Intern. J. Fuzzy Mathematical Archive
Vol. 10, No. 1, 2016, 49-69
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 4 February 2016
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Fuzzy Dot Subalgebras and Fuzzy Dot Ideals of Distributive Implication Groupoids

***S.V. Tchoffo Foka*[1], *Marcel Tonga*[1], *T. Senapati*[2] and *M. Chandramouleeswaran*[3]**

[1]Department of Mathematics, University of Yaoundé 1,
P.O. Box 812, Yaoundé, Cameroon
E-mail: tchoffofoka88@yahoo.fr; tongamarcel@yahoo.fr

[2]Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore 721102 India
E-mail: math.tapan@gmail.com

[3]Saiva Bhanu Kshatriya College, Aruppukottai, 626101, India
E-mail: moulee59@gmail.com

Received 12 January 2016; accepted 25 January 2016

Abstract. In this paper, the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras, fuzzy dot ideals and fuzzy implicative dot ideals of distributive implication groupoids are introduced, and some of their properties are investigated. Fuzzy dot subalgebras generated by fuzzy points are described.

Keywords: Fuzzy subset, fuzzy point, distributive implication groupoid, fuzzy dot subalgebra, fuzzy normal dot subalgebra, fuzzy dot ideal, fuzzy implicative dot ideal.

AMS Mathematics Subject Classification (2010): 03E72, 46K15

1. Introduction

In 1965, Zadeh (see, [14]) introduced the notion of fuzzy sets. A few years later, many researchers fuzzified algebraic structures (see, [11]). In 2001, Hong (see, [10]) introduced the notion of fuzzy dot subalgebras of BCH-algebras as a generalization of the notion of fuzzy subalgebras of BCH-algebras. In 2008, Peng Jia-yin (see, [12]) introduced the notion of fuzzy dot ideals of BCH-algebras as a generalization of the notion of fuzzy ideals of BCH-algebras. In 50-ties Henkin and Skolem first introduced the notion of Hilbert algebra (see, [9]) as an algebraic counterpart of intuitionistic logic. In 2007, Chajda and Halas (see, [7]) introduced the notion of distributive implication groupoids which is a generalization of the implication reduct of intuitionistic logic. Between 2014 and 2015, Bandaru (see, [4,5,6]) introduced the notions of fuzzy ideals, fuzzy implicative ideals, fuzzy subalgebras and fuzzy normal subalgebras of distributive implication groupoids. In this paper, the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras, fuzzy dot ideals and fuzzy implicative dot ideals are introduced respectively as generalizations of the notions of fuzzy subalgebras, fuzzy normal subalgebras, fuzzy ideals and fuzzy implicative ideals, and some of their properties are investigated.

2. Preliminaries

In this section, we recall some definitions that are required in the sequel.

Definition 2.1. (see, [7]) An algebra $(A; *, 1)$ of type $(2, 0)$, denoted by $\mathcal{A}$, is called a distributive implication groupoid if it satisfies the following identities:

i. $x * x = 1$,
ii. $1 * x = x$,
iii. $x * (y * z) = (x * y) * (x * z)$ (left distributivity).

Example 2. The five elements groupoid given by the following Cayley table is a distributive implication groupoid.

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	a	1	1	d
c	1	a	1	1	d
d	1	1	c	c	1

In every distributive implication groupoid, one can introduce the so called induced relation $\leq$ by the setting $x \leq y$ if and only if $x * y = 1$. $\leq$ is a quasiorder and the relationship $x \leq 1$ and $x \leq y * x$ are satisfied.

Notation 2. For any $x_1, \cdots, x_n, a \in A$, we define $\prod_{i=1}^{n} x_i * a = x_n * (\cdots (x_1 * a) \cdots)$.

In what follows, let $\mathcal{A}$ denote a distributive implication groupoid unless otherwise specified.

Definition 2.2. A nonempty subset S of A is called a subalgebra of $\mathcal{A}$ if for any $x, y \in A$, $x \in S$ and $y \in S$ imply $x * y \in S$.

Definition 2.3. (see, [3]) A subset I of A is called an ideal of $\mathcal{A}$ if it satisfies the following conditions:

i. $1 \in I$,
ii. $x \in A$ and $y \in I$ imply $x * y \in I$,
iii. $x \in A$ and $y_1, y_2 \in I$ imply $(y_2 * (y_1 * x)) * x \in I$.

Definition 2.4. (see, [3]) A subset D of A is called a deductive system of $\mathcal{A}$ if it satisfies the following conditions:

i. $1 \in D$,
ii. $x \in D$ and $x * y \in D$ imply $y \in D$.

Theorem 2. (see, [3]) A subset D of A is an ideal of $\mathcal{A}$ if and only if it is a deductive system of $\mathcal{A}$.

Definition 2.5. (see, [4]) A subset I of A is called an implicative ideal of $\mathcal{A}$ if it satisfies the following conditions:

i. $1 \in I$,
ii. $z * ((x * y) * x) \in I$ and $z \in I$ imply $x \in I$; for all $x, y, z \in A$.

Definition 2.6. Let $\mathcal{A}$ and $\mathcal{B}$ be two distributive implication groupoids. A mapping f from A to B is said to be a homomorphism of distributive implication groupoids if $f(x * y) = f(x) * f(y)$ for all $x, y \in A$. Note that $f(1) = 1$.

Definition 2.7. A fuzzy subset of A is a function $\mu : A \to [0,1]$ from A to the real unit interval [0, 1].
We define on the set $F(A)$ of all fuzzy subsets of A the binary operations $\wedge$ and $\vee$ respectively by:
$(\mu \wedge \nu)(x) = min\{\mu(x), \nu(x)\}$ and $(\mu \vee \nu)(x) = max\{\mu(x), \nu(x)\}$ for all $x \in A$. We also define on $F(A)$ the partial order $\leq$ by:

$$\mu \leq \nu \text{ if and only if } \mu(x) \leq \nu(x) \text{ for all } x \in A.$$

Definition 2.8. A fuzzy relation on A is a fuzzy subset of $A \times A$.

3. Fuzzy dot subalgebras of distributive implication groupoids

In this section, fuzzy dot subalgebras of distributive implication groupoids are defined and some of their properties are investigated.

Definition 3.1. (see, [6]) Let μ be a fuzzy subset of A. μ is called a fuzzy subalgebra of $\mathcal{A}$ if it satisfies the following condition:

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\} \text{ for all } x, y \in A.$$

Definition 3.2. Let μ be a fuzzy subset of A. μ is called a fuzzy dot subalgebra of $\mathcal{A}$ if it satisfies the following condition:

$$\mu(x * y) \geq \mu(x) \cdot \mu(y) \text{ for all } x, y \in A.$$

Example 3.1. Consider the distributive implication groupoid of the example 2. and the fuzzy subset ξ defined by $\xi(1) = \xi(a) = \xi(c) = 0.3$ and $\xi(b) = \xi(d) = 0.5$. Then ξ is a fuzzy dot subalgebra.
Note that every fuzzy subalgebra is a fuzzy dot subalgebra, but the converse is not necessarily true. In fact, the fuzzy dot subalgebra ξ of the example 3.1 is not a fuzzy subalgebra, since

$$\xi(d * b) = \xi(c) = 0.3 \ngeq 0.5 = \min\{0.5,0.5\} = \min\{\xi(d), \xi(b)\}.$$

Proposition 3.1. Every fuzzy dot subalgebra μ of $\mathcal{A}$ satisfies the inequality

$$\mu(1) \geq \mu(x)^2 \text{ for all } x \in A.$$

Proof. For any fuzzy dot subalgebra μ of $\mathcal{A}$ and $x \in A$, we have

$$\mu(1) = \mu(x * x) \geq \mu(x) \cdot \mu(x) = \mu(x)^2.$$

Theorem 3.1. Let μ be a fuzzy dot subalgebra of $\mathcal{A}$. If there exists a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} \mu(x_n)^2 = 1$, then $\mu(1) = 1$.
Proof. Assume that there exists a sequence $\{x_n\}$ in A such that $\lim_{n\to\infty} \mu(x_n)^2 = 1$. By the proposition 3.1, $\mu(1) \geq \mu(x_n)^2$ for every positive integer n. Thus, $1 \geq \mu(1) \geq \lim_{n\to\infty} \mu(x_n)^2 = 1$. Hence, $\mu(1) = 1$.

Proposition 3.2. Let μ be a fuzzy subset of A and m be a positive integer. If μ is a fuzzy dot subalgebra of $\mathcal{A}$, then the fuzzy subset μ^m of A, defined by
$\mu^m(x) = \mu(x)^m$ for all $x \in A$, is a fuzzy dot subalgebra of $\mathcal{A}$.

Proof. Assume that μ is a fuzzy dot subalgebra of $\mathcal{A}$. Since $\mu(x * y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in A$, we have $\mu(x * y)^m \geq (\mu(x) \cdot \mu(y))^m$ for all $x, y \in A$; i.e., $\mu(x * y)^m \geq \mu(x)^m \cdot \mu(y)^m$ for all $x, y \in A$; i.e., $\mu^m(x * y) \geq \mu^m(x) \cdot \mu^m(y)$ for all $x, y \in A$. Hence, μ^m is a fuzzy dot subalgebra of $\mathcal{A}$.

Theorem 3.2. Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy dot subalgebras of $\mathcal{A}$. Then $inf_{i \in I}\mu_i$ is a fuzzy dot subalgebra of $\mathcal{A}$.

Proof. For any $x, y \in A$,

$$\begin{aligned}(inf_{i\in I}\mu_i)(x * y) &= inf_{i\in I}\, \mu_i(x * y) \\ &\geq inf_{i\in I}[\mu_i(x) \cdot \mu_i(y)] \\ &\geq inf_{i\in I}[\left(inf_{i\in I}\mu_i(x)\right) \cdot \left(inf_{i\in I}\mu_i(y)\right)] \\ &= \left(inf_{i\in I}\mu_i(x)\right) \cdot \left(inf_{i\in I}\mu_i(y)\right) \\ &= (inf_{i\in I}\mu_i)(x) \cdot (inf_{i\in I}\mu_i)(y).\end{aligned}$$

Hence, $inf_{i\in I}\mu_i$ is a fuzzy dot subalgebra of $\mathcal{A}$.

