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A Comparative Study of Two Factor ANOVA Model Under Fuzzy Environments Using Trapezoidal Fuzzy Numbers

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Abstract. This paper deals with the problem of two factor ANOVA test using Trapezoidal Fuzzy Numbers (TFNs). The proposed ANOVA test is analysed under various types of trapezoidal fuzzy models such as Alpha Cut Interval, Membership Function, Ranking Function, Total Integral Value and Graded Mean Integration Representation. Finally a comparative view of the conclusions obtained from various test is given. Moreover, two numerical examples having different conclusions have been given for a concrete comparative study.

Keywords: Trapezoidal Fuzzy Numbers, Alpha Cut, Membership Function, Ranking Function, Total Integral Value, Graded Mean Integration Representation

AMS Mathematics Subject Classification (2010): 62A86, 62F03, 97K80

1. Introduction

Fuzzy set theory [26] has been applied to many areas which need to manage uncertain and vague data. Such areas include approximate reasoning, decision making, optimization, control and so on. In traditional statistical testing [10], the observations of sample are crisp and a statistical test leads to the binary decision. However, in the real life, the data sometimes cannot be recorded or collected precisely. The statistical hypotheses testing under fuzzy environments has been studied by many authors using the fuzzy set theory concepts introduced by Zadeh [26].Viertl [21] investigated some methods to construct confidence intervals and statistical tests for fuzzy data. Wu [24] proposed some approaches to construct fuzzy confidence intervals for the unknown fuzzy parameter. A new approach to the problem of testing statistical hypotheses is introduced by Chachi et al. [6]. Mikihiko Konishi et al. [14] proposed a method of ANOVA for the fuzzy interval data by using the concept of fuzzy sets. Hypothesis testing of one factor ANOVA model for fuzzy data was proposed by Wu [23, 25] using the h-level set and the notions of pessimistic degree and optimistic degree by solving optimization problems. Gajivaradhan and Parthiban analysed one-way ANOVA test using alpha cut interval method for trapezoidal fuzzy numbers [8].

Liou and Wang ranked fuzzy numbers with total integral value [13]. Wang et al. presented the method for centroid formulae for a generalized fuzzy number [22]. Iuliana Carmen BĂRBĂCIORU dealt with the statistical hypotheses testing using membership function of fuzzy numbers [11]. Salim Rezvani analysed the ranking functions with trapezoidal fuzzy numbers [17]. Wang arrived some different approach for ranking trapezoidal fuzzy numbers [22]. Thorani et al. approached the ranking function of a trapezoidal fuzzy number with some modifications [18]. Salim Rezvani and Mohammad Molani presented the shape function and Graded Mean Integration Representation for trapezoidal fuzzy numbers [16]. Liou and Wang proposed the Total Integral Value of the trapezoidal fuzzy number with the index of optimism and pessimism [13].

In this paper, the two-factor ANOVA model for trapezoidal fuzzy numbers (TFNs) using α-cut interval method is analysed with two different numerical examples. And the same test is proposed using membership function of TFNs [11] ranking grades of TFNs [17, 18], Graded Mean Integration Representation (GMIR) of TFNs [16] and Total Integral Value (TIV) of TFNs [13]. And finally, a comparative study of all these methods is given and ends with conclusion. **In order to present this paper in nutshell, we only present the necessary data and explanations by avoiding elementary, surplus mathematical calculations and repetitive tables.**

2. Preliminaries

Definition 2.1. Generalized fuzzy number

A generalized fuzzy number $\widetilde{A}$ is described as any fuzzy subset of the real line $\mathbb{R}$, whose membership function $\mu_{\widetilde{A}}(x)$ satisfies the following conditions:

i. $\mu_{\widetilde{A}}(x)$ is a continuous mapping from $\mathbb{R}$ to the closed interval $[0, \omega],\ 0 \leq \omega \leq 1$,

ii. $\mu_{\widetilde{A}}(x) = 0$, for all $x \in (-\infty, a]$,

iii. $\mu_L(x) = L_{\widetilde{A}}(x)$ is strictly increasing on $[a, b]$,

iv. $\mu_{\widetilde{A}}(x) = \omega$, for all $[b, c]$, as ω is a constant and $0 < \omega \leq 1$,

v. $\mu_R(x) = R_{\widetilde{A}}(x)$ is strictly decreasing on $[c, d]$,

vi. $\mu_{\widetilde{A}}(x) = 0$, for all $x \in [d, \infty)$.

where a, b, c, d are real numbers such that $a < b \leq c < d$.

Definition 2.2. A fuzzy set $\widetilde{A}$ is called **normal** fuzzy set if there exists an element (member) 'x' such that $\mu_{\widetilde{A}}(x) = 1$. A fuzzy set $\widetilde{A}$ is called **convex** fuzzy set if $\mu_{\widetilde{A}}(\alpha x_1 + (1 - \alpha) x_2) \geq \min\{\mu_{\widetilde{A}}(x_1), \mu_{\widetilde{A}}(x_2)\}$ where $x_1, x_2 \in X$ and $\alpha \in [0, 1]$. The set $\widetilde{A}_\alpha = \{x \in X / \mu_{\widetilde{A}}(x) \geq \alpha\}$ is said to be the α - cut of a fuzzy set $\widetilde{A}$.

Definition 2.3. A fuzzy subset $\widetilde{A}$ of the real line $\mathbb{R}$ with **membership function** $\mu_{\widetilde{A}}(x)$ such that $\mu_{\widetilde{A}}(x):\mathbb{R}\to[0, 1]$, is called a fuzzy number if $\widetilde{A}$ is normal, $\widetilde{A}$ is fuzzy convex, $\mu_{\widetilde{A}}(x)$ is upper semi-continuous and $\text{Supp}(\widetilde{A})$ is bounded, where $\text{Supp}(\widetilde{A}) = \text{cl}\{x \in \mathbb{R} : \mu_{\widetilde{A}}(x) > 0\}$ and 'cl' is the closure operator.

If $\widetilde{A} = (a, b, c, d)_n$, then[1-4],

$\widetilde{A}_\alpha = \left[\widetilde{A}_L(\alpha), \widetilde{A}_U(\alpha)\right] = \left[a + (b - a)\sqrt[n]{\alpha},\ d - (d - c)\sqrt[n]{\alpha}\right];\ \alpha \in [0, 1]$.

When $n = 1$ and $b = c$, we get a triangular fuzzy number. The conditions $r = 1, a = b$ and $c = d$ imply the closed interval and in the case $r = 1, a = b = c = d = t$ (some constant), we can get a crisp number 't'. Since a trapezoidal fuzzy number is completely characterized by $n = 1$ and four real numbers $a \le b \le c \le d$, it is often denoted as $\widetilde{A} = (a, b, c, d)$. And the family of trapezoidal fuzzy numbers will be denoted by $F^T(\mathbb{R})$. Now, for $n = 1$ we have a normal trapezoidal fuzzy number $\widetilde{A} = (a, b, c, d)$ and the corresponding α - cut is defined by

$\widetilde{A}_\alpha = [a + \alpha(b - a),\ d - \alpha(d - c)];\ \alpha \in [0, 1] --- (2.3)$. And we need the following results which can be found in [10, 12].

Result 2.1. Let $D = \{[a, b],\ a \le b \text{ and } a, b \in \mathbb{R}\}$, the set of all closed, bounded intervals on the real line $\mathbb{R}$.

Result 2.2. Let $A = [a, b]$ and $B = [c, d]$ be in D. Then $A = B$ if $a = c$ and $b = d$.

