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All the Solutions of the Diophantine Equation $p^3 + q^2 = z^2$

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Abstract. In this short article, it is established for the title equation: (i) No solutions exist when $p = 2$. (ii) Exactly two solutions exist when $p = 3$. In both solutions q is prime. (iii) Exactly two solutions exist for each and every prime $p > 3$ in which q is composite. Some numerical solutions are also exhibited.

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1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solutions, or how many solutions. In most cases, we are reduced to study individual equations, rather than classes of equations.

The literature contains a very large number of articles on non-linear such individual equations involving primes and powers of all kinds. Among them are for example [2, 3, 4, 5, 7, 9]. The title equation stems from $p^x + q^y = z^2$.

Whereas in most articles, the values x, y are investigated for the solutions of the equation, in this paper these values are fixed positive integers. In the equation $p^3 + q^2 = z^2$ we consider all primes $p \geq 2$ and $q > 1$. We are interested in particular: how many solutions exist for a given prime or primes p, and also when is q prime or composite.

2. The main result

In this section, we determine all the solutions for each and every prime $p \geq 2$ with the respective value q. This is done in Theorem 2.1.

Theorem 2.1. Suppose that $p \geq 2$ is prime and $q > 1$. Then the equation

$$p^3 + q^2 = z^2 \tag{1}$$

has:

(a) No solutions when $p = 2$.

(b) Exactly two solutions when $p = 3$ with q prime.

(c) For each and every prime $p > 3$, exactly two solutions in which q is composite.

Proof: **(a)** Suppose that $p = 2$ in (1). We have

$$2^3 = z^2 - q^2 = (z - q)(z + q)$$

implying the two possibilities, namely : $z - q = 1$, $z + q = 8$, and $z - q = 2$, $z + q = 4$. If $z - q = 1$ then $z = q + 1$, and hence $z + q = 2q + 1 = 8$ is impossible. If $z - q = 2$ then $z = q + 2$, thus $z + q = 2q + 2 = 4$ yields $q = 1$. This is in contradiction that $q > 1$. Therefore, equation (1) has no solutions when $p = 2$.

Part **(a)** is complete.

(b) Suppose that $p = 3$ in (1). Then

$$3^3 = z^2 - q^2 = (z - q)(z + q).$$

The two possible cases are:

$$z - q = 1, \quad z + q = 27, \text{ and } \quad z - q = 3, \quad z + q = 9.$$

If $z - q = 1$ then $z = q + 1$, and hence $z + q = 2q + 1 = 27$ which yields $q = 13$ prime, and $z = 14$. If $z - q = 3$ then $z = q + 3$, and $z + q = 2q + 3 = 9$. Thus, $q = 3 = p$ and $z = 6$.

As asserted, when $p = 3$, equation (1) has exactly two solutions in which q is prime.

This completes part **(b)**.

(c) Suppose that $p > 3$ in (1). Then

$$p^3 = z^2 - q^2 = (z - q)(z + q).$$

Two possibilities exist, namely:

$$z - q = 1, \quad z + q = p^3, \text{ and } \quad z - q = p, \quad z + q = p^2.$$

If $z - q = 1$ then $z = q + 1$, and therefore $z + q = 2q + 1 = p^3$. Thus,

$$q = \frac{p^3 - 1}{2} = \frac{(p-1)}{2} \cdot (p^2 + p + 1), \qquad z = \frac{p^3 + 1}{2} = \frac{(p+1)}{2} \cdot (p^2 - p + 1). \qquad (2)$$

For every prime $p > 3$ in (2), the value q is composite. The first solution of equation (1) has been determined in the form of an identity valid for every $p > 3$.

If $z - q = p$ and $z + q = p^2$, then $(z + q) - (z - q)$ and $(z - q) + (z + q)$ yield respectively $2q = p^2 - p$ and $2z = p^2 + p$, or

$$q = \frac{p(p-1)}{2}, \qquad z = \frac{p(p+1)}{2}. \qquad (3)$$

For every prime $p > 3$ in (3), the value q is composite. The second solution of equation (1) has been established in the form of an identity valid for every $p > 3$.

This concludes the proof of part **(c)** and of Theorem 2.1. □

For the first four consecutive primes $p > 3$, the two types of solutions described in (2) and (3) are now demonstrated in the respective two tables.

p	$q = (p^3 - 1)/2$	$z = q + 1 = (p^3 + 1)/2$
5	62	63
7	171	172
11	665	666
13	1098	1099

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p	***q* = *p*(*p* – 1)/2**	***z* = *q* + *p* = *p*(*p* + 1)/2**
5	10	15
7	21	28
11	55	66
13	78	91

Final remark. In Theorem 2.1., it has been shown that $p^3 + q^2 = z^2$ has two solutions for each and every prime $p \geq 3$. Thus, the equation has infinitely many solutions.

Other interesting equations may also stem from the equation $p^x + q^y = z^2$.

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