Definition 3.3. Let μ be a fuzzy subset of A. The smallest fuzzy dot subalgebra of $\mathcal{A}$ which contains μ is said to be the fuzzy dot subalgebra of $\mathcal{A}$ generated by μ, and will be denoted by $\langle \mu \rangle$.

Notation 3.1. We denote the set of all fuzzy dot subalgebras of $\mathcal{A}$ by $Fs(\mathcal{A})$.
For any $\mu, \nu \in Fs(\mathcal{A})$, we define the meet of μ and ν (denoted by $\mu \sqcap \nu$) by $\mu \sqcap \nu = \mu \wedge \nu$ and the join of μ and ν (denoted by $\mu \sqcup \nu$) by $\mu \sqcup \nu = \langle \mu \vee \nu \rangle$.

Notation 3.2. Let $\alpha, \beta \in [0, 1]$, $B \subseteq A$ and $\langle B \rangle$ be the subalgebra of $\mathcal{A}$ generated by B. B_β^α and $[B_\beta^\alpha]$ denote the fuzzy subsets of A respectively defined by:

$$B_\beta^\alpha(x) = \begin{cases} \alpha \text{ if } x \in B, \\ \beta \text{ otherwise.} \end{cases} \text{ and } [B_\beta^\alpha](x) = \begin{cases} \alpha \text{ if } x \in \langle B \rangle, \\ \beta \text{ otherwise.} \end{cases} \text{ for all } x \in A.$$

Lemma 3.1. Let B be a nonempty subset of A and $\alpha, \beta \in [0, 1]$ such that $\beta \leq \alpha$. Then $[B_\beta^\alpha]$ is a fuzzy dot subalgebra of $\mathcal{A}$.

Proof. Let $x, y \in A$.
If $x * y \notin \langle B \rangle$, then $x \notin \langle B \rangle$ or $y \notin \langle B \rangle$; thus, $[B_\beta^\alpha](x) = \beta$ or $[B_\beta^\alpha](y) = \beta$; thus, $[B_\beta^\alpha](x * y) = \beta \geq \max\{\beta^2, \alpha \cdot \beta\} \geq [B_\beta^\alpha](x) \cdot [B_\beta^\alpha](y)$.
If $x * y \in \langle B \rangle$, then $[B_\beta^\alpha](x * y) = \alpha \geq \max\{\alpha^2, \beta^2, \alpha \cdot \beta\} \geq [B_\beta^\alpha](x) \cdot [B_\beta^\alpha](y)$.
Hence, $[B_\beta^\alpha]$ is a fuzzy dot subalgebra of $\mathcal{A}$.
If B is a subalgebra of $\mathcal{A}$, then B_β^α is a fuzzy dot subalgebra of $\mathcal{A}$; but the converse is not necessarily true. In fact, consider the distributive implication groupoid of the example 2.; $\{1, b, d\}_{0.3}^{0.5}$ is a fuzzy dot subalgebra but $\{1, b, d\}$ is not a subalgebra, because $b * d = c \notin \{1, b, d\}$.

Theorem 3.3. Let B be a nonempty subset of A and $\beta \in [0, 1]$. Then $[B_\beta^1]$ is the fuzzy dot subalgebra of $\mathcal{A}$ generated by B_β^1; i.e., $\langle B_\beta^1 \rangle = [B_\beta^1]$.

Proof. Since $B_\beta^1(x) \leq 1 = [B_\beta^1]\,(x)$ for all $x \in \langle B \rangle$ and $B_\beta^1(x) = \beta = [B_\beta^1](x)$ for all $x \notin \langle B \rangle$, it follows that $B_\beta^1(x) \leq [B_\beta^1](x)$ for all $x \in A$; i.e., $[B_\beta^1]$ contains B_β^1. More, $[B_\beta^1]$ is a fuzzy dot subalgebra of $\mathcal{A}$ by the lemma 3.1, because $\beta \leq 1$. It suffices now to show that $[B_\beta^1]$ is the smallest fuzzy dot subalgebra of $\mathcal{A}$ containing B_β^1. So, let ν

be a fuzzy dot subalgebra of $\mathcal{A}$ containing B^1_β. For any $x \notin \langle B \rangle$, we have $[B^1_\beta](x) = \beta = B^1_\beta(x) \leq \nu(x)$. For any $x \in \langle B \rangle$, there is an n-ary term t in the language of distributive implication groupoids and $x_1, \dots, x_n \in B$ such that $x = t^{\mathcal{A}}(x_1, \dots, x_n)$; thus, there is a positive integer k such that $\nu(x) \geq (\nu(x_1) \cdots \nu(x_n))^k$; thus, $\nu(x) \geq \left(B^1_\beta(x_1) \cdots B^1_\beta(x_n)\right)^k = (1 \cdots 1)^k = 1^k = 1 = \left[B^1_\beta\right](x)$. Therefore, $\nu(x) \geq [B^1_\beta](x)$ for all $x \in A$; i.e., ν contains B^1_β. Hence, $[B^1_\beta]$ is the smallest fuzzy dot subalgebra of $\mathcal{A}$ containing B^1_β; i.e., $[B^1_\beta]$ is the fuzzy dot subalgebra of $\mathcal{A}$ generated by B^1_β.

Corollary 3.1. Let B be a nonempty subset of A and $\beta \in [0,1]$. Then B^1_β is a fuzzy dot subalgebra of $\mathcal{A}$ if and only B is a subalgebra of $\mathcal{A}$.
Proof. B^1_β is a fuzzy dot subalgebra of $\mathcal{A}$ if and only if $\langle B^1_\beta \rangle = B^1_\beta$; i.e., $[B^1_\beta] = B^1_\beta$; i.e., $\langle B \rangle = B$; i.e., B is a subalgebra of $\mathcal{A}$.

Notation 3.3. χ_B denotes the characteristic function of a subset B of A.
Corollary 3.2. Let B be a nonempty subset of A. Then χ_B is a fuzzy dot subalgebra of $\mathcal{A}$ if and only if B is a subalgebra of $\mathcal{A}$.
Proof. Straightforward, because $\chi_B = B^1_0$.

Theorem 3.4. $\left(Fs(\mathcal{A}); \sqcap, \sqcup; \underline{0}, \underline{1}\right)$ is a complete lattice; where $\underline{0}$ and $\underline{1}$ are the fuzzy subsets of A with values 0 and 1 respectively.
Proof. Since $0 \leq 0$ and $1 \leq 1$, $\underline{0} = B^0_0$ and $\underline{1} = B^1_1$ (for all subalgebra B of $\mathcal{A}$) are fuzzy dot subalgebras of $\mathcal{A}$. Therefore, $\left(Fs(\mathcal{A}); \sqcap, \sqcup; \underline{0}, \underline{1}\right)$ is a complete lattice by the theorem 3.2.

Definition 3.4. For any fuzzy subset μ of A and $\alpha \in [0,1]$, the subset $\{x \in A : \mu(x) \geq \alpha\}$ of A, denoted by $U(\mu; \alpha)$, is called a level subset of μ.

Proposition 3.3. (see, [6]) A fuzzy subset of A is a fuzzy subalgebra of $\mathcal{A}$ if and only if all its nonempty level subsets are subalgebras of $\mathcal{A}$.
There is a fuzzy dot subalgebra of a distributive implication groupoid with a nonempty level subset which is not a subalgebra. Consider the fuzzy dot subalgebra ξ of the example 3.1; $U(\xi; 0.5) = \{b, d\}$ is not a subalgebra, since $1 \notin U(\xi; 0.5)$.

Theorem 3.5. Let μ be a fuzzy subset of A. If μ is a fuzzy dot subalgebra of $\mathcal{A}$, then $U(\mu; 1)$ is either empty or a subalgebra of $\mathcal{A}$.
Proof. Assume that μ is a fuzzy dot subalgebra of $\mathcal{A}$ and $U(\mu; 1)$ is a nonempty subset of A. If x and y belong to $U(\mu; 1)$, then $\mu(x * y) \geq \mu(x) \cdot \mu(y) = 1 \cdot 1 = 1$; thus, $\mu(x * y) = 1$; i.e., $x * y \in U(\mu; 1)$.
Hence, $U(\mu; 1)$ is a subalgebra of $\mathcal{A}$.

Definition 3.5. Let $f : A \to B$ be a function from a set A to a set B and ν be a fuzzy subset of B. The preimage under f of ν, denoted by $f^{-1}[\nu]$, is the fuzzy subset of A defined by: $f^{-1}[\nu](x) = \nu(f(x))$ for all $x \in A$.

Theorem 3.6. Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of distributive implication groupoids and ν be a fuzzy subset of B. If ν is a fuzzy dot subalgebra of $\mathcal{B}$, then $f^{-1}[\nu]$ is a fuzzy dot subalgebra of $\mathcal{A}$.
Proof. Assume that ν is a fuzzy dot subalgebra of $\mathcal{B}$. For any $x, y \in A$,

$$\begin{aligned} f^{-1}[\nu](x * y) &= \nu(f(x * y)) = \nu(f(x) * f(y)) \\ &\geq \nu(f(x)) \cdot \nu(f(y)) \\ &= f^{-1}[\nu](x) \cdot f^{-1}[\nu](y). \end{aligned}$$

Hence, $f^{-1}[\nu]$ is a fuzzy dot subalgebra of $\mathcal{A}$.