3. ANOVA for two factors of classification

Let the N variate values $\{x_{ij}\}$ be classified according to two factors. Let there be 'h' rows (blocks) representing one factor of classification and 'k' columns representing the other factor, so that $N = hk$. We wish to test the null hypothesis H_0 that the rows and columns are homogeneous viz., there is no difference in the N variates between the various rows and between the various columns. Let x_{ij} be the variate value in the i^{th} row and j^{th} column. Let $\overline{x}$ be the general mean of all the N values, $\overline{x}_{i*}$ be the mean of 'k' values in the i^{th} row and $\overline{x}_{*j}$ be the mean of the 'h' values in the j^{th} column. Then the required formulae [10, 19] for two-way ANOVA test is given below:

Source of Variation (S.V.)	**Sum of Squares (S.S.)**	**Degrees of freedom (d.f.)**	**Mean Square (M.S.)**	**Variance ration** F_0
Between rows	Q_1	$(h\text{-}1)$	$M_1 = \frac{Q_1}{(h\text{-}1)}$	$F_{Rows} = \left[\frac{M_1}{Q_3/(h\text{-}1)(k\text{-}1)}\right]^{\pm 1}$
Between columns	Q_2	$(k\text{-}1)$	$M_2 = \frac{Q_2}{(k\text{-}1)}$	$F_{Col.} = \left[\frac{M_2}{Q_3/(h\text{-}1)(k\text{-}1)}\right]^{\pm 1}$
Residual	Q_3	$(h\text{-}1)(k\text{-}1)$	$\frac{Q_3}{(h\text{-}1)(k\text{-}1)}$	
Total	Q	$(hk\text{-}1)$		

Where $Q = \sum\sum x_{ij}^2 - \frac{T^2}{N}$; $Q_1 = \sum \frac{T_i^2}{k} - \frac{T^2}{N}$; $Q_2 = \sum \frac{T_j^2}{h} - \frac{T^2}{N}$;

$Q_3 = Q - (Q_1 + Q_2)$ and $T = \sum_i T_i = \sum_j T_j$

where Q = total variation; Q_1 = Sum of the squares due to the variations in the rows; Q_2 = that in the columns and Q_3 = that due to the residual variations.

4. Two-factor ANOVA test with tfns using alpha cut interval method

The fuzzy test of hypotheses of two-factor ANOVA model where the sample data are trapezoidal fuzzy numbers is proposed here. Using the relation, we transform the fuzzy ANOVA model to interval ANOVA model. Fetching the upper limit of the fuzzy interval, we construct upper level crisp ANOVA model and using the lower limit of the fuzzy interval, we construct the lower level crisp ANOVA model. Thus, in this proposed approach, two crisp ANOVA models are designated in terms of upper and lower levels. Finally, we analyse the lower and upper level models using crisp two-factor ANOVA technique. For lower level model, from α-cut intervals of TFNs we have, $a_{ij} + \alpha(b_{ij} - a_{ij})$ where $0 \le i \le h$; $0 \le j \le k$ and for upper level model, $d_{ij} - \alpha(d_{ij} - c_{ij})$ where $0 \le i \le h$; $0 \le j \le k$. The required formulae are given below:

For upper level model: $Q^L = \sum_i \sum_j \left[a_{ij} + \alpha(b_{ij} - a_{ij})\right]^2 - \frac{T^2}{N}$ where $0 \le i \le h$, $0 \le j \le k$; $T_i = \sum_j \left[a_{ij} + \alpha(b_{ij} - a_{ij})\right]$, i = 1, 2, .., h ; $T_j = \sum_i \left[a_{ij} + \alpha(b_{ij} - a_{ij})\right]$, j = 1, 2, .., k; $Q_1^L = \sum \frac{T_i^2}{k} - \frac{T^2}{N}$; $Q_2^L = \sum \frac{T_j^2}{h} - \frac{T^2}{N}$; $Q_3^L = Q^L - (Q_1^L + Q_2^L)$.

For lower level model: $Q^U = \sum_i \sum_j \left[d_{ij} - \alpha\left(d_{ij} - c_{ij}\right)\right]^2 - \frac{T^2}{N}$, $0 \le i \le h,\ 0 \le j \le k;$

$T_j = \sum_i \left[d_{ij} - \alpha\left(d_{ij} - c_{ij}\right)\right]$, j = 1, 2, .., k ; $Q_1^U = \sum \frac{T_i^2}{k} - \frac{T^2}{N}$; $Q_2^U = \sum \frac{T_j^2}{h} - \frac{T^2}{N}$;

$Q_3^U = Q^U - \left(Q_1^U + Q_2^U\right)$ and $T = \sum_i T_i = \sum_j T_j$.

The null hypothesis $\widetilde{H}_0 : \tilde{\mu}_1 = \tilde{\mu}_2 = \cdots = \tilde{\mu}_t$ against the alternative hypothesis $\widetilde{H}_A : \tilde{\mu}_1 \ne \tilde{\mu}_2 \ne \cdots \ne \tilde{\mu}_t$.

$\Rightarrow \left[\widetilde{H}_0\right] : \left[\tilde{\mu}_1\right] = \left[\tilde{\mu}_2\right] = \cdots = \left[\tilde{\mu}_t\right]$ against $\left[\widetilde{H}_A\right] : \left[\tilde{\mu}_1\right] \ne \left[\tilde{\mu}_2\right] \ne \cdots \ne \left[\tilde{\mu}_t\right]$.

$\Rightarrow \left[H_0^L, H_0^U\right] : \left[\mu_1^L,\ \mu_1^U\right] = \left[\mu_2^L,\ \mu_2^U\right] = \cdots = \left[\mu_t^L,\ \mu_t^U\right]$ against

$\left[H_A^L, H_A^U\right] : \left[\mu_1^L,\ \mu_1^U\right] \ne \left[\mu_2^L,\ \mu_2^U\right] \ne \cdots \ne \left[\mu_t^L,\ \mu_t^U\right]$ where t = h for rows t = k for columns. The following two sets of hypotheses can be obtained

Between rows:

The null hypothesis for lower level model:

H_0^L: $\mu_1^L = \mu_1^L = \cdots = \mu_h^L$ against the alternative hypothesis H_A^L: $\mu_1^L \ne \mu_1^L \ne \cdots \ne \mu_h^L$.

The null hypothesis for upper level model:

H_0^U: $\mu_1^U = \mu_1^U = \cdots = \mu_h^U$ against the alternative hypothesis H_A^U: $\mu_1^U \ne \mu_1^U \ne \cdots \ne \mu_h^U$.

Between columns:

The null hypothesis for lower level model:

H_0^L: $\mu_1^L = \mu_1^L = \cdots = \mu_k^L$ against the alternative hypothesis H_A^L: $\mu_1^L \ne \mu_1^L \ne \cdots \ne \mu_k^L$.

The null hypothesis for upper level model:

H_0^U: $\mu_1^U = \mu_1^U = \cdots = \mu_k^U$ against the alternative hypothesis H_A^U: $\mu_1^U \ne \mu_1^U \ne \cdots \ne \mu_k^U$.

Decision rules

Lower level model:

(i) If $F_{Row}^L < F_t$ at 'r' level of significance with $\left((h\text{-}1), (h\text{-}1)(k\text{-}1)\right)$ degrees of freedom, then the null hypothesis H_0^L is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^L is accepted.

(ii) If $F_{Col.}^L < F_t$ at 'r' level of significance with $\left((k\text{-}1), (h\text{-}1)(k\text{-}1)\right)$ degrees of freedom, then the null hypothesis H_0^L is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H_A^L is accepted.

Upper level model:

(i) If $F^{U}_{Row} < F_t$ at 'r' level of significance with $((h\text{-}1),(h\text{-}1)(k\text{-}1))$ degrees of freedom, then the null hypothesis H^{U}_{0} is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^{U}_{A} is accepted.

(ii) If $F^{U}_{Col.} < F_t$ at 'r' level of significance with $((k\text{-}1),(h\text{-}1)(k\text{-}1))$ degrees of freedom, then the null hypothesis H^{U}_{0} is accepted for certain value of $\alpha \in [0,1]$, otherwise the alternative hypothesis H^{U}_{A} is accepted.

Example 1.
Three varieties of crop are tested in a randomized block design with four replications, the layout being trapezoidal fuzzy numbers due to some work congestion given as below: The yields are given in kilograms. And we analyse for significance.

C(45, 46, 49, 51)	A(46, 48, 51, 53)	B(48, 50, 52, 53)	A(47, 49, 51, 52)
A(42, 46, 49, 51)	B(45, 48, 49, 52)	C(47, 49, 51, 53)	C(48, 50, 52, 54)
B(44, 47, 50, 51)	C(48, 50, 51, 53)	A(45, 48, 50, 51)	B(44, 46, 49, 50)

Example 2.
The following data represent the number of units of production per day turned out by 5 different workers using 4 different types of machines. Due to some inevitable situations, the obtained data are in terms of trapezoidal fuzzy numbers.

