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Multi-Fuzzy BG-ideals in BG-algebra

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Abstract. Multi-fuzzy set theory is an extension of fuzzy set theory, which deals with the multi-dimensional fuzziness. In this paper, we apply the concept of multi-fuzzy sets to ideals in BG-algebra and introduce the notion of multi-fuzzy BG-ideals, the multi-level subset of BG-ideals. And also we discuss some related properties of multi-fuzzy BG-ideals based on level subset of it. Also we define the inverse homomorphic images of multi-fuzzy BG-ideals and present some of its properties.

Keywords: BG-algebra, BG-ideal, Fuzzy BG-ideal , Multi-fuzzy BG-ideal, Multi-level subset of multi-fuzzy BG-ideal, Homomorphism.

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1. Introduction

The notion of a fuzzy subset was initially introduced by Zadeh [8] in 1965, for representing uncertainity. In 2000, Sabu and Ramakrishnan [9,10] proposed the theory of multi-fuzzy sets in terms of multi-dimensional membership functions and investigated some properties of multi-level fuzziness. Theory of multi-fuzzy set is an extension of theory of fuzzy sets. Complete characterization of many real life problems can be done by multi-fuzzy membership functions of the objects involved in the problem.

Imai and Iseki introduced two classes of abstract algebras: BCK algebras and BCI-algebras [1,2,3]. It is shown that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. Neggers and Kim [4] introduced a new notion, called a B-algebra. In 2005, Kim and Kim [6] introduced the notion of a BG-algebra which is a generalization of B-algebras. With these ideas, fuzzy subalgebras of BG-algebra were developed by Ahn and Lee [7]. Muthuraj et al. [11] presented fuzzy ideals in BG-algebra in 2010. Muthuraj and Devi [12] introduced the concept of multi-fuzzy subalgebra of BG-algebra in 2016. In this paper, we define a new algebraic structure of multi-fuzzy ideals in BG-algebra and discuss some of their related properties based on level subsets. Also, we investigate the properties of multi-fuzzy BG-ideals of BG-algebra under homomorphism.

2. Preliminaries

In this section, the basic definitions of a BG-algebra, BG-ideal, multi-fuzzy sets are recalled. We start with

Definition 2.1. A non-empty set X with a constant 0 and a binary operation " $*$ " is called a BG-algebra if it satisfies the following axioms:

1. $x * x = 0$
2. $x * 0 = x$
3. $(x * y) * (0 * y) = x \ \forall \ x, y \in X$.

Example 2.2. Let X = { 0, 1, 2 } be a set with the following table :

$*$	0	1	2
0	0	1	2
1	1	0	1
2	2	2	0

Table 1

Then $(X ; *, 0)$ is a BG-algebra.

Definition 2.3. Let S be a non-empty subset of a BG-algebra X, then S is called a subalgebra of X if $x * y \in S$ for all $x, y \in S$.

Definition 2.4. Let X be a BG-algebra and I be a subset of X . Then I is called a BG-ideal of X if it satisfies the following conditions:

(i). $0 \in I$
(ii). $x * y \in I$ and $y \in I \Rightarrow x \in I$
(iii). $x \in I$ and $y \in X \Rightarrow x * y \in I$

Definition 2.5. Let μ be a fuzzy set in a BG-algebra X. Then μ is called a fuzzy subalgebra of X if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in X$.

Definition 2.6. Let μ be a fuzzy set in a BG-algebra X. Then μ is called a fuzzy BG-ideal of X if it satisfies the following inequalities :

(i). $\mu(0) \geq \mu(x)$
(ii). $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$
(iii). $\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \ \forall \ x, y \in X$

Definition 2.7. A mapping $f : X \rightarrow Y$ of a BG-algebra is called a homomorphism if $f(x * y) = f(x) * f(y) \quad \forall \ x, y \in X$.

Remark 2.1. If $f : X \rightarrow Y$ is a homomorphism of BG-algebra then $f(0) = 0$.

Definition 2.8. Let X be a non-empty set. A multi-fuzzy set A in X is defined as a set of ordered sequences:

$A = \{(x, \mu_1(x), \mu_2(x), ..., \mu_i(x), ...) : x \in X\}$, where $\mu_i : X \rightarrow [0,1]$ for all i .

Remark 2.2.

(i). If the sequences of the membership functions have only k-terms (finite number of terms), k is called the dimension of A.

(ii). The set of all multi-fuzzy sets in X of dimension k is denoted by $M^kFS(X)$.