Theorem 3.7. Let $f : \mathcal{A} \to \mathcal{B}$ be an onto homomorphism of distributive implication groupoids and ν be a fuzzy subset of B. If $f^{-1}[\nu]$ is a fuzzy dot subalgebra of $\mathcal{A}$, then ν is a fuzzy dot subalgebra of $\mathcal{B}$.
Proof. For any $y, z \in B$ and $(a, b) \in f^{-1}(y) \times f^{-1}(z)$,

$$\begin{aligned} \nu(y * z) &= \nu(f(a) * f(b)) = \nu(f(a * b)) = f^{-1}[\nu](a * b) \\ &\geq f^{-1}[\nu](a) \cdot f^{-1}[\nu](b) \\ &= \nu(f(a)) \cdot \nu(f(b)) \\ &= \nu(y) \cdot \nu(z). \end{aligned}$$

Hence, ν is a fuzzy dot subalgebra of $\mathcal{B}$.

Definition 3.6. Let $f : A \to B$ be a function from a set A to a set B and μ be a fuzzy subset of A. The image under f of μ, denoted by $f[\mu]$, is the fuzzy subset of B defined by: $f[\mu](y) = \begin{cases} sup_{a \epsilon f^{-1}(y)} \mu(a) \ if \ f^{-1}(y) \neq \emptyset, \\ 0 \ otherwise. \end{cases}$ for all $y \in B$.

Theorem 3.8. Let $f : \mathcal{A} \to \mathcal{B}$ be an onto homomorphism of distributive implication groupoids and μ be a fuzzy subset of A. If μ is a fuzzy dot subalgebra of $\mathcal{A}$, then $f[\mu]$ is a fuzzy dot subalgebra of $\mathcal{B}$.
Proof. Assume that μ is a fuzzy dot subalgebra of $\mathcal{A}$.
Let $y_1, y_2 \in B$, $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1 * y_2)$. Consider the set $A_1 * A_2 = \{x \in A : x = a_1 * a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2\}$.
If $x \in A_1 * A_2$, then $x = a_1 * a_2$ for some $a_1 \in A_1$ and $a_2 \in A_2$; so that,
$f(x) = f(a_1 * a_2) = f(a_1) * f(a_2) = y_1 * y_2$; that is, $x \in f^{-1}(y_1 * y_2) = A_{12}$
Hence, $A_1 * A_2 \subseteq A_{12}$. It follows that

$$\begin{aligned} f[\mu](y_1 * y_2) &= sup_{a \epsilon f^{-1}(y_1 * y_2)} \mu(a) = sup_{a \epsilon A_{12}} \mu(a) \\ &\geq sup_{a \epsilon A_1 * A_2} \mu(a) = sup_{a_1 \epsilon A_1, a_2 \epsilon A_2} \mu(a_1 * a_2) \\ &\geq sup_{a_1 \epsilon A_1, a_2 \epsilon A_2} \mu(a_1) \cdot \mu(a_2). \end{aligned}$$

Since $\cdot : [0,1] \times [0,1] \to [0,1]$ is continuous, for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\alpha_1 \geq sup_{a_1 \epsilon A_1} \mu(a_1) - \delta$ and $\alpha_2 \geq sup_{a_2 \epsilon A_2} \mu(a_2) - \delta$, then

$$\alpha_1 \cdot \alpha_2 \geq sup_{a_1 \epsilon A_1} \mu(a_1) \cdot sup_{a_2 \epsilon A_2} \mu(a_2) - \varepsilon.$$

Choose $x_1 \in A_1$ and $x_2 \in A_2$ such that

$$\mu(x_1) \geq sup_{a_1 \epsilon A_1} \mu(a_1) - \delta \text{ and } \mu(x_2) \geq sup_{a_2 \epsilon A_2} \mu(a_2) - \delta, \text{ then}$$

$$\mu(x_1) \cdot \mu(x_2) \geq sup_{a_1 \epsilon A_1} \mu(a_1) \cdot sup_{a_2 \epsilon A_2} \mu(a_2) - \varepsilon.$$

Consequently, $f[\mu](y_1 * y_2) \geq sup_{a_1 \epsilon A_1, a_2 \epsilon A_2} \mu(a_1) \cdot \mu(a_2)$
$\geq sup_{a_1 \epsilon A_1} \mu(a_1) \cdot sup_{a_2 \epsilon A_2} \mu(a_2)$
$= f[\mu](y_1) \cdot f[\mu](y_2)$.

Hence, $f[\mu]$ is a fuzzy dot subalgebra of $\mathcal{B}$.

Theorem 3.9. Let μ and ν be two fuzzy dot subalgebras of $\mathcal{A}$. The fuzzy subset $\mu \times \nu$ of $A \times A$, defined by $(\mu \times \nu)(x, y) = \mu(x) \cdot \nu(y)$ for all $(x, y) \in A \times A$, is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$.

Proof. For any $(x_1, y_1), (x_2, y_2) \in A \times A$,

$$\begin{aligned}(\mu \times \nu)\big((x_1, y_1) * (x_2, y_2)\big) &= (\mu \times \nu)(x_1 * x_2, y_1 * y_2)\\ &= \mu(x_1 * x_2) \cdot \nu(y_1 * y_2)\\ &\geq \big(\mu(x_1) \cdot \mu(x_2)\big) \cdot \big(\nu(y_1) \cdot \nu(y_2)\big)\\ &= \big(\mu(x_1) \cdot \nu(y_1)\big) \cdot \big(\mu(x_2) \cdot \nu(y_2)\big)\\ &= (\mu \times \nu)(x_1, y_1) \cdot (\mu \times \nu)(x_2, y_2).\end{aligned}$$

Hence, $\mu \times \nu$ is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$.

Theorem 3.10. Let σ be a fuzzy subset of A. The strong σ-relation μ_σ on A, defined by $\mu_\sigma(x, y) = \sigma(x) \cdot \sigma(y)$ for all $(x, y) \in A \times A$, is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$ if and only if σ is a fuzzy dot subalgebra of $\mathcal{A}$.

Proof. (⇒) Assume that μ_σ is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$.

For any $x, y \in A$,
$$\begin{aligned}\sigma(x * y) &= \sqrt{\sigma(x * y) \cdot \sigma(x * y)}\\ &= \sqrt{\mu_\sigma(x * y, x * y)}\\ &= \sqrt{\mu_\sigma\big((x, x) * (y, y)\big)}\\ &\geq \sqrt{\mu_\sigma(x, x) \cdot \mu_\sigma(y, y)}\\ &= \sqrt{(\sigma(x) \cdot \sigma(x)) \cdot (\sigma(y) \cdot \sigma(y))}\\ &= \sqrt{(\sigma(x) \cdot \sigma(y)) \cdot (\sigma(x) \cdot \sigma(y))}\\ &= \sigma(x) \cdot \sigma(y).\end{aligned}$$

Hence, σ is a fuzzy dot subalgebra of $\mathcal{A}$.

(⇐) Assume that σ is a fuzzy dot subalgebra of $\mathcal{A}$.

For any $(x_1, y_1), (x_2, y_2) \in A \times A$,
$$\begin{aligned}\mu_\sigma\big((x_1, y_1) * (x_2, y_2)\big) &= \mu_\sigma(x_1 * x_2, y_1 * y_2) = \sigma(x_1 * x_2) \cdot \sigma(y_1 * y_2)\\ &\geq \big(\sigma(x_1) \cdot \sigma(x_2)\big) \cdot \big(\sigma(y_1) \cdot \sigma(y_2)\big)\\ &= \big(\sigma(x_1) \cdot \sigma(y_1)\big) \cdot \big(\sigma(x_2) \cdot \sigma(y_2)\big)\\ &= \mu_\sigma(x_1, y_1) \cdot \mu_\sigma(x_2, y_2).\end{aligned}$$

Hence, μ_σ is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$.

Definition 3.7. Let σ be a fuzzy subset of A. A fuzzy relation μ on A is called a σ-product relation on A if $\mu(x, y) \geq \sigma(x) \cdot \sigma(y)$ for all $x, y \in A$.

Definition 3.8. Let σ be a fuzzy subset of A. A fuzzy relation μ on A is called a left fuzzy relation on σ if $\mu(x, y) = \sigma(x)$ for all $x, y \in A$. Similarly, we can define a right fuzzy relation on σ.

Note that a left (resp. right) fuzzy relation on σ is a fuzzy σ-product relation on A.

Theorem 3.11. Let μ be a left fuzzy relation on a fuzzy subset σ of A. If μ is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$, then σ is a fuzzy dot subalgebra of $\mathcal{A}$.
Proof. Assume that μ is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$.
For any $x_1, x_2, y_1, y_2 \in A$, $\sigma(x_1 * x_2) = \mu(x_1 * x_2, y_1 * y_2) = \mu\big((x_1, y_1) * (x_2, y_2)\big)$
$$\geq \mu(x_1, y_1) \cdot \mu(x_2, y_2) = \sigma(x_1) \cdot \sigma(x_2).$$
Hence, σ is a fuzzy dot subalgebra of $\mathcal{A}$.

Theorem 3.12. Let $z \in A$ and μ be a fuzzy relation on A satisfying the inequality $\mu(x, y) \leq \mu(x, 1)$ for all $x, y \in A$. If μ is a fuzzy dot subalgebra of $\mathcal{A}$, then the fuzzy subset σ_z of A, defined by $\sigma_z(x) = \mu(x, z)$ for all $x \in A$, is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$.
Proof. Assume that μ is a fuzzy dot subalgebra of $\mathcal{A}$.
For any $x, y \in A$, $\sigma_z(x * y) = \mu(x * y, z) = \mu(x * y, 1 * z) = \mu\big((x, 1) * (y, z)\big)$
$$\geq \mu(x, 1) \cdot \mu(y, z) \geq \mu(x, z) \cdot \mu(y, z) = \sigma_z(x) \cdot \sigma_z(y).$$
Hence, σ_z is a fuzzy dot subalgebra of $\mathcal{A}$.