		Machine Type			
		A	**B**	**C**	**D**
Workers	1	(41, 43, 45, 46)	(31, 36, 38, 40)	(41, 43, 46, 48)	(30, 32, 35, 37)
	2	(43, 45, 47, 49)	(37, 38, 41, 44)	(47, 48, 53, 55)	(38, 40, 42, 45)
	3	(29, 31, 34, 36)	(32, 35, 38, 39)	(38, 41, 43, 45)	(27, 30, 33, 35)
	4	(38, 40, 44, 46)	(30, 33, 35, 38)	(39, 43, 46, 47)	(28, 32, 34, 35)
	5	(32, 35, 37, 39)	(39, 41, 44, 45)	(43, 46, 47, 50)	(34, 37, 39, 42)

(a) We test whether the five men differ with respect to mean productivity.
(b) We test the mean productivity is the same for the four different machine types.

5. Two-way ANOVA test using alpha cut interval method

Example 5.1. Let us consider Example 1, using relation (2.3), the interval model for the TFNs using α-cut method is

Blocks	**Varieties of Crop**		
	A	**B**	**C**
1	$[42+4\alpha, 51\text{-}2\alpha]$	$[44+3\alpha, 51\text{-}\alpha]$	$[45+\alpha, 51\text{-}2\alpha]$
2	$[46+2\alpha, 53\text{-}2\alpha]$	$[45+3\alpha, 52\text{-}3\alpha]$	$[48+2\alpha, 53\text{-}2\alpha]$
3	$[45+3\alpha, 51\text{-}\alpha]$	$[48+2\alpha, 53\text{-}\alpha]$	$[47+2\alpha, 53\text{-}2\alpha]$
4	$[47+2\alpha, 52\text{-}\alpha]$	$[44+2\alpha, 50\text{-}\alpha]$	$[48+2\alpha, 54\text{-}2\alpha]$

The lower level model:

Blocks	**Varieties of Crop**		
	A	**B**	**C**
1	[42+4α]	[44+3α]	[45+α]
2	[46+2α]	[45+3α]	[48+2α]
3	[45+3α]	[48+2α]	[47+2α]
4	[47+2α]	[44+2α]	[48+2α]

The upper level model:

Blocks	**Varieties of Crop**		
	A	**B**	**C**
1	[51-2α]	[51-α]	[51-2α]
2	[53-2α]	[52-3α]	[53-2α]
3	[51-α]	[53-α]	[53-2α]
4	[52-α]	[50-α]	[54-2α]

The 2-way ANOVA table for lower level model:

S.V.	**S.S.**	**d. f.**	**M.S.**	**Variance Ratio** F_0^L
Between rows	$Q_1^L = (8\alpha^2 - 64\alpha + 211)/12$	3	$M_1^L = \dfrac{Q_1^L}{(h\text{-}1)}$	$F_{Rows}^L = \left[\dfrac{M_1^L}{Q_3^L/(h\text{-}1)(k\text{-}1)}\right]^{\pm 1}$
Between columns	$Q_2^L = (13\alpha^2 - 54\alpha + 57)/6$	2	$M_2^L = \dfrac{Q_2^L}{(k\text{-}1)}$	$F_{Col.}^L = \left[\dfrac{M_2^L}{Q_3^L/(h\text{-}1)(k\text{-}1)}\right]^{\pm 1}$
Residual	$Q_3^L = (23\alpha^2 - 34\alpha + 79)/6$	6	$Q_3^L / 6$	

Between Rows: $F_{Rows}^L = \left[\dfrac{8\alpha^2 - 64\alpha + 211}{23\alpha^2 - 34\alpha + 79}\right]$ where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((h\text{-}1),(h\text{-}1)(k\text{-}1)) = (3,6)$ degrees of freedom is 4.76. That is, $F_{t(5\%)} = 4.76$. Here, $F_{Rows}^L < F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ We accept the null hypothesis H_0^L at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at lower level model. Hence the difference between rows is not significant. **Therefore, the blocks do not differ significantly with respect to the yield.**

Between Columns: $F_{Col.}^L = \left[\dfrac{23\alpha^2 - 34\alpha + 79}{3(13\alpha^2 - 54\alpha + 57)}\right]$ where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((h\text{-}1)(k\text{-}1),(k\text{-}1)) = (6,2)$ degrees of freedom is 19.33. That is, $F_{t(5\%)} = 19.33$. Here, $F_{Rows}^L < F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ We accept the null hypothesis H_0^L at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at lower level model. Hence the difference between columns is not significant. **Therefore, the varieties of crop do not differ significantly with respect to the yield.**

The 2-way ANOVA table for upper level model:

S. V.	S. S.	d. f.	M. S.	Variance Ratio F_0^U
Between rows	$Q_1^U = (6\alpha^2\text{-}6\alpha+14)/3$	3	$M_1^U = \dfrac{Q_1^U}{(h\text{-}1)}$	$F_{Rows}^U = \left[\dfrac{M_1^U}{Q_3^U/(h\text{-}1)(k\text{-}1)}\right]^{\pm 1}$
Between columns	$Q_2^U = (4\alpha^2\text{-}18\alpha+21)/6$	2	$M_2^U = \dfrac{Q_2^U}{(k\text{-}1)}$	$F_{Col.}^U = \left[\dfrac{M_2^U}{Q_3^U/(h\text{-}1)(k\text{-}1)}\right]^{\pm 1}$
Residual	$Q_3^U = (12\alpha^2\text{-}6\alpha+47)/6$	6	$Q_3^U/6$	

Between Rows: $F_{Rows}^U = \left[\dfrac{4(6\alpha^2 - 6\alpha+14)}{12\alpha^2 - 6\alpha+47}\right]$ where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((h\text{-}1),(h\text{-}1)(k\text{-}1)) = (3,6)$ degrees of freedom is 4.76. That is, $F_{t(5\%)} = 4.76$. Here, $F_{Rows}^U < F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ We accept the null hypothesis H_0^U at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at upper level model. Hence the difference between rows is not significant. **Therefore, the blocks do not differ significantly with respect to the yield.**

Between Columns: $F_{Col.}^U = \left[\dfrac{3(4\alpha^2 - 18\alpha+21)}{12\alpha^2 - 6\alpha+47}\right]$ where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((k\text{-}1),(h\text{-}1)(k\text{-}1)) = (2,6)$ degrees of freedom is 5.14. That is, $F_{t(5\%)} = 5.14$. Here, $F_{Rows}^U < F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ We accept the null hypothesis H_0^U at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at upper level model. Hence the difference between columns is not significant. **Therefore, the varieties of crop do not differ significantly with respect to the yield.**

Thus, considering the decisions obtained in lower level and upper level model, we conclude that the null hypothesis H_0 is accepted $\forall\alpha$, $\alpha \in [0,1]$ for difference between rows and columns. Hence, the blocks do not differ significantly with respect to the yield and the varieties of crop do not differ significantly with respect to the yield.

Example 5.2. Let us consider Example 2, and using relation (2.3), the interval model for the TFNs using α-cut method is

Workers	Machine Type			
	A	B	C	D
1	[41+2α, 46-α]	[31+5α, 40-2α]	[41+2α, 48-2α]	[30+2α, 37-2α]
2	[43+2α, 49-2α]	[37+α, 44-3α]	[47+α, 55-2α]	[38+2α, 45-3α]
3	[29+2α, 36-2α]	[32+3α, 39-α]	[38+3α, 45-2α]	[27+3α, 35-2α]
4	[38+2α, 46-2α]	[30+3α, 38-3α]	[39+4α, 47-α]	[28+4α, 35-α]
5	[32+3α, 39-2α]	[39+2α, 45-α]	[43+3α, 30-3α]	[34+3α, 42-3α]

The upper and lower level models can be tabulated as per example 5.1.