(iii). The multi-fuzzy membership function μ_A is a function from X to $[0,1]^k$ such that for all x in X, $\mu_A(x) = (\mu_1(x), \mu_2(x),\ldots,\mu_k(x))$

(iv). For the sake of simplicity, we denote the multi-fuzzy set

$A = \{(x, \mu_1(x), \mu_2(x),\ldots,\mu_k(x)): x \in X\}$ as $A=(\mu_1, \mu_2,\ldots,\mu_k)$.

Definition 2.9. Let k be a positive integer and let A and B in $M^kFS(X)$, where $A = (\mu_1, \mu_2,\ldots,\mu_k)$ and $B = (\nu_1, \nu_2,\ldots, \nu_k)$, then we have the following relations and operations :

i). $A \subseteq B$ if and only if $\mu_i \leq \nu_i$, for all i=1,2,………..,k;

ii). $A = B$ if and only if $\mu_i = \nu_i$, for all i=1,2,………...,k;

iii). $A \cup B = (\mu_1 \cup \nu_1,\ldots\mu_k \cup \nu_k) = \{(x , \max(\mu_1(x),\nu_1(x)), \ldots\max(\mu_k(x),\nu_k(x))) : x \in X \}$

iv). $A \cap B = (\mu_1 \cap \nu_1,\ldots\mu_k \cap \nu_k) = \{(x , \min(\mu_1(x),\nu_1(x)), \ldots\min(\mu_k(x),\nu_k(x))) : x \in X\}$

Definition 2.10. Let A be a multi-fuzzy set of a BG-algebra X. For any $t = (t_1,t_2,\ldots\ldots t_k)$ where $t_i \in [0,1]$, for all i , the set $U(A ; t) = \{ x \in X / A(x) \geq t \}$ is called the multi-level subset of A.

Definition 2.11. Let A be a multi-fuzzy set in a BG-algebra X . Then A is called a multi-fuzzy subalgebra of X if $A(x * y) \geq \min \{ A(x) , A(y) \}$ $\forall x , y \in X$.

3. Multi-fuzzy BG-ideal

In this section, the notion of multi-fuzzy BG-ideal is introduced and some of its properties are discussed.

Definition 3.1. Let A be a multi-fuzzy set in X. Then A is called a multi-fuzzy BG-ideal in X if it satisfies the following conditions:

(i). $A(0) \geq A(x)$

(ii). $A(x) \geq \min \{ A(x * y) , A(y) \}$

(iii). $A(x * y) \geq \min\{ A(x) , A(y) \}$ $\forall x , y \in X$.

Example 3.2. Consider a BG-algebra X = { 0 , 1 , 2 } with the table 1 in Example 2.2. Define a multi-fuzzy set $A : X\rightarrow[0,1]$ by $A(0) = A(1) = (r_1 , r_2)$ and $A(2) = (s_1, s_2)$ where $r_1, r_2, s_1, s_2 \in [0,1]$ with $r_1< s_1$ and $r_2 < s_2$. Then A is a multi-fuzzy BG-ideal in X.

Theorem 3.3. Let X be a BG-algebra. Then A is a multi-fuzzy BG-ideal of X if and only if A is a multi-fuzzy subalgebra of X.

Proof: Every multi-fuzzy BG-ideal of a BG-algebra X is a multi-fuzzy subalgebra of X. Conversely, let A be a multi-fuzzy subalgebra in X.

Let $x , y \in X$.

i) $A (0) = A (x * x)$

$\geq \min \{ A(x) , A(x) \} = A(x) \quad \forall x \in X$

ii) $A (x) = A((x * y) * (0 * y))$

$\geq \min \{ A (x * y) , A(0 * y) \}$

$\geq$ min { A (x ∗ y) , min { A(0) , A(y) } }

$\geq$ min { A (x ∗ y) , A(y) }

Hence A is a multi-fuzzy BG-ideal in X.

Theorem 3.4. Let A_1 and A_2 be two multi-fuzzy BG-ideals of a BG-algebra X . Then $A_1 \cap A_2$ is also a multi-fuzzy BG-ideal in X.