Theorem 3.13. Let μ be a fuzzy subset of $A \times A$ satisfying $\mu(x, 1) = \mu(1, x) = 1$ for all $x \in A$. If μ is a fuzzy dot subalgebra of $\mathcal{A} \times \mathcal{A}$ then, the fuzzy subset σ_μ of A, defined by $\sigma_\mu(x) = inf_{z \epsilon A} \mu(x, z) \cdot \mu(z, x)$ for all $x \in A$, is a fuzzy dot subalgebra of $\mathcal{A}$.
Proof. For any $x, y, z \in A$, $\mu(x * y, z) = \mu(x * y, 1 * z) = \mu\big((x, 1) * (y, z)\big)$
$$\geq \mu(x, 1) \cdot \mu(y, z) = 1 \cdot \mu(y, z) = \mu(y, z)$$
and
$$\mu(z, x * y) = \mu(1 * z, x * y) = \mu\big((1, x) * (z, y)\big)$$
$$\geq \mu(1, x) \cdot \mu(z, y) = 1 \cdot \mu(z, y) = \mu(z, y);$$
It follows that $\mu(x * y, z) \cdot \mu(z, x * y) \geq \mu(y, z) \cdot \mu(z, y)$
$$\geq \big(\mu(x, z) \cdot \mu(z, x)\big) \cdot \big(\mu(y, z) \cdot \mu(z, y)\big).$$
So that, for any $x, y \in A$,
$$\begin{aligned}
\sigma_\mu(x * y) &= inf_{z \epsilon A} \mu(x * y, z) \cdot \mu(z, x * y) \\
&\geq inf_{z \epsilon A}\big(\mu(x, z) \cdot \mu(z, x)\big) \cdot \big(\mu(y, z) \cdot \mu(z, y)\big) \\
&\geq inf_{z \epsilon A}\big(inf_{z \epsilon A} \mu(x, z) \cdot \mu(z, x)\big) \cdot \big(inf_{z \epsilon A} \mu(y, z) \cdot \mu(z, y)\big) \\
&= \big(inf_{z \epsilon A} \mu(x, z) \cdot \mu(z, x)\big) \cdot \big(inf_{z \epsilon A} \mu(y, z) \cdot \mu(z, y)\big) \\
&= \sigma_\mu(x) \cdot \sigma_\mu(y).
\end{aligned}$$
Hence, σ_μ is a fuzzy dot subalgebra of $\mathcal{A}$.

Definition 3.9. (see, [8]) A fuzzy map f from a set A to a set B is an ordinary map $f : A \to F(B)$ from A to the set $F(B)$ of all fuzzy subsets of B satisfying the following conditions:

i. for any $x \in A$, there exists $y_x \in B$ such that $f(x)(y_x) = 1$;
ii. for any $x \in A$ and $y_1, y_2 \in B$, $f(x)(y_1) = f(x)(y_2)$ implies $y_1 = y_2$.

One observes that a fuzzy map f from A to B gives rise to a unique ordinary map μ_f from $A \times B$ to $[0, 1]$, given by $\mu_f(x, y) = f(x)(y)$ for all $x \in A$ and $y \in B$. One also notes that a fuzzy map f from A to B gives rise to a unique ordinary map f_1 from A to B, given by $f_1(x) = y_x$.

Theorem 3.14. Let $f : \mathcal{A} \to \mathcal{B}$ be a fuzzy homomorphism of distributive implication groupoids; i.e., $\mu_f(x_1 * x_2, y) = sup_{y=y_1*y_2}\mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2)$ for all $x_1, x_2 \in A$ and $y \in B$. Then

a. $\mu_f(x_1 * x_2, y_1 * y_2) \geq \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2)$ for all $x_1, x_2 \in A$, $y_1, y_2 \in B$;

b. $\mu_f(1,1) = 1$.

Proof. a. For any $x_1, x_2 \in A$ and $y_1, y_2 \in B$,

$$\mu_f(x_1 * x_2, y_1 * y_2) = sup_{y_1*y_2=z_1*z_2}\mu_f(x_1, z_1) \cdot \mu_f(x_2, z_2)$$
$$\geq \mu_f(x_1, y_1) \cdot \mu_f(x_2, y_2).$$

b. For any $x \in A$, there exists $y_x \in B$ such that $\mu_f(x, y_x) = 1$; thus,
$\mu_f(1,1) = \mu_f(x * x, y_x * y_x) \geq \mu_f(x, y_x) \cdot \mu_f(x, y_x) = 1 \cdot 1 = 1$;
so, $\mu_f(1,1) = 1$.

Definition 3.10. For any fuzzy subsets μ and ν of A, $\mu * \nu$ is the fuzzy subset of A defined by: $(\mu * \nu)(x) = sup_{x=a*b}\mu(a) \cdot \nu(b)$ for all $x \in A$.

Theorem 3.15. Let μ be a fuzzy subset of A. Then the following are equivalent:

a. μ is a fuzzy dot subalgebra of $\mathcal{A}$.

b. $\mu * \mu \leq \mu$.

Proof. $(\boldsymbol{a.} \Rightarrow \boldsymbol{b.})$ Assume that μ is a fuzzy dot subalgebra of $\mathcal{A}$. For any $x \in A$, we have $\mu(x) = \mu(a * b) \geq \mu(a) \cdot \mu(b)$ for all $a, b \in A$ such that $x = a * b$; thus, $\mu(x) \geq sup_{x=a*b}\mu(a) \cdot \mu(b)$; i.e., $\mu(x) \geq (\mu * \mu)(x)$. Therefore, $\mu * \mu \leq \mu$.
$(\boldsymbol{b.} \Rightarrow \boldsymbol{a.})$ Assume that $\mu * \mu \leq \mu$. Then $(\mu * \mu)(a * b) \leq \mu(a * b)$ for all $a, b \in A$; i.e., $sup_{a*b=u*v}\mu(u) \cdot \mu(v) \leq \mu(a * b)$ for all $a, b \in A$; thus, $\mu(a) \cdot \mu(b) \leq \mu(a * b)$ for all $a, b \in A$; i.e., μ is a fuzzy dot subalgebra of $\mathcal{A}$.

Definition 3.11. For any $x \in A$ and $\alpha \in [0, 1]$, the fuzzy subset x_α of A, defined by $x_\alpha(x) = \alpha$ and $x_\alpha(y) = 0$ for all $y \neq x$, is called a fuzzy point of A.

Theorem 3.16. Let x_α and y_β be two fuzzy points of A. Then $x_\alpha * y_\beta = (x * y)_{\alpha\cdot\beta}$.
Proof. For any $z = a * b \neq x * y$, we have $a \neq x$ or $b \neq y$; i.e., $x_\alpha(a) = 0$ or $y_\beta(b) = 0$; thus, $x_\alpha(a) \cdot y_\beta(b) \leq \min\{x_\alpha(a), y_\beta(b)\} = 0$; so, $x_\alpha(a) \cdot y_\beta(b) = 0$. Thus, $(x_\alpha * y_\beta)(z) = sup_{z=a*b}x_\alpha(a) \cdot y_\beta(b) = sup_{z=a*b}0 = 0$ for all $z \neq x * y$. More, $(x_\alpha * y_\beta)(x * y) = sup_{x*y=a*b}x_\alpha(a) \cdot y_\beta(b) = x_\alpha(x) \cdot y_\beta(y) = \alpha \cdot \beta$.
Hence, $(x_\alpha * y_\beta)(z) = (x * y)_{\alpha\cdot\beta}(z)$ for all $z \in A$; i.e., $x_\alpha * y_\beta = (x * y)_{\alpha\cdot\beta}$.

Corollary 3.3. Let x_α, y_β, z_γ and 1_δ be four fuzzy points of A. Then

a. $x_\alpha * x_\alpha = 1_{\alpha^2}$,

b. $1_\delta * x_\alpha = x_{\delta\cdot\alpha}$,

c. $(x_\alpha * y_\beta) * (x_\alpha * z_\gamma) \leq x_\alpha * (y_\beta * z_\gamma)$.

Proof. a. $x_\alpha * x_\alpha = (x * x)_{\alpha\cdot\alpha} = 1_{\alpha^2}$.
b. $1_\delta * x_\alpha = (1 * x)_{\delta\cdot\alpha} = x_{\delta\cdot\alpha}$.
c. $x_\alpha * (y_\beta * z_\gamma) = x_\alpha * (y * z)_{\beta\cdot\gamma}$

$$= \left(x * (y * z)\right)_{\alpha\cdot(\beta\cdot\gamma)}$$
$$= ((x * y) * (x * z))_{(\alpha\cdot\beta)\cdot\gamma}$$

$\geq ((x * y) * (x * z))_{(\alpha\cdot\beta)\cdot(\alpha\cdot\gamma)}$
$= (x * y)_{\alpha\cdot\beta} * (x * z)_{\alpha\cdot\gamma}$
$= (x_\alpha * y_\beta) * (x_\alpha * z_\gamma)$.

Hence, $(x_\alpha * y_\beta) * (x_\alpha * z_\gamma) \leq x_\alpha * (y_\beta * z_\gamma)$.

We can also establish the following equalities:

a) $x_\alpha * 1_\delta = 1_{\alpha\cdot\delta}$,
b) $x_\alpha * (y_\beta * x_\alpha) = 1_{\beta\cdot\alpha^2}$,
c) $(y_\beta * z_\gamma) * \left((x_\alpha * y_\beta) * (x_\alpha * z_\gamma)\right) = 1_{(\alpha\cdot\beta\cdot\gamma)^2}$,
d) $\left(x_\alpha * (y_\beta * z_\gamma)\right) * \left(y_\beta * (x_\alpha * z_\gamma)\right) = 1_{(\alpha\cdot\beta\cdot\gamma)^2}$,
e) $(x_\alpha * y_\beta) * \left((y_\beta * z_\gamma) * (x_\alpha * z_\gamma)\right) = 1_{(\alpha\cdot\beta\cdot\gamma)^2}$.