The 2-way ANOVA table for lower level model:

S. V.	S. S.	d. f.	M. S.	Variance Ratio F_0^L
Between rows	$Q_1^L = (68\alpha^2 - 624\alpha + 2153)/10$	4	$M_1^L = \frac{Q_1^L}{(h-1)}$	$F_{Rows}^L = \left[\frac{M_1^L}{Q_3^L/(h-1)(k-1)}\right]^{\pm 1}$
Between columns	$Q_2^L = (24\alpha^2 - 320\alpha + 5763)/20$	3	$M_2^L = \frac{Q_2^L}{(k-1)}$	$F_{Col.}^L = \left[\frac{M_2^L}{Q_3^L/(h-1)(k-1)}\right]^{\pm 1}$
Residual	$Q_3^L = (108\alpha^2 - 240\alpha + 1231)/10$	12	$Q_3^L/6$	

Between Rows: $F_{Rows}^L = \left[\frac{3(68\alpha^2 - 624\alpha + 2153)}{108\alpha^2 - 240\alpha + 1231}\right]$ since $M_1^L > \frac{Q_3^L}{(h-1)(k-1)}$

where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((h-1),(h-1)(k-1)) = (4,12)$ degrees of freedom is 3.26. That is, $F_{t(5\%)} = 3.26$. Here, $F_{Rows}^L > F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ The null hypothesis H_0^L is rejected at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at lower level model. $\Rightarrow$ The difference between rows is significant. **Therefore, the 5 workers differ significantly with respect to mean productivity.**

Between Columns: $F_{Col.}^L = \left[\frac{2(24\alpha^2 - 320\alpha + 5763)}{108\alpha^2 - 240\alpha + 1231}\right]$ since $M_2^L > \frac{Q_3^L}{(h-1)(k-1)}$

where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((k-1),(h-1)(k-1)) = (3,12)$ degrees of freedom is 3.49. That is, $F_{t(5\%)} = 3.49$. Here, $F_{Rows}^L > F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ The null hypothesis H_0^L is rejected at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at lower level model. $\Rightarrow$ There is a significant difference between columns. **Therefore, the 4 machine types also differ significantly with respect to mean productivity.**

The 2-way ANOVA table for upper level model:

S. V.	S. S.	d. f.	M. S.	Variance Ratio F_0^U
Between rows	$Q_1^U = (20\alpha^2 - 350\alpha+1957)/10$	4	$M_1^U = \frac{Q_1^U}{(h-1)}$	$F_{Rows}^U = \left[\frac{M_1^U}{Q_3^U/(h-1)(k-1)}\right]^{\pm1}$
Between columns	$Q_2^U = (8\alpha^2+176\alpha+5691)/20$	3	$M_2^U = \frac{Q_2^U}{(k-1)}$	$F_{Col.}^U = \left[\frac{M_2^U}{Q_3^U/(h-1)(k-1)}\right]^{\pm1}$
Residual	$Q_3^U = (76\alpha^2+122\alpha+1007)/10$	12	$Q_3^U / 6$	

Between Rows: $F_{Rows}^U = \left[\frac{4(6\alpha^2 - 6\alpha+14)}{12\alpha^2 - 6\alpha+47}\right]$ where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((h-1),(h-1)(k-1)) = (4,12)$ degrees of freedom is 3.26. That is, $F_{t(5\%)} = 3.26$. Here, $F_{Rows}^U > F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ We reject the null hypothesis H_0^U at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at upper level model. Hence the difference between rows is significant. **Therefore, the 5 workers differ significantly with respect to mean productivity.**

Between Columns: $F_{Col.}^U = \left[\frac{3(4\alpha^2 - 18\alpha+21)}{12\alpha^2 - 6\alpha+47}\right]$ where $0 \le \alpha \le 1$. Now, the tabulated value of F at 5% level of significance with $((k-1),(h-1)(k-1)) = (3,12)$ degrees of freedom is 3.49. That is, $F_{t(5\%)} = 3.49$. Here, $F_{Rows}^U > F_{t(5\%)}$ $\forall\alpha$, $(0 \le \alpha \le 1)$. $\Rightarrow$ We reject the null hypothesis H_0^U at 5% l.o.s. $\forall\alpha$, $(0 \le \alpha \le 1)$ at upper level model. Hence the difference between columns is significant. **Therefore, the 4 machine types also differ significantly with respect to mean productivity.**

Note: Here, $\alpha \in [0,1]$ is the degree of optimism of the decision maker. That is, the index of optimism α represents a decision maker's attitude. A larger α indicates a higher degree of optimism. More specifically, the left and right relative values in the α-cut interval are used to reflect the decision maker's pessimistic for $\alpha = 0$ and optimistic view point for $\alpha = 1$ respectively. In addition, when $\alpha = 0.5$ represents a moderate decision maker's view point.

6. Two-way ANOVA model using membership function

Proposition 6.1. (a) If $\widetilde{A} = (a, b, c, d; w)$ is a generalized trapezoidal fuzzy number and 'k' be a scalar with $k \geq 0,\ y = kA$ then $\tilde{y} = k\widetilde{A}$ is a fuzzy number with $(ka, kb, kc, kd; w)$.(b)If $\widetilde{A} = (a, b, c, d; w)$ is a generalized trapezoidal fuzzy number and 'k' be a scalar with $k < 0,\ y = kA$ then $\tilde{y} = k\widetilde{A}$ is a fuzzy number with $(kd, kc, kb, ka; w)$.

Proof: (a) When $k \geq 0$, with the transformation $y = kA$ we can find the membership function of fuzzy set $\tilde{y} = k\widetilde{A}$ by α-cut method. Now, the α-cut interval of $\widetilde{A}$ is $\widetilde{A}_\alpha = \left[\widetilde{A}_L(\alpha), \widetilde{A}_U(\alpha)\right]$. That is $\widetilde{A}_\alpha = \left[a + \frac{\alpha}{w}(b - a), d - \frac{\alpha}{w}(d - c)\right]$. The lo α-cut of $\widetilde{A}$ is $\widetilde{A}_L(\alpha) = a + \frac{\alpha}{w}(b - a)$ and the upper level α-cut of $\widetilde{A}$ is $\widetilde{A}_U(\alpha) = d - \frac{\alpha}{w}(d - c)$.

Hence, $A \in \left[a + \frac{\alpha}{w}(b - a), d - \frac{\alpha}{w}(d - c)\right]$.

So, $y(= kA) \in \left[ka + \frac{\alpha}{w}(kb - ka),\ kd - \frac{\alpha}{w}(kd - kc)\right]$. So, $\frac{\alpha}{w}(kb - ka) = y - ka.$

$\Rightarrow \alpha = w\left(\frac{y - ka}{kb - ka}\right); ka \leq y \leq kb$---(1) and $\frac{\alpha}{w}(kd - kc) = kd - y$

$\Rightarrow \alpha = w\left(\frac{y - kd}{kc - kd}\right); kc \leq y \leq kd$---(2) From (1) and (2), we have the membership function of $\tilde{y} = k\widetilde{A}$ as follows:

$\mu_{\tilde{y}}(y) = w\left(\frac{y - ka}{kb - ka}\right)$ for $ka \leq y \leq kb;$ w for $kb \leq y \leq kc;$ $w\left(\frac{y - kd}{kc - kd}\right)$ for $kc \leq y \leq kd;$ and 0, otherwise.---(3)

Similarly we can prove (b) if $y = kA$, $k < 0$ then $\tilde{y} = (kd, kc, kb, ka; w)$ is a fuzzy number with membership function,

$\mu_{\tilde{y}}(y) = w\left(\frac{y - kd}{kc - kd}\right)$ for $kd \leq y \leq kc;$ w for $kc \leq y \leq kb;$ $w\left(\frac{y - ka}{kb - ka}\right)$ for $kb \leq y \leq ka;$ and 0, otherwise.---(4)

And for a normalized trapezoidal number, we put w = 1 in equations (3) and (4).

