All the solutions of the Diophantine Equation $p^x + (p+4)^y = z^2$ when $p$, $(p + 4)$ are Primes and $x + y = 2$, 3, 4

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Received 18 January 2018; accepted 24 January 2018

Abstract. In this paper we consider the Diophantine equation $p^x + (p+4)^y = z^2$ when $p > 2$, $(p + 4)$ are primes, and $x$, $y$, $z$ are positive integers. All the possibilities of $x + y = 2$, 3, 4 are examined, and it is established that the equation has the unique solution $(p, x, y, z) = (3, 2, 1, 4)$.

Keywords: Diophantine equations, Cousin primes

AMS Mathematics Subject Classification (2010): 11D61

1. Introduction
A prime gap is the difference between two consecutive primes. Numerous articles have been written on prime gaps, a very minute fraction of which is brought [3,4] here. In 1849, A. de Polignac conjectured that for every positive integer $k$, there are infinitely many primes $p$ such that $p + 2k$ is prime too. Many questions and conjectures on the above still remain unanswered and unsolved.

When $k = 1$, the pairs $(p, p + 2)$ are known as Twin Primes. The first four such pairs are: $(3, 5), (5, 7), (11, 13), (17, 19)$. The Twin Prime conjecture stating that there are infinitely many such pairs remains unproved. When $k = 2$, the pairs $(p, p + 4)$ are called Cousin Primes. The first six pairs are: $(3, 7), (7, 11), (13, 17), (19, 23), (37, 41), (43, 47)$.

In this paper, the known Diophantine equation $p^x + q^y = z^2$ [see 1, 5, 6, 7] is considered when $p$ and $q$ are Cousin Primes i.e.,

$$p^x + (p+4)^y = z^2,$$

(1)

and $x$, $y$, $z$ are positive integers. We examine all the possibilities of $x + y = 2, 3, 4$ for solutions of equation (1). This is done in Section 2.

2. The equation $p^x + (p+4)^y = z^2$
In this section we prove the following result.

Theorem 2.1. Suppose that $p > 2$, $(p + 4)$ are any two primes, and $x$, $y$, $z$ are positive integers. If $x + y = 2, 3, 4$, then the equation $p^x + (p+4)^y = z^2$ has the unique solution

$$3^2 + 7^1 = 4^2.$$

241
Nechemia Burshtein

**Proof:** For \( x + y = 2, 3, 4 \), we examine all possible values \( x, y \). These are:

- **Case 1.** \( x + y = 2 \), \( x = 1, \ y = 1 \).
- **Case 2.** \( x + y = 3 \), \( x = 1, \ y = 2 \).
- **Case 3.** \( x + y = 3 \), \( x = 2, \ y = 1 \).
- **Case 4.** \( x + y = 4 \), \( x = 1, \ y = 3 \).
- **Case 5.** \( x + y = 4 \), \( x = 2, \ y = 2 \).
- **Case 6.** \( x + y = 4 \), \( x = 3, \ y = 1 \).

Each of these cases is considered separately. The value \( z^2 \) is even, hence \( z \) is even. Thus \( z^2 \) is a multiple of 4.

**Case 1.** Suppose in equation (1) \( x = 1 \) and \( y = 1 \). We then obtain \( p + (p + 4) = z^2 \) or \( 2(p + 2) = z^2 \).

But, the left-hand side of (2) is a multiple of 2 only, whereas the right-hand of (2) is a multiple of 4. Since this is impossible, equation (1) has no solution in this case.

**Case 2.** Suppose in equation (1) \( x = 1 \) and \( y = 2 \). We have

\[
p^1 + (p + 4)^2 = z^2
\]

implying \( p^2 + 9p + 16 = z^2 \) or \( p(p + 9) = (z - 4)(z + 4) \). (3)

From (3) it follows that \( p \) divides at least one of the values \( z - 4, \ z + 4 \).

If \( p | (z - 4) \), denote \( pA = z - 4 \) where \( A \) is an even integer. Thus, \( p(p + 9) = pA(pA + 8) \) implying \( p + 9 = A(pA + 8) \). (4)

For any prime \( p \), (4) clearly implies that all values \( A \) are impossible. Thus, (4) does not exist, and \( p \nmid (z - 4) \).

