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Convergence of Interpolatory Polynomial Between Lagrange and Hermite

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Abstract. The aim of this paperis to study an interpolation problem, which is an intermediate problem between Lagrange and Hermite. We consider this problem on the nodes obtained by projecting vertically the zeros of$(1-x^2)P_n(x)$ onto the unit circle, where $P_n(x)$ stands for n^{th} Legendre polynomial. We prove the regularity of the problem, give explicit forms and establish a convergence theorem for the same.

Keywords: Legendre polynomial, Explicit representation, Convergence

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1. Introduction

In 1990, Tian Liang Tu [7] obtained the divergence and mean convergence of the Hermite interpolation operator. In 1993, Prasad [6] studied the Hermite-Fejer interpolation of higher order on the n-distinct zeros of $(1-x^2)P_{n-2}(x)$, where $P_n(x)$ is the n^{th} Legendre polynomial. In 1998, Bahadur and Mathur [3] proved the convergence of Quasi-Hermite interpolation on the nodes obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ onto the unit circle, where $P_n(x)$ is the n^{th} Legendre polynomial. In 2001, Daruls and Gonzalezvera [4] gave an extension to the unit circle of the classical Hermite-Fejer Theorem. In 2011, Bahadur [1] presented a method for computing the convergence of Hermite interpolation polynomial onto the unit circle.

In 2016, Bahadur and Bano [2] considered modified Hermite interpolation onto the unit circle. Later on, several mathematician have considered Hermite and Lagrange interpolation on different set of nodes. These have motivated us to consider a problem between Lagrange and Hermite interpolation.

In this paper, we consider an interpolation problem on the nodes, which are obtained by projecting vertically the zeros of $(1-x^2)P_n(x)$ onto the unit circle. We prescribe the functional value at ±1, whereas first derivative at all other points.

In section 2, we give some preliminaries and in section 3, we describe the problem and give the existence theorem of the interpolatory polynomials, whereas in section 4, we give the explicit formulae of theinterpolatory polynomials. Lastly in section 5 and 6, we give estimates and convergence ofinterpolatory polynomials respectively.

2. Preliminaries

In this section, we shall give some well knownresults.

$\{ z_0 = 1, z_{2n+1} = -1, z_k = \cos\theta_k + i\sin\theta_k, z_{n+k} = -z_k, k = 1(1)n \}$ (2.1)

be the vertical projections on unit circle of the zeroes of $(1-x^2)P_n(x)$, where $P_n(x)$ stands for n^{th} Legendre polynomial having zeros,

$x_k = cos\theta_k$, $k = 1(1)n$ $s.t.$ $1 > x_1 > x_2 > \cdots > x_n > -1$ (2.2)

W(z)= $\prod_{k=1}^{2n}(z - z_k) = K_n P_n\left(\frac{1+z^2}{2z}\right) z_n$ (2.3)

$$K_n = \frac{(2^n n!)}{(2n-1)!!} \tag{2.3}$$

$$R(z) = (z^2-1)W(z) \tag{2.4}$$

The differential equation satisfied by $P_n(x)$ is

$$(1-x^2)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0 \tag{2.5}$$

Fundamental polynomials of Lagrange interpolation based on the zeros of $W(z)$ and $R(z)$are given by

$$L_{1k}(z) = \frac{R(z)}{(z-z_k)R'(z_k)} k = 0(1)2n+1 \tag{2.6}$$

$$L_k(z) = \frac{W(z)}{(z-z_k)W'(z_k)} k = 1(1)2n \tag{2.7}$$

For $-1 \le x \le 1$ we have ,

$$|z^2-1| = 2\sqrt{1-x^2} \tag{2.8}$$

$$(1-x^2)^{\frac{1}{4}}|P_n(x)| \le \sqrt{\frac{2}{\pi n}} \tag{2.9}$$

$$|P_n(x)| \le 1 \tag{2.10}$$

Let $x_k's$ be the zeroes of $P_n(x)$, then

$$\left(1-x_k^2\right)^{-1} \sim \left(\frac{k}{n}\right)^{-2} \tag{2.11}$$

$$|P_n{}'(x_k)| \ge ck^{-\frac{3}{2}}n^2 \tag{2.12}$$

3. The problem and the regularity

Let $Z_n = \{z_k; k = 0(1)2n+1\}$ satisfying (2.1),

Here, we are interested in determining the interpolatory polynomial $L_n(z)$ of degree ≤ 2n+1 satisfying the following conditions.

$$\begin{cases} L_n(f, z_k) = \alpha_k & , k = 0, 2n+1 \\ L_n{}'(f, z_k) = \beta_k & , k = 1(1)2n \end{cases} \tag{3.1}$$

where, α_k and β_k are arbitrary constants.