Calculation of membership function of TFNs

The membership grades for a normalized TFN $\tilde{y} = (a, b, c, d; 1)$ is calculated by the relation [11] $\int_{Supp(\tilde{y})} \mu_{\tilde{y}}(y)dy = \int_a^b \left(\frac{y - b}{b - a}\right)dy + \int_b^c dy + \int_c^d \left(\frac{y - d}{c - d}\right)dy - - - (6.2)$

Example 6.1. Let us consider Example 1, since for a normalized TFN $\widetilde{A}$, $\mu_{\tilde{A}} : X \to [0,1]$, we transform the TFNs in problem (1) by multiplying each members with "0.01" using proposition-6.1 and the first member will be, for i = 1 and j= 1, $\tilde{y}_{11} = (0.42,\ 0.46, 0.49, 0.51; 1)$. The membership value is calculated as follows:

$$\int_{Supp(\tilde{y}_{11})} \mu_{\tilde{y}_{11}}(y)dy = \int_{0.42}^{0.46} \left(\frac{y - 0.42}{0.04}\right)dy + \int_{0.46}^{0.49} dy + \int_{0.49}^{0.51} \left(\frac{y - 0.51}{-0.02}\right)dy = 0.06 = I_{11}$$

Similarly we can calculate the membership grades of all other entries using $\int_{Supp(\tilde{y}_{ij})} \mu_{\tilde{y}_{ij}}(y)dy = I_{ij}$ i = 1, 2, 3, 4; j = 1, 2, 3 for the given TFNs which has been tabulated below.

$I_{ij} = \int_{Supp(\tilde{y}_{ij})} \mu_{\tilde{y}_{ij}}(y)dy$; i = 1, 2, 3, 4; j = 1, 2, 3			
	Varieties of crop(j)		
Blocks(i)	**A**	**B**	**C**
1	0.06	0.05	0.045
2	0.05	0.04	0.03
3	0.04	0.035	0.04
4	0.035	0.045	0.04

The ANOVA table values of tfns using membership grades

Here, $Q_1 = 0.000325$, $h\text{-}1=3$; $Q_2 = 0.000125$, $k\text{-}1=2$; $Q_3 = 0.000275$, $(h\text{-}1)(k\text{-}1)=6$; $M_1 = 0.0001083$, $M_2 = 0.0000625$, $Q_3 / (h\text{-}1)(k\text{-}1) = 0.00004583$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now, $F_{Row} = 2.36$ and $F_{t\,(5\%)}(v_1=3, v_2=6) = 4.76$. And $F_{Row} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The blocks do not differ significantly with respect to the yield.** And $F_{Col.} = 1.36$, $F_{t\,(5\%)}(v_1=2, v_2=6) = 5.14$. And $F_{Col.} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The varieties of crop do not differ significantly with respect to the yield.**

Example 6.2. Let us consider Example 2, the calculated membership grades using are:

$$I_{ij} = \int_{Supp(\tilde{y}_{ij})} \mu_{\tilde{y}_{ij}}(y)\,dy \text{ ; } i = 1, 2, 3, 4, 5;\ j = 1, 2, 3, 4$$

Workers(i)	Machine type(j)			
	A	**B**	**C**	**D**
1	0.035	0.055	0.05	0.05
2	0.04	0.05	0.065	0.045
3	0.05	0.05	0.045	0.055
4	0.06	0.05	0.055	0.045
5	0.045	0.045	0.04	0.05

The ANOVA table values of tfns using membership grades

Here, $Q_1 = 0.00013$, $h\text{-}1=4$; $Q_2 = 0.00007$, $k\text{-}1=3$; $Q_3 = 0.00073$, $(h\text{-}1)(k\text{-}1)=12$; $M_1 = 0.0000325$, $M_2 = 0.0000233$, $Q_3/(h\text{-}1)(k\text{-}1) = 0.000061$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now, $F_{Row} = 1.88$ and $F_{t\,(5\%)}(v_1=12,\ v_2=4) = 5.91$. And $F_{Row} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\tilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The 5 workers do not differ significantly with respect to mean productivity.** And $F_{Col.} = 2.62$, $F_{t\,(5\%)}(v_1=12,\ v_2=3) = 8.74$. And $F_{Col.} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\tilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The 4 machine types do not differ significantly with respect to mean productivity.**

7. Rezvani's ranking function of TFNs

The centroid of a trapezoid is considered as the balancing point of the trapezoid. Divide the trapezoid into three plane figures. These three plane figures are a triangle (APB), a rectangle (BPQC) and a triangle (CQD) respectively. Let the centroids of the three plane figures be G_1, G_2 and G_3 respectively. The incenter of these centroids G_1, G_2 and G_3 is taken as the point of reference to define the ranking of generalized trapezoidal fuzzy numbers. The reason for selecting this point as a point of reference is that each centroid point are **balancing points** of each individual plane figure and the incenter of these centroid points is much more balancing point for a generalized trapezoidal fuzzy number. Therefore, this point would be a better reference point than the centroid point of the trapezoid.

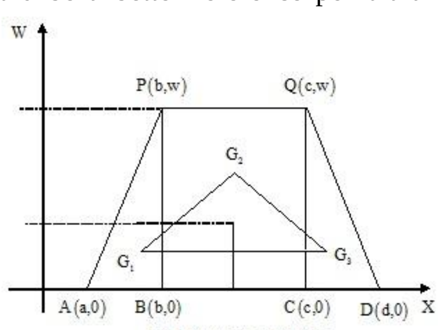

Fig.1 Centroid of centroids

Consider a generalized trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; w)$. The centroids of the three plane figures are:

$$G_1=\left(\frac{a+2b}{3},\frac{w}{3}\right),\ G_2=\left(\frac{b+c}{2},\frac{w}{2}\right) \text{ and } G_3=\left(\frac{2c+d}{3},\frac{w}{3}\right)---(7.1)$$

Equation of the line G_1G_3 is $y=\frac{w}{3}$ and G_2 does not lie on the line G_1G_3. Therefore, G_1, G_2 and G_3 are non-collinear and they form a triangle. We define the incenter $I(\overline{x}_0, \overline{y}_0)$ of the triangle with vertices G_1, G_2 and G_3 of the generalized fuzzy number $\widetilde{A}=(a, b, c, d; w)$ as [17]

$$I_{\widetilde{A}}(\overline{x}_0, \overline{y}_0)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma},\ \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right]--(7.2)$$

where $\alpha=\frac{\sqrt{(c-3b+2d)^2+w^2}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3}, \gamma=\frac{\sqrt{(3c-2a-b)^2+w^2}}{6}$

And ranking function of the trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of all real numbers $\left[\text{i.e. } R:[\widetilde{A}]\to\mathbb{R}\right]$ is defined as $R(\widetilde{A})=\sqrt{\overline{x}_0^2+\overline{y}_0^2}---(7.3)$ which is the Euclidean distance from the incenter of the centroids. For a normalized TFN, we put w = 1 in equations (1), (2) and (3) so we have,

$$G_1=\left(\frac{a+2b}{3},\frac{1}{3}\right),\ G_2=\left(\frac{b+c}{2},\frac{1}{2}\right) \text{ and } G_3=\left(\frac{2c+d}{3},\frac{1}{3}\right)---(7.4)$$

$$I_{\widetilde{A}}(\overline{x}_0, \overline{y}_0)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma},\ \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right]---(7.5)$$

where $\alpha=\frac{\sqrt{(c-3b+2d)^2+1}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3}$ and $\gamma=\frac{\sqrt{(3c-2a-b)^2+1}}{6}$

And ranking function of the trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; 1)$ is defined as $R(\widetilde{A})=\sqrt{\overline{x}_0^2+\overline{y}_0^2}---(7.6)$.

8. Two-way ANOVA test using Rezvani's ranking function

We now analyse the two-way ANOVA test by assigning rank for each normalized trapezoidal fuzzy numbers and based on the ranking grades the decisions are observed.

Example8.1.Let us consider Example 1, using the above relations (7.4), (7.5) and (7.6), we get the ranks of each TFNs $\widetilde{A}$ as below:

Blocks	**Varieties of crop**		
	$R(\widetilde{A})$	$R(\widetilde{B})$	$R(\widetilde{C})$
1	47.5011	48.5008	47.5024
2	49.5018	48.5018	50.5017
3	49	51.0007	50.0017
4	50.0007	47.5012	51.0017

Based on this rank of TFNs, we propose two-factor ANOVA test.