Definition 3.12. A fuzzy point x_α of A is said to be contained in a fuzzy subset μ of A, denoted by $x_\alpha \in \mu$, if $\mu(x) \geq \alpha$.

Theorem 3.17. The following are equivalent:

a. μ is a fuzzy dot subalgebra of $\mathcal{A}$.
b. For any $x_\alpha \in \mu$ and $y_\beta \in \mu$, $x_\alpha * y_\beta \in \mu$.

Proof. (***a.*** $\Rightarrow$ ***b.***) Assume that μ is a fuzzy dot subalgebra of $\mathcal{A}$. For any $x_\alpha \in \mu$ and $y_\beta \in \mu$, we have $\mu(x) \geq \alpha$ and $\mu(y) \geq \beta$; thus, $\mu(x)\cdot\mu(y) \geq \alpha\cdot\beta$; thus, $\mu(x * y) \geq \alpha\cdot\beta$; i.e., $(x * y)_{\alpha\cdot\beta} \in \mu$; i.e., $x_\alpha * y_\beta \in \mu$.
(***b.*** $\Rightarrow$ ***a.***) Assume that $x_\alpha * y_\beta \in \mu$ for all $x_\alpha \in \mu$ and $y_\beta \in \mu$. For any $x, y \in A$, we have $x_{\mu(x)} * y_{\mu(y)} \in \mu$, since $x_{\mu(x)}, y_{\mu(y)} \in \mu$; thus, $(x * y)_{\mu(x)\cdot\mu(y)} \in \mu$; i.e., $\mu(x * y) \geq \mu(x)\cdot\mu(y)$.
Hence, μ is a fuzzy dot subalgebra of $\mathcal{A}$.

Notation 3.4. For any $x \in A \setminus \{1\}$ and $\alpha \in [0,1]$, $\widehat{x_\alpha}$ denotes the fuzzy subset of A defined by:

$$\widehat{x_\alpha}(t) = \begin{cases} \alpha^2 \; if \; t = 1, \\ \alpha \; if \; t = x, \\ 0 \; otherwise. \end{cases} \quad \text{for all } t \in A.$$

Theorem 3.18. Let $x \in A \setminus \{1\}$ and $\alpha \in [0,1]$. Then $\widehat{x_\alpha}$ is the fuzzy dot subalgebra of $\mathcal{A}$ generated by x_α; i.e., $\langle x_\alpha \rangle = \widehat{x_\alpha}$.

Proof. Since $x_\alpha(x) = \alpha = \widehat{x_\alpha}(x)$ and $x_\alpha(t) = 0 \leq \widehat{x_\alpha}(t)$ for all $t \neq x$, we have $x_\alpha(t) \leq \widehat{x_\alpha}(t)$ for all $t \in A$; i.e., $\widehat{x_\alpha}$ contains x_α. Next we show that $\widehat{x_\alpha}$ is a fuzzy dot subalgebra of $\mathcal{A}$.
For any $a, b \in A$ such that $a \notin \{1, x\}$ or $b \notin \{1, x\}$, we have $\widehat{x_\alpha}(a) = 0$ or $\widehat{x_\alpha}(b) = 0$; thus, $\widehat{x_\alpha}(a)\cdot\widehat{x_\alpha}(b) \leq \min\{\widehat{x_\alpha}(a), \widehat{x_\alpha}(b)\} = 0 \leq \widehat{x_\alpha}(a * b)$.
$\widehat{x_\alpha}(1 * 1) = \widehat{x_\alpha}(1) = \alpha^2 \geq \alpha^2\cdot\alpha^2 = \widehat{x_\alpha}(1)\cdot\widehat{x_\alpha}(1)$.
$\widehat{x_\alpha}(1 * x) = \widehat{x_\alpha}(x) = \alpha \geq \alpha^2\cdot\alpha = \widehat{x_\alpha}(1)\cdot\widehat{x_\alpha}(x)$.
$\widehat{x_\alpha}(x * x) = \widehat{x_\alpha}(1) = \alpha^2 = \alpha\cdot\alpha = \widehat{x_\alpha}(x)\cdot\widehat{x_\alpha}(x)$.
$\widehat{x_\alpha}(x * 1) = \widehat{x_\alpha}(1) = \alpha^2 \geq \alpha\cdot\alpha^2 = \widehat{x_\alpha}(x)\cdot\widehat{x_\alpha}(1)$.

Therefore, $\widehat{x_\alpha}(a * b) \geq \widehat{x_\alpha}(a) \cdot \widehat{x_\alpha}(b)$ for all $a, b \in A$; i.e., $\widehat{x_\alpha}$ is a fuzzy dot subalgebra of $\mathcal{A}$. Finally, we show that $\widehat{x_\alpha}$ is the smallest fuzzy dot subalgebra of $\mathcal{A}$ containing x_α. So, let ν be a fuzzy dot subalgebra of $\mathcal{A}$ containing x_α. Since

$\nu(1) \geq \nu(x)^2 \geq \left(x_\alpha(x)\right)^2 = \alpha^2 = \widehat{x_\alpha}(1)$, $\nu(x) \geq x_\alpha(x) = \alpha = \widehat{x_\alpha}(x)$ and $\nu(t) \geq 0 = \widehat{x_\alpha}(t)$ for all $t \in A \setminus \{1, x\}$, we have $\nu(t) \geq \widehat{x_\alpha}(t)$ for all $t \in A$; i.e., ν contains $\widehat{x_\alpha}$.

Hence, $\widehat{x_\alpha}$ is the smallest fuzzy dot subalgebra of $\mathcal{A}$ containing x_α.

We remark that $\langle 1_\alpha \rangle = 1_\alpha$ for all $\alpha \in [0, 1]$.

4. Fuzzy normal dot subalgebras of distributive implication groupoids

In this section, fuzzy normal dot subalgebras are defined and the relationship between fuzzy normal dot subalgebras, fuzzy normal subalgebras and fuzzy dot subalgebras are discussed.

Definition 4.1. (see, [6]) Let μ be a fuzzy subset of A. μ is called a fuzzy normal subalgebra of $\mathcal{A}$ if it satisfies the following condition:

$$\mu\big((x * a) * (y * b)\big) \geq \min\{\mu(x * y), \mu(a * b)\} \text{ for all } x, y, a, b \in A.$$

Definition 4.2. Let μ be a fuzzy subset of A. μ is called a fuzzy normal dot subalgebra of $\mathcal{A}$ if it satisfies the following condition:

$$\mu\big((x * a) * (y * b)\big) \geq \mu(x * y) \cdot \mu(a * b) \text{ for all } x, y, a, b \in A.$$

Example 4.1. The fuzzy subset ξ of the example 3.1 is a fuzzy normal dot subalgebra.

Note that every fuzzy normal subalgebra is a fuzzy normal dot subalgebra, but the converse is not necessarily true. In fact, the fuzzy normal dot subalgebra ξ of the example 4.1 is not a fuzzy normal subalgebra, since

$$\xi\big((b * c) * (d * d)\big) = \xi(1) = 0.3 \ngeq 0.5 = \min\{\xi(b * d), \xi(c * d)\}.$$

Theorem 4.1. Every fuzzy normal dot subalgebra of $\mathcal{A}$ is a fuzzy dot subalgebra of $\mathcal{A}$.

Proof. Let μ be a fuzzy normal dot subalgebra of $\mathcal{A}$. For any $x, y \in A$,

$$\mu(x * y) = \mu\big((1 * 1) * (x * y)\big) \geq \mu(1 * x) \cdot \mu(1 * y) = \mu(x) \cdot \mu(y).$$

Hence, μ is a fuzzy dot subalgebra of $\mathcal{A}$.

The converse of the theorem 4.1 is not necessarily true. Consider the distributive implication groupoid of the example 2. and the fuzzy subset ς defined by

$\varsigma(1) = \varsigma(a) = 0.7$, $\varsigma(b) = \varsigma(d) = 0.6$ and $\varsigma(c) = 0.4$. Then ς is a fuzzy dot subalgebra; but ς is not a fuzzy normal dot subalgebra, since

$$\varsigma\big((a * a) * (d * b)\big) = \varsigma(c) = 0.4 \ngeq 0.42 = \varsigma(1) \cdot \varsigma(b) = \varsigma(a * d) \cdot \varsigma(a * b).$$

Theorem 4.2. Let μ be a fuzzy normal dot subalgebra of $\mathcal{A}$. Then

$$\mu(x) \geq \mu(1)^2 \text{ for all } x \in A.$$

Proof. For any $x \in A$,

$$\begin{aligned} \mu(x) = \mu(1 * x) &= \mu\big((x * x) * (1 * x)\big) \\ &\geq \mu(x * 1) \cdot \mu(x * x) \\ &= \mu(1) \cdot \mu(1) \\ &= \mu(1)^2. \end{aligned}$$

Corollary 4. Let μ be a fuzzy normal dot subalgebra of $\mathcal{A}$. If $\mu(1) = 1$, then μ is the constant fuzzy subset of A with value 1.

Proof. Assume that $\mu(1) = 1$. We have $\mu(x) \geq \mu(1)^2 = 1^2 = 1$ for all $x \in A$; thus, $\mu(x) = 1$ for all $x \in A$; i.e., μ is the constant fuzzy subset of A with value 1.

5. Fuzzy dot ideals of distributive implication groupoids

In this section, fuzzy dot ideals are defined and some of their properties are investigated.

Definition 5.1. (see, [5]) A fuzzy subset μ of A is called a fuzzy ideal of $\mathcal{A}$ if it satisfies the following conditions:

i. $\mu(1) \geq \mu(x)$,
ii. $\mu(x) \geq \min\{\mu(y), \mu(y * x)\}$; for all $x, y \in A$.