We establish convergence theorem for the same.

Theorem 3.1. $L_n(z)$ is regular on Z_n.

Proof: It is sufficient,if we show the unique solution of (3.1) is $L_n(z) \equiv 0$

In this case, let us consider

$$L_n'(z_k) = 0 \quad k = 1(1)2n$$

$$L_n'(z) = a\, W(z) \qquad using\ (2.2)$$

$$L_n(z) = a\int_{-1}^{z} W(t)dt$$

Now,satisfying conditions in (3.1) we have,

$$L_n(1) = a\int_{-1}^{1} W(t)dt = 0$$

As, $\int_{-1}^{1} W(t)dt \neq 0$, which implies $a = 0$.
So, $L_n(z) \equiv 0$. Hence the theorem follows.

4. Explicit representation of interpolatory polynomials

We shall write,

$$L_n(z) = \sum_{k=0,2n+1} \propto_k A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z) \tag{4.1}$$

where, $A_k(z)$ *and* $B_k(z)$ are unique polynomial, each of degree atmost $2n+1$ satisfying the conditions,

$$\begin{cases} A_k(z_j) = \delta_{kj} & j,k = 0,2n+1 \\ A'_k(z_j) = 0 & k = 0,2n+1 \quad j = 1(1)2n \end{cases} \tag{4.2}$$

$$\begin{cases} B_k(z_j) = 0 & j = 0,2n+1 \quad ,k = 1(1)2n \\ B'_k(z_j) = \delta_{kj} & j,k = 1(1)2n \end{cases} \tag{4.3}$$

Theorem 4.1. For $k = 1(1)2n$, we have

$$B_k(z) = (z^2-1)L_k(z) - \int_{-1}^{z}(t^2-1)L'_k(t)dt + (1-2z_k)\int_{-1}^{z} L_k(t)dt - S_k(1)\frac{\int_{-1}^{z} W(t)dt}{\int_{-1}^{1} W(t)dt} \tag{4.4}$$

Proof: Let, $B_k(z) = (z^2-1)L_k(z) + S_k(z) + a\int_{-1}^{z} W(t)dt$
where, $B_k(z)$ is atmost of degree $2n+1$ satisfying the conditions given in (4.3) provides us with

$$a = -\frac{S_k(1)}{\int_{-1}^{1} W(t)dt}$$

Applying the conditions given in (4.3) for ${B_k}'(z)$, we have

$$S_k(z) = -\int_{-1}^{z}(t^2-1)L'_k(t)dt + a_k\int_{-1}^{z} L_k(t)dt$$

where, $a_k = 1 - 2z_k$
Hence, we have theorem 4.1.

Theorem 4.2. For $k = 0,2n+1$

$$A_k(z) = (-1)^k\left(\frac{\int_{-z_k}^{z} W(t)dt}{\int_{-1}^{1} W(t)dt}\right) \tag{4.5}$$

Proof: Let $A_k(z) = (-1)^k a_k \int_{-z_k}^{z} W(t)dt$
where, $A_k(z)$ is atmost of degree $2n+1$ satisfying the conditions given in (4.2), we get

$$a_k = \frac{1}{\int_{-1}^{1} W(t)dt}$$

Hence the theorem follows.

5. Estimation of fundamental polynomials

Lemma 5.1. Let $B_k(z)$ be given by (4.4), then

$$\sum_{k=1}^{2n} |B_k(z)| \le clogn$$

where, c is a constant independent of z.

Proof: Let $\sum_{k=1}^{2n} |B_k(z)| \le I_1 + I_2 + I_3$

where, $I_1 = \sum_{k=1}^{2n} |(z^2-1)L_k(z)|$

Using (2.2), (2.7), (2.8), (2.9) and (2.11), we have

$$I_1 \le clogn \tag{5.1}$$

Now, $I_2 = \sum_{k=1}^{2n} \left| -\int_{-1}^{z} (t^2-1)L_k'(t)dt \; + (1-2z_k)\int_{-1}^{z} L_k(t)dt \right|$

$$I_2 \le \sum_{k=1}^{2n} \int_{-1}^{z} |(t^2-1)||L_k{}'(t)|dt + (1+2|z_k|)\int_{-1}^{z} |L_k(t)|dt$$

Using (2.2), (2.7), (2.8), (2.9), (2.10) and (2.11), we have

$$I_2 \le clogn \tag{5.2}$$

Now, $I_3 = \sum_{k=1}^{2n} \left| S_k(1) \frac{\int_{-1}^{z} W(t)dt}{\int_{-1}^{1} W(t)dt} \right|$

Using (2.2) and (2.10), we have

$$I_3 \le c \tag{5.3}$$

Combining (5.1), (5.2) and (5.3), our desired lemma follows.