The ANOVA table values of tfns using Rezvani's ranking grades:

Here, $Q_1 = 19.227$, $h-1=3$; $Q_2 = 1.7953$, $k-1=2$; $Q_3 = 9.3723$, $(h-1)(k-1)=6$; $M_1 = 2.6866$, $M_2 = 0.8976$, $Q_3/(h-1)(k-1) = 1.5621$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3.Now, $F_{Row} = 1.27$ and $F_{t\,(5\%)}(v_1=3,\ v_2=6) = 4.76$. And $F_{Row} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The blocks do not differ significantly with respect to the yield.** And $F_{Col.} = 1.74$, $F_{t\,(5\%)}(v_1=6,\ v_2=2) = 19.33$. And $F_{Col.} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The varieties of crop do not differ significantly with respect to the yield.**

Example 8.2. Let us consider Example 2, using the above relations (7.4), (7.5) and (7.6), we get the ranks of each TFNs $\widetilde{A}$ as below:

Workers	**Machine Type**			
	$R(\widetilde{A})$	$R(\widetilde{B})$	$R(\widetilde{C})$	$R(\widetilde{D})$
1	44.0009	37.0008	44.5019	33.5026
2	46.0019	39.5032	50.5020	41.0028
3	32.5027	36.5014	42.0014	31.5023
4	42.0021	34.0026	44.5006	33.0004
5	36.0017	42.5015	46.5019	38.0023

The ANOVA table values of tfns using Rezvani's ranking grades:

Here, $Q_1 = 160.649$, $h-1=4$; $Q_2 = 283.422$, $k-1=3$; $Q_3 = 112.153$, $(h-1)(k-1)=12$; $M_1 = 40.1622$, $M_2 = 94.474$, $Q_3/(h-1)(k-1) = 9.34608$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now,

$F_{Row} = 4.30$ and $F_{t(5\%)}(v_1=4, v_2=12) = 3.26$. And $F_{Row} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between rows is significant. $\Rightarrow$ **The 5 workers differ significantly with respect to mean productivity.** And $F_{Col.} = 10.1084$, $F_{t(5\%)}(v_1=3, v_2=12) = 3.49$. And $F_{Col.} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between columns is significant. $\Rightarrow$ **The 4 machine types differ significantly with respect to mean productivity.**

9. Thorani's centroid point and ranking method

As per the description in Salim Rezvani's ranking method, Y. L. P. Thorani et al. [18] presented a different kind of centroid point and ranking function of TFNs. The incenter $I_{\widetilde{A}}\left(\overline{x}_0, \overline{y}_0\right)$ of the triangle [Fig. 1] with vertices G_1, G_2 and G_3 of the generalized TFN $\widetilde{A}=(a, b, c, d; w)$ is given by,

$$I_{\widetilde{A}}\left(\overline{x}_0, \overline{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{w}{3}\right)+\beta\left(\frac{w}{2}\right)+\gamma\left(\frac{w}{3}\right)}{\alpha+\beta+\gamma}\right] --(9.1)$$

where $\alpha=\frac{\sqrt{(c-3b+2d)^2+w^2}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3}, \gamma=\frac{\sqrt{(3c-2a-b)^2+w^2}}{6}$

And the ranking function of the generalized TFN $\widetilde{A}=(a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as $R\left(\widetilde{A}\right)=x_0\times y_0 ---(9.2)$.

For a normalized TFN, we put w = 1 in equations (1) and (2) so we have,

$$I_{\widetilde{A}}\left(\overline{x}_0, \overline{y}_0\right)=\left[\frac{\alpha\left(\frac{a+2b}{3}\right)+\beta\left(\frac{b+c}{2}\right)+\gamma\left(\frac{2c+d}{3}\right)}{\alpha+\beta+\gamma}, \frac{\alpha\left(\frac{1}{3}\right)+\beta\left(\frac{1}{2}\right)+\gamma\left(\frac{1}{3}\right)}{\alpha+\beta+\gamma}\right] ---(9.3)$$

where $\alpha=\frac{\sqrt{(c-3b+2d)^2+1}}{6}, \beta=\frac{\sqrt{(2c+d-a-2b)^2}}{3}$ and $\gamma=\frac{\sqrt{(3c-2a-b)^2+1}}{6}$

And for $\widetilde{A}=(a, b, c, d; 1)$, $R\left(\widetilde{A}\right)=x_0\times y_0 ---(9.4)$

10. Two-way ANOVA test using Thorani's ranking function

Example 10.1. Let us consider Example 1, using the above relations (9.3) and (9.4), we get the ranks of each TFNs $\widetilde{A}$ which are tabulated below:

Blocks	Varieties of crop		
	$R(\tilde{A})$	$R(\tilde{B})$	$R(\tilde{C})$
1	19.7869	20.2018	19.785
2	20.6189	20.1959	21.0204
3	20.4054	21.2364	20.823
4	20.82	19.7845	21.2394

The two-way ANOVA table for the above ranking function is given below

The ANOVA table values of tfns using Thorani's ranking grades:

Here, $Q_1 = 1.3800$, $h\text{-}1=3$; $Q_2 = 0.3062$, $k\text{-}1=2$; $Q_3 = 1.6160$, $(h\text{-}1)(k\text{-}1)=6$; $M_1 = 0.4600$, $M_2 = 0.1531$, $Q_3/(h\text{-}1)(k\text{-}1) = 0.2693$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3.Now, $F_{Row} = 1.7081$ and $F_{t\,(5\%)}(v_1=3,\ v_2=6) = 4.76$. And $F_{Row} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\tilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The blocks do not differ significantly with respect to the yield.** And $F_{Col.} = 1.7590$, $F_{t\,(5\%)}(v_1=6,\ v_2=2) = 19.33$. And $F_{Col.} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\tilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The varieties of crop do not differ significantly with respect to the yield.**

Example10.2. Let us consider Example 2, using the above relations (9.3) and (9.4), we get the ranks of each TFNs $\tilde{A}$ which are tabulated below:

Workers	Machine Type			
	$R(\tilde{A})$	$R(\tilde{B})$	$R(\tilde{C})$	$R(\tilde{D})$
1	18.3215	15.4112	18.5362	13.9542
2	19.1571	16.4538	21.0385	17.0765
3	13.5377	15.2033	17.4925	13.1215
4	17.4966	14.1618	18.5362	13.743
5	14.9935	17.7019	19.3631	15.8279

The ANOVA table values of tfns using Thorani's ranking grades:

Here, $Q_1 = 27.8632$, $h\text{-}1=4$; $Q_2 = 49.1746$, $k\text{-}1=3$; $Q_3 = 19.4485$, $(h\text{-}1)(k\text{-}1)=12$; $M_1 = 6.9658$, $M_2 = 16.3915$, $Q_3/(h\text{-}1)(k\text{-}1) = 1.6207$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now, $F_{Row} = 4.2980$ and $F_{t\,(5\%)}(v_1=4,\ v_2=12) = 3.26$. And $F_{Row} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\tilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between rows is significant. $\Rightarrow$ **The 5 workers differ significantly with respect to mean productivity.** And $F_{Col.} = 10.1138$, $F_{t\,(5\%)}(v_1=3,\ v_2=12) = 3.49$. And $F_{Col.} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between columns is significant. $\Rightarrow$ **The 4 machine types differ significantly with respect to mean productivity.**

11. Graded mean integration representation (GMIR)

Let $\widetilde{A}=(a, b, c, d; w)$ be a generalized trapezoidal fuzzy number, then the **GMIR**[16]of $\widetilde{A}$ is defined by $P(\widetilde{A}) = \int_0^w h\left[\frac{L^{-1}(h)+R^{-1}(h)}{2}\right]dh \,/ \int_0^w hdh$.

Theorem 11.1. Let $\widetilde{A}=(a, b, c, d; 1)$ be a TFN with normal shape function, where a, b, c, d are real numbers such that $a < b \le c < d$. Then the graded mean integration representation (GMIR) of $\widetilde{A}$ is $P(\widetilde{A}) = \frac{(a+d)}{2} + \frac{n}{2n+1}(b - a - d + c)$.

Proof : For a trapezoidal fuzzy number $\widetilde{A}=(a, b, c, d; 1)_n$, we have $L(x) = \left(\frac{x-a}{b-a}\right)^n$ and $R(x) = \left(\frac{d-x}{d-c}\right)^n$ Then, $h = \left(\frac{x-a}{b-a}\right)^n \Rightarrow L^{-1}(h) = a + (b-a)h^{1/n}$;

$h = \left(\frac{d-x}{d-c}\right)^n \Rightarrow R^{-1}(h) = d - (d-c)h^{1/n}$

$$\therefore P(\widetilde{A}) = \left(\frac{1}{2}\int_0^1 h\left[\left(a+(b-a)h^{1/n}\right)+\left(d-(d-c)h^{1/n}\right)\right]dh\right) / \int_0^1 hdh$$

$$= \left(\frac{1}{2}\left[\frac{(a+d)}{2} + \frac{n}{2n+1}(b - a - d + c)\right]\right) / \left(1/2\right)$$

Thus, $P(\widetilde{A}) = \frac{(a+d)}{2} + \frac{n}{2n+1}(b - a - d + c)$ Hence the proof.