Definition 5.2. A fuzzy subset μ of A is called a fuzzy dot ideal of $\mathcal{A}$ if it satisfies the following conditions:

i. $\mu(1) \geq \mu(x)$,
ii. $\mu(x) \geq \mu(y) \cdot \mu(y * x)$; for all $x, y \in A$.

Example 5.1. Let $A = \{1, a, b, c, d\}$ be a set with the following Cayley table:

∗	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	1	1	1	d
d	1	a	b	c	1

Then $\mathcal{A} = (A; *, 1)$ is a distributive implication groupoid. Define a fuzzy subset ζ of A by $\zeta(1) = \zeta(b) = 0.6$, $\zeta(a) = 0.5$ and $\zeta(c) = \zeta(d) = 0.4$. Then ζ is a fuzzy dot ideal of $\mathcal{A}$.

Note that every fuzzy ideal of $\mathcal{A}$ is a fuzzy dot ideal of $\mathcal{A}$, but the converse is not necessarily true. In fact, the fuzzy dot ideal ζ of the example 5.1 is not a fuzzy ideal, since $\zeta(c) = 0.4 \ngeq 0.5 = \min\{0.5, 0.6\} = \min\{\zeta(a), \zeta(1)\} = \min\{\zeta(a), \zeta(a * c)\}$.

Proposition 5.1. Let μ be a fuzzy subset of A. If μ is a fuzzy dot ideal of $\mathcal{A}$, then μ is a fuzzy dot subalgebra of $\mathcal{A}$.

Proof. Assume that μ is a fuzzy dot ideal of $\mathcal{A}$.

For any $x, y \in A$,
$$\begin{aligned}\mu(x * y) &\geq \mu(y) \cdot \mu\big(y * (x * y)\big)\\ &= \mu(y) \cdot \mu(1)\\ &\geq \mu(y) \cdot \mu(x)\\ &= \mu(x) \cdot \mu(y).\end{aligned}$$

Hence, μ is a fuzzy dot subalgebra of $\mathcal{A}$.

Proposition 5.2. Let μ be a fuzzy dot ideal of $\mathcal{A}$. If the inequality $x \leq y$ holds in $\mathcal{A}$, then $\mu(y) \geq \mu(x) \cdot \mu(1)$ for all $x, y \in A$.

Proof. Let $x, y \in A$ such that $x \leq y$. Then $x * y = 1$, and thus
$\mu(y) \geq \mu(x) \cdot \mu(x * y) = \mu(x) \cdot \mu(1)$.

Corollary 5.1. Every fuzzy dot ideal μ of $\mathcal{A}$ such that $\mu(1) = 1$ is order-preversing.

Proof. Straightforward.

Proposition 5.3. Let μ be a fuzzy dot ideal of $\mathcal{A}$. Then

$$z \leq x * y \text{ implies } \mu(y) \geq \mu(x) \cdot \mu(z) \cdot \mu(1) \text{ for all } x, y, z \in A.$$

Proof. Let $x, y, z \in A$ such that $z \leq x * y$. Then $z * (x * y) = 1$, and thus
$\mu(y) \geq \mu(x) \cdot \mu(x * y) \geq \mu(x) \cdot \mu(z) \cdot \mu(z * (x * y)) = \mu(x) \cdot \mu(z) \cdot \mu(1)$.

Corollary 5.2. Let μ be a fuzzy subset of A such that $\mu(1) = 1$. Then μ is a fuzzy dot ideal of $\mathcal{A}$ if and only if

$$z \leq x * y \text{ implies } \mu(y) \geq \mu(x) \cdot \mu(z) \text{ for all } x, y, z \in A.$$

Proof. Assume that μ is a fuzzy dot ideal of $\mathcal{A}$. Then for any $x, y, z \in A$ such that $z \leq x * y$, we have

$$\mu(y) \geq \mu(x) \cdot \mu(z) \cdot \mu(1) = \mu(x) \cdot \mu(z) \cdot 1 = \mu(x) \cdot \mu(z).$$

Conversely, assume that $z \leq x * y$ implies $\mu(y) \geq \mu(x) \cdot \mu(z)$ for all $x, y, z \in A$.
For any $x \in A$, $\mu(1) = 1 \geq \mu(x)$. For any $x, y \in A$, we have $y * x \leq y * x$; thus, $\mu(x) \geq \mu(y) \cdot \mu(y * x)$.
Hence, μ is a fuzzy dot ideal of $\mathcal{A}$.

Proposition 5.4. Let μ be a fuzzy subset of A and m be a positive integer. If μ is a fuzzy dot ideal of $\mathcal{A}$, then μ^m is a fuzzy dot ideal of $\mathcal{A}$.

Proof. Assume that μ is a fuzzy dot ideal of $\mathcal{A}$. Since $\mu(1) \geq \mu(x)$ and $\mu(x) \geq \mu(y) \cdot \mu(y * x)$ for all $x, y \in A$, we have $\mu(1)^m \geq \mu(x)^m$ and $\mu(x)^m \geq (\mu(y) \cdot \mu(y * x))^m$ for all $x, y \in A$; i.e., $\mu(1)^m \geq \mu(x)^m$ and $\mu(x)^m \geq \mu(y)^m \cdot \mu(y * x)^m$; i.e., $\mu^m(1) \geq \mu^m(x)$ and $\mu^m(x) \geq \mu^m(y) \cdot \mu^m(y * x)$ for all $x, y \in A$. Hence, μ^m is a fuzzy dot ideal of $\mathcal{A}$.

Proposition 5.5. Let μ be a fuzzy subset of A. If μ and the fuzzy subset μ^c of A (defined by $\mu^c(x) = 1 - \mu(x)$ for all $x \in A$) are fuzzy dot ideals of $\mathcal{A}$, then μ is the constant fuzzy subset of A with value $\mu(1)$.

Proof. Assume that μ and μ^c are fuzzy dot ideals of $\mathcal{A}$. Since $\mu(1) \geq \mu(x)$ and $\mu^c(1) \geq \mu^c(x)$ for all $x \in A$, we have $\mu(1) \geq \mu(x)$ and $1 - \mu(1) \geq 1 - \mu(x)$ for all $x \in A$; i.e., $\mu(1) \geq \mu(x)$ and $\mu(1) \leq \mu(x)$ for all $x \in A$; i.e., $\mu(x) = \mu(1)$ for all $x \in A$; i.e., μ is the constant fuzzy subset of A with value $\mu(1)$.

Theorem 5.1. Let $\{\mu_i\}_{i \in I}$ be a family of fuzzy dot ideals of $\mathcal{A}$. Then $inf_{i \in I}\mu_i$ is a fuzzy dot ideal of $\mathcal{A}$.

Proof. For any $x \in A$, $(inf_{i \in I}\mu_i)(1) = inf_{i \in I}\mu_i(1) \geq inf_{i \in I}\mu_i(x) = (inf_{i \in I}\mu_i)(x)$.
For any $x, y \in A$,

$$\begin{aligned}(inf_{i \in I}\mu_i)(x) &= inf_{i \in I}\mu_i(x) \\ &\geq inf_{i \in I}[\mu_i(y) \cdot \mu_i(y * x)] \\ &\geq inf_{i \in I}[\big(inf_{i \in I}\mu_i(x)\big) \cdot \big(inf_{i \in I}\mu_i(y * x)\big)] \\ &= \big(inf_{i \in I}\mu_i(x)\big) \cdot \big(inf_{i \in I}\mu_i(y * x)\big) \\ &= (inf_{i \in I}\mu_i)(x) \cdot (inf_{i \in I}\mu_i)(y * x).\end{aligned}$$

Hence, $inf_{i \in I}\mu_i$ is a fuzzy dot ideal of $\mathcal{A}$.

Definition 5.3. Let μ be a fuzzy subset of A. The smallest fuzzy dot ideal of $\mathcal{A}$ which contains μ is said to be the fuzzy dot ideal of $\mathcal{A}$ generated by μ, and will be denoted by $(\mu\rangle$.

Notation 5.1. We denote the set of all fuzzy dot ideals of $\mathcal{A}$ by $Fi(\mathcal{A})$.
For any $\mu, \nu \in Fi(\mathcal{A})$, we define the meet of μ and ν (denoted by $\mu \sqcap \nu$) by $\mu \sqcap \nu = \mu \wedge \nu$ and the join of μ and ν (denoted by $\mu \sqcup \nu$) by $\mu \sqcup \nu = (\mu \vee \nu\rangle$.

Notation 5.2. Let $\alpha, \beta \in [0, 1]$, $I \subseteq A$ and $(I]$ be the ideal of $\mathcal{A}$ generated by I. $(I_\beta^\alpha]$ denotes the fuzzy subset of A defined by: $(I_\beta^\alpha](x) = \begin{cases} \alpha\ if\ x \in (I], \\ \beta\ otherwise. \end{cases}$ for all $x \in A$.

Lemma 5.1. Let I be a nonempty subset of A and $\alpha, \beta \in A$ such that $\beta \leq \alpha$. Then $(I_\beta^\alpha]$ is a fuzzy dot ideal of $\mathcal{A}$.
Proof. Since $1 \in (I]$, we have $(I_\beta^\alpha](1) = \alpha = \max\{\alpha, \beta\} \geq (I_\beta^\alpha](x)$ for all $x \in A$. Let $x, y \in A$.
If $x \notin (I]$, then $y \notin (I]$ or $y * x \notin (I]$; thus, $(I_\beta^\alpha](y) = \beta$ or $(I_\beta^\alpha](y * x) = \beta$; thus, $(I_\beta^\alpha](x) = \beta \geq \max\{\beta^2, \alpha \cdot \beta\} \geq (I_\beta^\alpha](y) \cdot (I_\beta^\alpha](y * x)$.
If $x \in (I]$, then $(I_\beta^\alpha](x) = \alpha \geq \max\{\alpha^2, \beta^2, \alpha \cdot \beta\} \geq (I_\beta^\alpha](y) \cdot (I_\beta^\alpha](y * x)$.
Hence, $(I_\beta^\alpha]$ is a fuzzy dot ideal of $\mathcal{A}$.
If I is an ideal of $\mathcal{A}$, then I_β^α is a fuzzy dot ideal of $\mathcal{A}$; but the converse is not necessarily true. In fact, consider the distributive implication groupoid $\mathcal{A}$ of the example 5.1; $\{1, a\}_{0.4}^{0.5}$ is a fuzzy dot ideal of $\mathcal{A}$ but $\{1, a\}$ is not an ideal of $\mathcal{A}$, since $a \in \{1, a\}$, $a * b = 1 \in \{1, a\}$ and $b \notin \{1, a\}$.