Lemma 5.2. Let $A_k(z)$ be given in theorem (4.2), then

$$\sum_{k=0,2n+1} |A_k(z)| \le c$$

where, c is a constant independent of z and n .

Proof: $\sum_{k=0,2n+1} |A_k(z)| = |A_0(z)| + |A_{2n+1}(z)|$

$$|A_0(z)| \le \left(\frac{1}{\left| \int_{-1}^{1} W(t)dt \right|} \right) \left| \int_{-1}^{z} W(t)dt \right|$$

Using (2.2) and (2.10), we have

$$|A_0(z)| \le c$$

Similarly,

$$|A_{2n+1}(z)| \le c$$

Combining these two equations, we get lemma 5.2.

6. Convergence

Theorem 6.1. Let $f(z)$ be continuous for $|z| \le 1$ and analytic for $|z| < 1$. Let the arbitrary β_k's be such that

$$|\beta_k| = O\left(n\omega_2\left(f, \frac{1}{n}\right)\right) k = 1(1)2n \tag{6.1}$$

Then $\{L_n(z)\}$ defined by

$$L_n(z) = \sum_{k=0,2n+1} f(z_k)A_k(z) + \sum_{k=1}^{2n} \beta_k B_k(z) \tag{6.2}$$

satisfies the relation,

$$|L_n(z) - f(z)| = O(\omega_2(f, n^{-1})logn), \tag{6.3}$$

where $\omega_2(f, n^{-1})$ be the modulus of continuity of second kind of $f(z)$.

Remark 6.1. Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$ and $f' \,\mathcal{E}\, Lip\alpha$, $\alpha > 0$, then the sequence $\{L_n(z)\}$ converges uniformly to $f(z)$ in $|z| \leq 1$, which follows from (6.3) as

$$\omega_2(f, n^{-1}) \leq n^{-1}\omega_2(f', n^{-1}) = O(n^{-1-\alpha}), \tag{6.4}$$

To prove the theorem (6.1), we shall need following

Let $f(z)$ be continuous for $|z| \leq 1$ and analytic for $|z| < 1$. Then thereexists a polynomial $F_n(z)$ of degree $\leq 2n + 1$ satisfying Jackson's inequality.

$$|f(z) - F_n(z)| \leq c\omega_2(f, n^{-1}), \qquad z = e^{i\theta}(0 \leq \theta < 2\pi) \tag{6.5}$$

and also an inequality due to Kiŝ [3].

$$\left|F_n^{(m)}(z)\right| \leq c\, n^m \omega_2(f, n^{-1}), m \,\mathcal{E}\, I^+, \text{ where } c \text{ is a constant.} \tag{6.6}$$

Proof: Since, $L_n(z)$ be the uniquely determined polynomial of degree $\leq 2n + 1$ and the polynomial $F_n(z)$ satisfying (6.5) and (6.6) can be expressed as

$$F_n(z) = \sum\nolimits_{k=0,2n+1} F_n(z_k)A_k(z) + \sum_{k=1}^{2n} F_n'(z_k)B_k(z)$$

$$\begin{aligned} |L_n(z) - f(z)| &\leq |L_n(z) - F_n(z)| + |F_n(z) - f(z)| \\ &\leq \sum_{k=0,2n+1} |f(z_k) - F_n(z_k)||A_k(z)| \\ &\quad + \sum_{k=1}^{2n} (|\beta_k| + |F_n'(z_k)|)\,|B_k(z)| \\ &\quad + |F_n(z) - f(z)| \end{aligned}$$

Using (6.1), (6.4), (6.5), lemma 5.2 and lemma 5.3, we have the theorem 6.1.

7. Conclusion

In this paper, we have defined an intermediate interpolation problem between Lagrange and Hermite on some set of nodes on the unit circle and established a convergence theorem in same regard.

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