Result 11.1. If n =1 in the above theorem, we have $P(\widetilde{A}) = \frac{a + 2b + 2c + d}{6}$

12. Two-way ANOVA using GMIR of TFNs

Example 12.1. Let us consider Example 1, using the result-11.1 from above theorem-11.1, we get the GMIR of each TFNs $\widetilde{A}$ which are tabulated below:

Blocks	**Varieties of crop**		
	$P(\widetilde{A})$	$P(\widetilde{B})$	$P(\widetilde{C})$
1	47.1667	48.1667	47.6667
2	49.5	48.5	50.5
3	48.6667	50.8333	50
4	49.8333	47.3333	51

The ANOVA table values of tfns using GMIR:

Here, Q_1 =8.5066, h-1=3; Q_2 =2.9075, k-1=2; Q_3 =8.9999, $(h-1)(k-1)=6$; $M_1 = 2.8355$, $M_2 = 1.4537$, $Q_3/(h-1)(k-1) = 1.5$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now, $F_{Row} = 1.8903$ and $F_{t\,(5\%)}(v_1=3,\ v_2=6) = 4.76$. And $F_{Row} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The blocks do not differ significantly with respect to the yield.** And $F_{Col.} = 1.0318$, $F_{t\,(5\%)}(v_1=6,\ v_2=2) = 19.33$. And $F_{Col.} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The varieties of crop do not differ significantly with respect to the yield.**

Example12.2. Let us consider Example 2, using the above result-11.1, we get the ranks of each TFNs $\widetilde{A}$ in problem-1 which are tabulated below:

Workers	**Machine Type**			
	$P(\widetilde{A})$	$P(\widetilde{B})$	$P(\widetilde{C})$	$P(\widetilde{D})$
1	43.8333	36.5	44.5	33.5
2	46	39.8333	50.6667	41.1667
3	32.5	36.1667	41.8333	31.3333
4	42	34	44	32.5
5	35.8333	42.3333	46.5	38

The ANOVA table values of tfns using GMIR:

Here, Q_1 =174.6199, h-1=4; Q_2 =284.1945, k-1=3; Q_3 =110.0697, $(h-1)(k-1)=12$; $M_1 = 43.6550$, $M_2 = 94.7315$, $Q_3/(h-1)(k-1) = 9.1725$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now, $F_{Row} = 4.7593$ and $F_{t\,(5\%)}(v_1=4,\ v_2=12) = 3.26$. And $F_{Row} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between rows is significant. $\Rightarrow$ **The 5 workers differ significantly with respect to mean productivity.** And $F_{Col.} = 10.3278$, $F_{t\,(5\%)}(v_1=3,\ v_2=12) = 3.49$. And $F_{Col.} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between columns is significant. $\Rightarrow$ **The 4 machine types differ significantly with respect to mean productivity.**

13. LIOU and WANG'S centroid point method

Liou and Wang [13] ranked fuzzy numbers with total integral value. For a fuzzy number defined by definition (2.3), the total integral value is defined as

$I_T^{\alpha}(\tilde{A}) = \alpha I_R(\tilde{A}) + (1-\alpha) I_L(\tilde{A})$ ---(13.1)

where $I_R(\tilde{A}) = \int_{Supp(\tilde{A})} R_{\tilde{A}}(x)dx$---(13.2) and $I_L(\tilde{A}) = \int_{Supp(\tilde{A})} L_{\tilde{A}}(x)dx$ ---(13.3) are the right and left integral values of $\tilde{A}$ respectively and $0 \le \alpha \le 1$.

(i) $\alpha \in [0,1]$ is the **index of optimism** which represents the **degree of optimism** of a decision maker. (ii) If $\alpha = 0$, then the total value of integral represents a **pessimistic decision maker's view point** which is equal to left integral value. (iii) If $\alpha = 1$, then the total integral value represents an **optimistic decision maker's view point** and is equal to the right integral value.(iv)If $\alpha = 0.5$ then the total integral value represents a **moderate decision maker's view point** and is equal to the mean of right and left integral values. For a decision maker, the larger the value of α is, the higher is the degree of optimism.

13. The ANOVA test using LIOU and WANG'S centroid point method

Example 14.1. Let us consider Example 1, using the above equations (13.1), (13.2) and (13.3), we get the centroid point of first member as follows:

$I_L(\tilde{A}_{11}) = \int_{42}^{46}\left(\frac{x-42}{4}\right)dx = 2;\ I_R(\tilde{A}_{11}) = \int_{49}^{51}\left(\frac{x-51}{-2}\right)dx = 1$ Therefore $I_T^{\alpha}(\tilde{A}_{11}) = 2-\alpha$

Similarly we can find $I_T^{\alpha}(\tilde{A}_{ij})$; for i = 1, 2, 3, 4; j = 1, 2, 3 and the calculated values are tabulated below:

$I_T^{\alpha}(\tilde{A}_{ij})$; for i = 1, 2, 3, 4; j = 1, 2, 3			
	Varieties of crop (j)		
Blocks (i)	**A**	**B**	**C**
1	$(2-\alpha)$	$(1.5-\alpha)$	$(0.5+0.5\alpha)$
2	1	1.5	1
3	$(1.5-\alpha)$	$(1-0.5\alpha)$	1
4	$(1-0.5\alpha)$	$(1-0.5\alpha)$	1

The ANOVA table values of tfns using Liou and Wang's Centroid Point:

Here, $Q_1 = \frac{1}{6}(3\alpha^2 - \alpha + 1)$, h-1=3; $Q_2 = \frac{1}{24}(31\alpha^2 - 40\alpha + 13)$, k-1=2;

$Q_3 = \frac{1}{24}(21\alpha^2 - 32\alpha + 23)$, $(h-1)(k-1)=6$; $M_1 = = \frac{1}{18}(3\alpha^2 - \alpha + 1)$,

$M_2 = \frac{1}{48}(31\alpha^2 - 40\alpha + 13)$, $Q_3/(h-1)(k-1) = \frac{1}{144}(21\alpha^2 - 32\alpha + 23)$ and variance ratio of F can be calculated as per section-3.Now, $F_{Row} = \frac{8(3\alpha^2 - \alpha + 1)}{(21\alpha^2 - 32\alpha + 23)};\ 0 \le \alpha \le 1$

and $F_{t\,(5\%)}(v_1=3,\ v_2=6) = 4.76$. And $F_{Row} < F_{t(5\%)}\ \forall\alpha,\ \alpha \in [0,1] \Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The blocks do not differ significantly with respect to the yield.** And

$$F_{Col.} = \frac{21\alpha^2 - 32\alpha + 23}{3\left(31\alpha^2 - 40\alpha + 13\right)};\ 0 \le \alpha \le 1,\ F_{t\,(5\%)}(v_1=6,\ v_2=2) = 19.33.$$

Here $F_{Col.} < F_{t(5\%)}$ for $\alpha \in [0, 0.5]$ and $\alpha \in [0.7, 1] \Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted for all α except $\alpha = 0.6$ at 5% level of significance. $\Rightarrow$ The difference between columns is not significant for all α but $\alpha = 0.6$. $\Rightarrow$ **The varieties of crop do not differ significantly with respect to the yield** for all α but $\alpha = 0.6$**.**

Example14.2. Let we consider Example 2, using the above equations (13.1), (13.2) and (13.3), we get the centroid points of TFNs and tabulated as follows:

$I_T^\alpha\left(\widetilde{A}_{ij}\right)$; for i = 1, 2, 3, 4, 5; j = 1, 2, 3, 4				
Workers(i)	**Machine type(j)**			
	A	**B**	**C**	**D**
1	[1-0.5α]	[2.5-1.5α]	1	1
2	1	[0.5+α]	[0.5+0.5α]	[1+0.5α]
3	1	[1.5-α]	[1.5-0.5α]	[1.5-0.5α]
4	1	1.5	[2-1.5α]	[2-1.5α]
5	[1.5-0.5α]	[1-0.5α]	1.5	1.5