If \( p | (z + 4) \), denote \( pB = z + 4 \) where \( B \) is an even integer. Then from (3) we have \( p(p + 9) = (pB - 8)pB \) or \( p + 9 = B(pB - 8) \) which yields

\[
p = \frac{9 + 8B}{B^2 - 1}
\]

Consequently, one can see that the right-hand side of (5) is never equal to an integer \( p \) implying that (5) is impossible, and \( p \nmid (z + 4) \).

Hence, Case 2 does not yield a solution of equation (1).

**Case 3.** Suppose in equation (1) \( x = 2 \) and \( y = 1 \). We have

\[
p^2 + (p + 4)^1 = z^2
\]

and

\[
p(p + 1) = z^2 - 4 = (z - 2)(z + 2).
\]

Thus, from (6), \( p \) divides at least one of the values \( z - 2, \ z + 2 \).

If \( p | (z - 2) \), denote \( pC = z - 2 \) and \( pC + 4 = z + 2 \) where \( C \) is an even integer. From (6) we then have

\[
p + 1 = C(pC + 4)
\]

which is clearly impossible for all values \( C \).

If \( p | (z + 2) \), denote \( pD = z + 2 \) and \( pD - 4 = z - 2 \) where \( D \) is an even integer. From (6) we have

242
All the solutions of the Diophantine Equation \( p^x + (p+4)^y = z^2 \) when \( p, (p + 4) \) are Primes and \( x + y = 2, 3, 4 \)

\[
p + 1 = (PD - 4)D
\]

which yields the two smallest possible values \( p = 3 \) and \( D = 2 \) as a solution of (7).

Hence

\[
3^2 + 7 = 4^2
\]

is a solution of equation (1).

**Case 4.** Suppose in equation (1) \( x = 1 \) and \( y = 3 \). We obtain

\[
p^1 + (p + 4)^3 = p + (p^3 + 12p^2 + 48p + 64) = z^2.
\]

Thus

\[
p + p^3 + 4(3p^2 + 12p + 16) = z^2.
\]

Since \( z^2 \) is a multiple of 4, it follows from (8) that \( 1 + p^2 \) must be a multiple of 4.

Every prime \( p \geq 3 \) is of the form \( 4N + 3 \) (\( N \geq 0 \)) or \( 4N + 1 \) (\( N \geq 1 \)). It is then easily seen in either case, that the value \( 1 + p^2 \) is never a multiple of 4.

Hence, Case 4 is impossible, and does not contribute a solution to equation (1).

**Case 5.** Suppose in equation (1) \( x = 2 \) and \( y = 2 \). We have

\[
p^2 + (p + 4)^2 = p^2 + (p^2 + 8p + 16) = z^2,
\]

hence

\[
2p^2 + (8p + 16) = z^2.
\]

But, \( z^2 \) and \( (8p + 16) \) are multiples of 4, whereas \( 2p^2 \) is not. Thus, equality (9) is impossible, implying that equation (1) has no solutions in this case.

**Case 6.** Suppose in equation (1) \( x = 3 \) and \( y = 1 \). We have

\[
p^3 + (p + 4)^1 = z^2.
\]

As in Case 4, the value \( p^3 + p \) must be a multiple of 4 since \( 4 \) and \( z^2 \) are multiples of 4. Using the argument in Case 4, it follows that \( 4 \nmid (p^3 + p) \).

Equality (10) is therefore impossible, and no solution of equation (1) exists in this case.

Thus, \( 3^2 + 7^1 = 4^2 \) is the unique solution as asserted.

The proof of Theorem 2.1 is complete. \[ \square \]

### 3. Conclusion

In this paper, we have established for any two primes \( p > 2, (p + 4) \), and positive integers \( x, y, z \) where \( x + y = 2, 3, 4 \), that the equation \( p^x + (p+4)^y = z^2 \) has a unique solution \( (p, x, y, z) = (3, 2, 1, 4) \). The following question may now be raised.

**Question 1.** Let \( p \geq 3, (p + 4) \) be any two primes, and \( x, y, z \) are positive integers. If \( x, y \) satisfy \( x + y > 4 \), does the equation \( p^x + (p+4)^y = z^2 \) have solutions?

### REFERENCES