Proof : Let x , y $\in A_1 \cap A_2$

Then x , y $\in A_1$ and x , y $\in A_2$

i) $A_1 \cap A_2(0) = A_1 \cap A_2(x * x)$

$= \min \{ A_1(x * x) , A_2(x * x) \}$

$\geq \min \{ \min \{ A_1(x) , A_1(x) \} , \min \{ A_2(x) , A_2(x) \} \}$

$= \min \{ A_1(x) , A_2(x) \}$

$= A_1 \cap A_2(x)$

ii) $A_1 \cap A_2(x) = \min \{ A_1(x) , A_2(x) \}$

$\geq \min \{ A_1(x * y) , A_1(y) \} , \min \{ A_2(x * y) , A_2(y) \} \}$

$= \min \{ A_1(x * y) , A_2(x * y) \} , \min \{ A_1(y) , A_2(y) \} \}$

$= \min \{ A_1 \cap A_2(x * y) , A_1 \cap A_2(y) \}$

iii) $A_1 \cap A_2(x*y) = \min \{ A_1(x * y) , A_2(x * y) \}$

$\geq \min \{ \min \{ A_1(x) , A_1(y) \} , \min \{ A_2(x) , A_2(y) \} \}$

$= \min \{ \min \{ A_1(x) , A_2(x) \} , \min \{ A_1(y) , A_2(y) \} \}$

$= \min \{ A_1 \cap A_2(x) , A_1 \cap A_2(y) \}$

Hence $A_1 \cap A_2$ is a multi-fuzzy BG-ideal in X.

Theorem 3.5. Let A be a multi-fuzzy BG-ideal of a BG-algebra X. If $x \leq y$ then $A(x) \geq A(y)$ i.e., order reversing.

Proof : Let x , y $\in$ X such that $x \leq y$.

Then $x * y = 0$

Since A is a multi-fuzzy BG-ideal in X , $A(x) \geq \min \{ A(x * y) , A(y) \}$

$= \min \{ A(0) , A(y) \}$

$= A(y)$

Hence it completes the proof.

Theorem 3.6. Let A be a multi-fuzzy BG-ideal of X . If the inequality $x * y \leq z$ holds in X , then $A(x) \geq \min \{ A(y) , A(z) \}$ for all x , y , z $\in$ X .

Proof: Assume the inequality $x * y \leq z$ holds in X.

Then $(x * y) * z = 0$.

$A(x) \geq \min \{ A(x * y) , A(y) \}$

$\geq \min \{ \min \{ A((x * y) * z) , A(z) \} , A(y) \}$

$= \min \{ \min \{ A(0) , A(z) \} , A(y) \} \}$

$= \min \{ A(y) , A(z) \}$

Definition 3.7. Let A be a multi-fuzzy set in a BG-algebra X. Then A is called multi-fuzzy closed ideal in X if it satisfies the following conditions :

i) $A(x) \geq \min \{ A(x * y) , A(y) \}$

ii) $A(0 * x) \geq A(x)$

Example 3.8. Consider a BG-algebra $X = \{ 0 , 1 , 2 , 3 \}$ with the following cayley table

∗	0	1	2	3
0	0	1	2	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Table 2:

Let $A : X \rightarrow I$ be a multi-fuzzy set defined by
$A(0) = A(1) = (0.6 , 0.8)$ and $A(2) = A(3) = (0.3 , 0.4)$
Then A is multi-fuzzy closed ideal in X.

Theorem 3.9. Every multi-fuzzy closed ideal is a multi-fuzzy ideal in X.

Proof: Let A be a multi-fuzzy closed ideal of X. It is enough to prove that $A(0) \geq A(x)$

Now , $A(0) \geq \min \{ A(0 * x) , A(x) \}$
$\geq \min \{ A(x) , A(x) \}$
$= A(x)$

Remark 3.3. The converse of the above theorem is not true in general.

Theorem 3.10. Every multi-fuzzy closed ideal of a BG-algebra is a multi-fuzzy BG-subalgebra of X.

Proof : Let A be a multi-fuzzy closed ideal of X.

Now, $A(x * y) \geq \min \{ A((x * y) * (0 * y)) , A(0 * y) \}$
$= \min \{ A(x) , A(0 * y) \}$
$\geq \min \{ A(x) , A(y) \}$

Hence the proof.

Theorem 3.11. If A is a multi-fuzzy BG-ideal in X, then the set $U(A ; t)$ is a BG-ideal in X for $t = (t_1,t_2,\ldots.t_k)$ where $t_i \in [0,1]$, for all i

Proof : Let A be a multi-fuzzy BG-ideal in X.

i) Since $A(0) \geq A(x) \geq t$, $0 \in U(A ; t)$

ii) Let $x * y \in U(A ; t)$ and $y \in U(A ; t)$
Then $A(x * y) \geq t$ and $A(y) \geq t$
Now, $A(x) \geq \min \{ A(x * y) , A(y) \}$
$\geq \min \{ t , t \}$
$= t$
This implies that $x \in U(A ; t)$.

iii). Let $x \in U(A ; t)$ and $y \in X$
Choose y in X such that $A(y) \geq t$
$A(x * y) \geq \min \{ A(x) , A(y) \}$

$\geq$ min { t , t }

= t

This implies that $x * y \in U(A ; t)$

Hence U(A ; t) is a BG-ideal in X.