The ANOVA table values of tfns using Liou and Wang's Centroid Point:

Here, $Q_1 = \frac{1}{10}\left(37\alpha^2 - 49\alpha + 17\right)$, h-1=4; $Q_2 = \frac{1}{10}\left(\alpha^2 - 3\alpha + 3\right)$, k-1=3;

$Q_3 = \frac{1}{10}\left(49\alpha^2 - 57\alpha + 27\right)$, $(h\text{-}1)(k\text{-}1)=12$; $M_1 = = \frac{1}{40}\left(37\alpha^2 - 49\alpha + 17\right)$,

$M_2 = \frac{1}{30}\left(\alpha^2 - 3\alpha + 3\right)$, $Q_3 / (h\text{-}1)(k\text{-}1) = \frac{1}{120}\left(49\alpha^2 - 57\alpha + 27\right)$ and variance ratio of F can be calculated as per section-3. Now, $F_{Row} = \frac{49\alpha^2 - 57\alpha + 27}{3\left(37\alpha^2 - 49\alpha + 17\right)};\ 0 \le \alpha \le 1$ and

$F_{t\,(5\%)}(v_1=12,\ v_2=4) = 5.91$. And $F_{Row} < F_{t(5\%)}\ \forall\alpha,\ \alpha \in [0,1] \Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The 5 workers do not differ significantly with respect to the mean productivity.** And $F_{Col.} = \frac{49\alpha^2 - 57\alpha + 27}{4\left(\alpha^2 - 3\alpha + 3\right)};\ 0 \le \alpha \le 1$, $F_{t\,(5\%)}(v_1=12,\ v_2=3) = 8.74$

.Here $F_{Col.} < F_{t(5\%)}$ for $\forall\alpha,\ \alpha \in [0,\ 1] \Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted for all α at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The 4 machines do not differ significantly with respect to the mean productivity.**

15. Wang's centroid point and ranking method

Wang et al. [22] found that the centroid formulae proposed by Cheng are incorrect and have led to some misapplications such as by Chu and Tsao. They presented the correct method for centroid formulae for a generalized fuzzy number $\tilde{A}=(a, b, c, d; w)$ as

$$\left(\bar{x}_0, \bar{y}_0\right)=\left[(a+b+c+d)-\left(\frac{dc-ab}{(d+c)-(a+b)}\right),\left(\frac{w}{3}\right)\left(1+\left(\frac{c-b}{(d+c)-(a+b)}\right)\right)\right]$$

(15.1)

And the ranking function associated with $\tilde{A}$ is $R(\tilde{A})=\sqrt{\bar{x}_0^2+\bar{y}_0^2}$ ---(15.2)

For a normalized TFN, we put w = 1 in equations (15.1) so we have,

$$\left(\bar{x}_0, \bar{y}_0\right)=\left[(a+b+c+d)-\left(\frac{dc-ab}{(d+c)-(a+b)}\right),\left(\frac{1}{3}\right)\left(1+\left(\frac{c-b}{(d+c)-(a+b)}\right)\right)\right]$$

--- (15.3)

And the ranking function associated with $\tilde{A}$ is $R(\tilde{A})=\sqrt{\bar{x}_0^2+\bar{y}_0^2}$ ---(15.4).

Let $\tilde{A}_i$ and $\tilde{A}_j$ be two fuzzy numbers, (i) $R(\tilde{A}_i) > R(\tilde{A}_j)$ then $\tilde{A}_i > \tilde{A}_j$ (ii) $R(\tilde{A}_i) < R(\tilde{A}_j)$ then $\tilde{A}_i < \tilde{A}_j$ (iii) $R(\tilde{A}_i) = R(\tilde{A}_j)$ then $\tilde{A}_i = \tilde{A}_j$

Example15.1. Let we consider Example 1, using the above relations (15.3) and (15.4), we obtain the ranks of TFNs which are tabulated below:

Blocks	Varieties of crop		
	$R(\tilde{A})$	$R(\tilde{B})$	$R(\tilde{C})$
1	140.751	143.8	143.333
2	148.5	145.5	151.5
3	145.25	152.143	150
4	149.143	141.667	153

The ANOVA table values of tfns using Wang's Centroid Point:

Here, Q_1 =80.32, h-1=3; Q_2 =34.86, k-1=2; Q_3 =79.82, $(h-1)(k-1)=6$; $M_1 = 26.77$, $M_2 = 17.43$, $Q_3/(h-1)(k-1) = 13.30$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3. Now, $F_{Row} = 2.012$ and $F_{t\,(5\%)}(v_1=3, v_2=6) = 4.76$. And $F_{Row} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\tilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between rows is not significant. $\Rightarrow$ **The blocks do not differ significantly with respect to the yield.** And

$F_{Col.} = 1.31$, $F_{t\,(5\%)}(v_1=2,\ v_2=6) = 5.14$. And $F_{Col.} < F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is accepted at 5% level of significance. $\Rightarrow$ The difference between columns is not significant. $\Rightarrow$ **The varieties of crop do not differ significantly with respect to the yield.**

Example15.2. Let we consider Example 2, using the above relations (15.3) and (15.4), we obtain the ranks of TFNs which are tabulated below:

Workers	**Machine Type**			
	$R(\widetilde{A})$	$R(\widetilde{B})$	$R(\widetilde{C})$	$R(\widetilde{D})$
1	131.144	108.273	133.501	100.5
2	138	120.2	152.308	123.889
3	97.5	107.8	125.111	93.636
4	126	102	130.909	96.333
5	107.111	126.667	139.5	114

The ANOVA table values of tfns using Wang's Centroid Point:

Here, $Q_1 = 1666.941$, $h\text{-}1=4$; $Q_2 = 2562.435$, $k\text{-}1=3$; $Q_3 = 987.25$, $(h\text{-}1)(k\text{-}1) = 12$; $M_1 = 416.735$, $M_2 = 854.145$, $Q_3/(h\text{-}1)(k\text{-}1) = 82.271$ and variance ratio of F can be calculated as per the description of the ANOVA table noted in section-3.Now, $F_{Row} = 5.07$ and $F_{t\,(5\%)}(v_1=4,\ v_2=12) = 3.26$. And $F_{Row} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between rows is significant. $\Rightarrow$ **The 5 workers differ significantly with respect to mean productivity.** And $F_{Col.} = 10.38$, $F_{t\,(5\%)}(v_1=3,\ v_2=12) = 3.49$. And $F_{Col.} > F_{t(5\%)}$. $\Rightarrow$ The null hypothesis $\widetilde{H}_0$ is rejected at 5% level of significance. $\Rightarrow$ The difference between columns is significant. $\Rightarrow$ **The 4 machine types also differ significantly with respect to mean productivity.**

16. Conclusion

ANOVA MODEL	DECISIONS OBTAINED	
	Example-A	Example-B
Alpha-cut interval method	$\widetilde{H}_0$ is accepted at both upper and lower level models	$\widetilde{H}_0$ is rejected at both upper and lower level models
Membership function method	$\widetilde{H}_0$ is accepted	$\widetilde{H}_0$ is accepted
Rezvani's ranking function	$\widetilde{H}_0$ is accepted	$\widetilde{H}_0$ is rejected
Thorani's centroid point and ranking function	$\widetilde{H}_0$ is accepted	$\widetilde{H}_0$ is rejected
GMIR	$\widetilde{H}_0$ is accepted	$\widetilde{H}_0$ is rejected
Liou & Wang's centroid point method	$\widetilde{H}_0$ is accepted	$\widetilde{H}_0$ is accepted
Wang's centroid point	$\widetilde{H}_0$ is accepted	$\widetilde{H}_0$ is rejected

Remark 16.1. Observing the above decisions concluded from various methods, the α-cut interval approach fits better than method of membership function and Liou & Wang's approaches meanwhile Rezvani, Thorani, GMIR and Wang's approaches exhibit parallel decisions in the light of the conclusion based on corresponding non-fuzzy problems available in [19, pp. 10.16 & 10.19].

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