairs (p, q) ?

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