Theorem 3.12. If X be a BG-algebra and U(A ; t) for $t = (t_1,t_2,\ldots.t_k)$ where $t_i \in [0,1]$, for all i is a BG-ideal in X , then A is a multi-fuzzy BG-ideal in X.

Proof: Let U(A ; t) be a BG-ideal in X.

Let x , y ∈ U(A ; t)

Then $A(x) \geq t$ and $A(y) \geq t$

i). Let A(x) = r and A(y) = s and such that $r \leq s$ where $r = (r_1 , r_2,\ldots.r_k)$ and $s = (s_1, s_2\ldots.s_k)$ for r_i and $s_i \in [0,1]$ for all i.

Since A(x) = r , x ∈ U(A ; r)

$x \in U(A ; r)$ and $y \in X$ implies $x * y \in U(A ; r)$

That is A(x) $\geq$ r

= min { r , s }

= min { A(x) , A(y) }

ii). A(0) = $A(x * x)$

$\geq$ min { A(x) , A(x) } by (i)

= A(x)

iii). A(x) = $A((x * y) * (0 * y))$

$\geq$ min { $A(x * y)$, $A(0 * y)$ } , by (i)

$\geq$ min { $A(x * y)$, min { A(0) , A(y) } }

= min { $A(x * y)$, A(y) }

Hence A is a multi-fuzzy BG-ideal in X .

4. Homomorphism of multi-fuzzy BG-ideals

In this section, the properties of multi-fuzzy BG-ideals are discussed under homomorphism.

Definition 4.1. Let $f : X \to Y$ be a mapping of BG-algebra and A be a multi-fuzzy set of Y then $f^{-1}(A)$ is the pre-image of A under f if $f^{-1}(A) = A(f(x))$ $\forall x \in X$.

Theorem 4.2. Let $f : X \to Y$ be a homomorphism of BG-algebra. If A is a multi-fuzzy BG-ideal of Y , then $f^{-1}(A)$ is a multi-fuzzy BG-ideal of X.

Proof : For any $x \in X$,

i) $f^{-1}(A)(x)$ = A(f(x))

$\leq$ A(0)

= A(f(0))

= $f^{-1}(A)(0)$

ii) $f^{-1}(A)(x)$ = A(f(x))

$\geq$ min { $A(f(x)) * A(f(y))$, A(f(y)) }

$= \min\{ A(f(x * y)), A(f(y)) \}$

$= \min\{ f^{-1}(A)(x * y), f^{-1}(A)(y) \}$

iii) $f^{-1}(A)(x * y) = A(f(x * y)) = A(f(x) * f(y))$

$\geq \min\{ A(f(x)), A(f(y)) \}$

$= \min\{ f^{-1}(A)(x), f^{-1}(A)(y) \}$

Hence the proof.

Theorem 4.3. Let $f : X \to Y$ be an epimorphism of a BG-algebra. If $f^{-1}(A)$ is a multi-fuzzy ideal in X then A is a multi-fuzzy ideal in Y.

Proof :

i) Let $y \in Y$ there exists $x \in X$ such that $f(x) = y$

$A(y) = A(f(x))$

$= f^{-1}(A)(x)$

$\leq f^{-1}(A)(0)$

$= A(f(0))$

$= A(0)$

That is $A(0) \geq A(y)$

ii) Let $x, y \in Y$ there exists $a, b \in X$ such that $f(a) = x$, $f(b) = y$

$A(x) = A(f(a))$

$= f^{-1}(A)(a)$

$\geq \min\{ f^{-1}(A)(a * b), f^{-1}(A)(b) \}$

$= \min\{ A(f(a * b)), A(f(b)) \}$

$= \min\{ A(f(a) * f(b)), A(f(b)) \}$

$= \min\{ A(x * y), A(y) \}$

iii). $A(x * y) = A(f(a) * f(b))$

$= A(f(a * b))$

$= f^{-1}(A)(a * b)$

$\geq \min\{ f^{-1}(A)(a), f^{-1}(A)(b) \}$

$= \min\{ A(f(a)), A(f(b)) \}$

$= \min\{ A(x), A(y) \}$

Hence A is a multi-fuzzy BG-ideal in Y.

5. Conclusion

In this paper, we introduced the concept of multi-fuzzy BG-ideals in BG-algebra and discussed some of its properties based on level sets and also presented some results under homomorphism.

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