Theorem 5.2. Let I be a nonempty subset of A and $\beta \in [0, 1]$. Then $(I_\beta^1]$ is the fuzzy dot ideal of $\mathcal{A}$ generated by I_β^1; i.e., $(I_\beta^1\rangle = (I_\beta^1]$.
Proof. Since $I_\beta^1(x) \leq 1 = (I_\beta^1](x)$ for all $x \in (I]$ and $I_\beta^1(x) = \beta = (I_\beta^1](x)$ for all $x \notin (I]$, it follows that $I_\beta^1(x) \leq (I_\beta^1](x)$ for all $x \in A$; i.e., $(I_\beta^1]$ contains I_β^1. More, $(I_\beta^1]$ is a fuzzy dot ideal of $\mathcal{A}$ by the lemma 5.1, because $\beta \leq 1$. It suffices now to show that $(I_\beta^1]$ is the smallest fuzzy dot ideal of $\mathcal{A}$ containing I_β^1. So, let ν be a fuzzy dot ideal of $\mathcal{A}$ containing I_β^1.
For any $x \notin (I]$, we have $(I_\beta^1](x) = \beta = I_\beta^1(x) \leq \nu(x)$.
$\nu(1) \geq \nu(y) \geq I_\beta^1(y) = 1 = (I_\beta^1](1)$ for all $y \in I$.
For any $x \in (I] \setminus \{1\}$, $\prod_{i=1}^n a_i * x = 1$ for some $a_1, \dots, a_n \in I$; thus,

$$\begin{aligned} \nu(x) &\geq \nu(a_1) \cdot \nu(a_1 * x) \\ &\geq \nu(a_1) \cdot \nu(a_2) \cdot \nu\big(a_2 * (a_1 * x)\big) \\ &\vdots \\ &\geq \nu(a_1) \cdot \nu(a_2) \cdot \dots \cdot \nu(a_n) \cdot \nu(\textstyle\prod_{i=1}^n a_i * x) \\ &= \nu(a_1) \cdot \nu(a_2) \cdot \dots \cdot \nu(a_n) \cdot \nu(1) \\ &\geq \nu(a_1) \cdot \nu(a_2) \cdot \dots \cdot \nu(a_n) \cdot \nu(y) \text{ for all } y \in I \\ &\geq I_\beta^1(a_1) \cdot I_\beta^1(a_2) \cdot \dots \cdot I_\beta^1(a_n) \cdot I_\beta^1(y) \text{ for all } y \in I \\ &= 1 \cdot 1 \cdot \dots \cdot 1 \cdot 1 \\ &= 1 \\ &= (I_\beta^1](x). \end{aligned}$$

Therefore, $\nu(x) \geq (I_\beta^1](x)$ for all $x \in A$; i.e., ν contains $(I_\beta^1]$.

Hence, $(I_\beta^1]$ is the smallest fuzzy dot ideal of $\mathcal{A}$ containing I_β^1; i.e., $(I_\beta^1]$ is the fuzzy dot ideal of $\mathcal{A}$ generated by I_β^1.

Corollary 5.3. Let I be a nonempty subset of A and $\beta \in [0,1]$. Then I_β^1 is a fuzzy dot ideal of $\mathcal{A}$ if and only if I is an ideal of $\mathcal{A}$.
Proof. I_β^1 is a fuzzy dot ideal of $\mathcal{A}$ if and only if $(I_\beta^1\rangle = I_\beta^1$; i.e., $(I_\beta^1] = I_\beta^1$; i.e., $(I] = I$; i.e., I is an ideal of $\mathcal{A}$.

Corollary 5.4. Let I be a nonempty subset of A. Then χ_I is a fuzzy dot ideal of $\mathcal{A}$ if and only if I is an ideal of $\mathcal{A}$.
Proof. Straightforward, because $\chi_I = I_0^1$.

Theorem 5.3. $\left(Fi(\mathcal{A});\sqcap,\sqcup;\ \underline{0},\underline{1}\right)$ is a complete lattice.
Proof. Since $0 \leq 0$ and $1 \leq 1$, $\underline{0} = I_0^0$ and $\underline{1} = I_1^1$ (for all ideal I of $\mathcal{A}$) are fuzzy dot ideals of $\mathcal{A}$. Therefore, $\left(Fi(\mathcal{A});\sqcap,\sqcup;\ \underline{0},\underline{1}\right)$ is a complete lattice by the theorem 5.1.

Proposition 5.6. (see, [5]) A fuzzy subset of A is a fuzzy dot ideal of $\mathcal{A}$ if and only if all its nonempty level subsets are ideals of $\mathcal{A}$.
There is a fuzzy dot ideal of a distributive implication groupoid with a nonempty level subset which is not an ideal. Consider the fuzzy dot ideal $\zeta$of the example 5.1;
$U(\zeta; 0.6) = \{1, b\}$ is not an ideal, since $b \in U(\zeta; 0.6)$, $b * a \in U(\zeta; 0.6)$ and
$a \notin U(\zeta; 0.6)$.

Theorem 5.4. Let μ be a fuzzy subset of A. If μ is a fuzzy dot ideal of $\mathcal{A}$, then $U(\mu, 1)$ is either empty or an ideal of $\mathcal{A}$.
Proof. Assume that μ is a fuzzy dot ideal of $\mathcal{A}$ and $U(\mu, 1)$ is a nonempty subset of A. For any $x, y \in$ A such that $y \in U(\mu, 1)$ and $y * x \in U(\mu, 1)$, we have
$\mu(x) \geq \mu(y) \cdot \mu(y * x) = 1 \cdot 1 = 1$; thus, $\mu(x) = 1$; i.e., $x \in U(\mu, 1)$.
Hence, $U(\mu, 1)$ is an ideal of $\mathcal{A}$.

Theorem 5.5. Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism of distributive implication groupoids and ν be a fuzzy subset of B. If ν is a fuzzy dot ideal of $\mathcal{B}$, then $f^{-1}[\nu]$ is a fuzzy dot ideal of $\mathcal{A}$.
Proof. Assume that ν is a fuzzy dot ideal of $\mathcal{B}$.
For any $x \in A$, $f^{-1}[\nu](1) = \nu\big(f(1)\big) = \nu(1) \geq \nu\big(f(x)\big) = f^{-1}[\nu](x)$.
For any $x, y \in A$,

$$\begin{aligned} f^{-1}[\nu](x) &= \nu\big(f(x)\big) \\ &\geq \nu\big(f(y)\big) \cdot \nu\big(f(y) * f(x)\big) \\ &= \nu\big(f(y)\big) \cdot \nu\big(f(y * x)\big) \\ &= f^{-1}[\nu](y) \cdot f^{-1}[\nu](y * x). \end{aligned}$$

Hence, $f^{-1}[\nu]$ is a fuzzy dot ideal of $\mathcal{A}$.

Theorem 5.6. Let $f : \mathcal{A} \to \mathcal{B}$ be an onto homomorphism of distributive implication groupoids and ν be a fuzzy subset of B. If $f^{-1}[\nu]$ is a fuzzy dot ideal of $\mathcal{A}$, then ν is a fuzzy dot ideal of $\mathcal{B}$.
Proof. Assume that $f^{-1}[\nu]$ is a fuzzy dot ideal of $\mathcal{A}$.
For any $y \in B$ and $a \in f^{-1}(y)$,

$$\nu(1) = \nu\big(f(1)\big) = f^{-1}[\nu](1) \geq f^{-1}[\nu](a) = \nu\big(f(a)\big) = \nu(y).$$

For any $y, z \in B$ and $(a, b) \in f^{-1}(y) \times f^{-1}(z)$,

$$\begin{aligned} \nu(y) &= \nu\big(f(a)\big) = f^{-1}[\nu](a) \\ &\geq f^{-1}[\nu](b) \cdot f^{-1}[\nu](b * a) \\ &= \nu\big(f(b)\big) \cdot \nu\big(f(b * a)\big) \\ &= \nu\big(f(b)\big) \cdot \nu\big(f(b) * f(a)\big) \\ &= \nu(z) \cdot \nu(z * y). \end{aligned}$$

Hence, ν is a fuzzy dot ideal of $\mathcal{B}$.

Theorem 5.7. Let μ and ν be two fuzzy dot ideals of $\mathcal{A}$. Then $\mu \times \nu$ is a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$.

Proof. For any $(x_1, y_1), (x_2, y_2) \in A \times A$,

$$(\mu \times \nu)(1,1) = \mu(1) \cdot \nu(1) \geq \mu(x_1) \cdot \mu(y_1) = (\mu \times \nu)(x_1, y_1)$$

and

$$\begin{aligned} (\mu \times \nu)(x_1, y_1) &= \mu(x_1) \cdot \nu(y_1) \\ &\geq \big(\mu(x_2) \cdot \mu(x_2 * x_1)\big) \cdot \big(\nu(y_2) \cdot \nu(y_2 * y_1)\big) \\ &= \big(\mu(x_2) \cdot \nu(y_2)\big) \cdot \big(\mu(x_2 * x_1) \cdot \nu(y_2 * y_1)\big) \\ &= (\mu \times \nu)(x_2, y_2) \cdot (\mu \times \nu)(x_2 * x_1, y_2 * y_1) \\ &= (\mu \times \nu)(x_2, y_2) \cdot (\mu \times \nu)\big((x_2, y_2) * (x_1, y_1)\big). \end{aligned}$$

Hence, $\mu \times \nu$ is a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$.

Theorem 5.8. Let σ be a fuzzy subset of A. The strong σ-relation μ_σ on A is a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$ if and only if σ is a fuzzy dot ideal of $\mathcal{A}$.

Proof. (⇒) Assume that μ_σ is a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$.

For any $x, y \in A$,

$$\sigma(1) = \sqrt{\sigma(1) \cdot \sigma(1)} = \sqrt{\mu_\sigma(1,1)} \geq \sqrt{\mu_\sigma(x, x)} = \sqrt{\sigma(x) \cdot \sigma(x)} = \sigma(x)$$

and

$$\begin{aligned} \sigma(x) = \sqrt{\sigma(x) \cdot \sigma(x)} &= \sqrt{\mu_\sigma(x, x)} \\ &\geq \sqrt{\mu_\sigma(y, y) \cdot \mu_\sigma((y, y) * (x, x))} \\ &= \sqrt{\mu_\sigma(y, y) \cdot \mu_\sigma(y * x, y * x)} \\ &= \sqrt{\big(\sigma(y) \cdot \sigma(y)\big) \cdot (\sigma(y * x) \cdot \sigma(y * x))} \\ &= \sqrt{\big(\sigma(y) \cdot \sigma(y * x)\big) \cdot (\sigma(y) \cdot \sigma(y * x))} \\ &= \sigma(y) \cdot \sigma(y * x). \end{aligned}$$

Hence, σ is a fuzzy dot ideal of $\mathcal{A}$.

(⇐) Assume that σ is a fuzzy dot ideal of $\mathcal{A}$.

For any $(x_1, y_1), (x_2, y_2) \in A \times A$,

$$\mu_\sigma(1,1) = \sigma(1) \cdot \sigma(1) \geq \sigma(x_1) \cdot \sigma(y_1) = \mu_\sigma(x_1, y_1)$$

and

$$\begin{aligned} \mu_\sigma(x_1, y_1) &= \sigma(x_1) \cdot \sigma(y_1) \\ &\geq \big(\sigma(x_2) \cdot \sigma(x_2 * x_1)\big) \cdot \big(\sigma(y_2) \cdot \sigma(y_2 * y_1)\big) \\ &= \big(\sigma(x_2) \cdot \sigma(y_2)\big) \cdot \big(\sigma(x_2 * x_1) \cdot \sigma(y_2 * y_1)\big) \\ &= \mu_\sigma(x_2, y_2) \cdot \mu_\sigma(x_2 * x_1, y_2 * y_1) \end{aligned}$$

$$= \mu_\sigma(x_2, y_2) \cdot \mu_\sigma((x_2, y_2) * (x_1, y_1)).$$

Hence, μ_σ is a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$.

Proposition 5.7. Let μ be a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$. Then
$\mu(x, y) \geq \mu(x, 1) \cdot \mu(1, y)$ and $\mu(x, y) \geq \mu(1, x) \cdot \mu(x, x * y)$ for all $x, y \in A$.

Proof. For any $x, y \in A$,

$$\begin{aligned} \mu(x, y) &\geq \mu(x, 1) \cdot \mu((x, 1) * (x, y)) \\ &= \mu(x, 1) \cdot \mu(x * x, 1 * y) \\ &= \mu(x, 1) \cdot \mu(1, y) \end{aligned}$$

and

$$\begin{aligned} \mu(x, y) &\geq \mu(1, x) \cdot \mu((1, x) * (x, y)) \\ &= \mu(1, x) \cdot \mu(1 * x, x * y) \\ &= \mu(1, x) \cdot \mu(x, x * y). \end{aligned}$$

Proposition 5.8. Let μ be a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$ such that $\mu(1,1) = 1$. Then
$\mu(x, 1) \geq \mu(x, y)$ and $\mu(1, y) \geq \mu(x, y)$ for all $x, y \in A$.

Proof. For any $x, y \in A$,

$$\begin{aligned} \mu(x, 1) &\geq \mu(x, y) \cdot \mu((x, y) * (x, 1)) \\ &= \mu(x, y) \cdot \mu(x * x, y * 1) \\ &= \mu(x, y) \cdot \mu(1,1) \\ &= \mu(x, y) \cdot 1 \\ &= \mu(x, y) \end{aligned}$$

and

$$\begin{aligned} \mu(1, y) &\geq \mu(x, y) \cdot \mu((x, y) * (1, y)) \\ &= \mu(x, y) \cdot \mu(x * 1, y * y) \\ &= \mu(x, y) \cdot \mu(1,1) \\ &= \mu(x, y) \cdot 1 \\ &= \mu(x, y). \end{aligned}$$

Corollary 5.5. Let $z \in A$ and μ be a fuzzy dot ideal of $\mathcal{A} \times \mathcal{A}$ such that $\mu(1,1) = 1$. Then the fuzzy subset σ_z of A, defined by $\sigma_z(x) = \mu(z, x)$ for all $x \in A$, is a fuzzy dot ideal of $\mathcal{A}$.

Proof. For any $x \in A$, $\sigma_z(1) = \mu(z, 1) \geq \mu(z, x) = \sigma_z(x)$.
For any $x, y \in A$, $\sigma_z(x) = \mu(z, x)$

$$\begin{aligned} &\geq \mu(1, y) \cdot \mu\big((1, y) * (z, x)\big) \\ &= \mu(1, y) \cdot \mu(1 * z, y * x) \\ &= \mu(1, y) \cdot \mu(z, y * x) \\ &\geq \mu(z, y) \cdot \mu(z, y * x) \\ &= \sigma_z(y) \cdot \sigma_z(y * x). \end{aligned}$$

Hence, σ_z is a fuzzy dot ideal of $\mathcal{A}$.

Theorem 5.9. Let μ be a fuzzy subset of A and, x_α and y_β be two fuzzy points. If μ is a fuzzy dot ideal of $\mathcal{A}$, then

$$x_\alpha \in \mu \text{ and } x_\alpha * y_\beta \in \mu \text{ imply } y_{\alpha^2 \cdot \beta} \in \mu.$$

Proof. Assume that μ is a fuzzy dot ideal of $\mathcal{A}$, $x_\alpha \in \mu$ and $x_\alpha * y_\beta \in \mu$. We have $x_\alpha \in \mu$ and $(x * y)_{\alpha\cdot\beta} \in \mu$; i.e., $\mu(x) \geq \alpha$ and $\mu(x * y) \geq \alpha \cdot \beta$; thus, $\mu(x) \cdot \mu(x * y) \geq \alpha \cdot (\alpha \cdot \beta) = \alpha^2 \cdot \beta$; thus, $\mu(y) \geq \alpha^2 \cdot \beta$; i.e., $y_{\alpha^2\cdot\beta} \in \mu$.

6. Fuzzy implicative dot ideals of distributive implication groupoids

In this section, fuzzy implicative dot ideals are defined and the relationship between fuzzy implicative dot ideals, fuzzy implicative ideals and fuzzy dot ideals are discussed.

Definition 6.1. (see, [4]) Let μ be a fuzzy subset of A. μ is called a fuzzy implicative ideal of $\mathcal{A}$ if it satisfies the following conditions:

i. $\mu(1) \geq \mu(x)$,

ii. $\mu(x) \geq \min\{\mu(z), \mu\left(z * \left((x * y) * x\right)\right)\}$; for all $x, y, z \in A$.

Definition 6.2. Let μ be a fuzzy subset of A. μ is called a fuzzy implicative dot ideal of $\mathcal{A}$ if it satisfies the following conditions:

i. $\mu(1) \geq \mu(x)$,

ii. $\mu(x) \geq \mu(z) \cdot \mu\left(z * \left((x * y) * x\right)\right)$; for all $x, y, z \in A$.

Example 6.1. Let $A = \{1, a, b, c, d, e, f, g\}$ be a set with the following Cayley table:

*	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	1	1	1	1	1	1	1	1
b	1	c	1	c	g	1	1	g
c	1	f	f	1	f	1	f	1
d	1	c	e	c	1	e	1	1
e	1	a	f	f	d	1	f	g
f	1	c	e	c	g	e	1	g
g	1	a	b	c	f	e	f	1

Then $\mathcal{A} = (A; *, 1)$ is a distributive implication groupoid. Define a fuzzy subset ϱ of A by $\varrho(1) = \varrho(a) = \varrho(b) = \varrho(c) = \varrho(d) = 0.7$ and $\varrho(e) = \varrho(f) = \varrho(g) = 0.5$. Then ϱ is a fuzzy implicative dot ideal of $\mathcal{A}$.

Note that every fuzzy implicative ideal is a fuzzy implicative dot ideal, but the converse is not necessarily true. In fact, the fuzzy implicative dot ideal of the example 6.1is not a fuzzy implicative ideal, since

$$\varrho(b * d) = \varrho(g) = 0.5 \ngeq 0.7 = \min\{0.7, 0.7\} = \min\{\varrho(b), \varrho(d)\}.$$

Theorem 6. Every fuzzy implicative dot ideal of $\mathcal{A}$ is a fuzzy dot ideal of $\mathcal{A}$.

Proof. Let μ be a fuzzy implicative dot ideal of $\mathcal{A}$.

For any $x, y \in A$,
$$\begin{aligned}\mu(x) &\geq \mu(y) \cdot \mu\left(y * \left((x * x) * x\right)\right)\\ &= \mu(y) \cdot \mu\left(y * (1 * x)\right)\\ &= \mu(y) \cdot \mu(y * x).\end{aligned}$$

Hence, μ is a fuzzy dot ideal of $\mathcal{A}$.

The converse of the theorem 6. is not necessarily true.

Example 6.2. Let $A = \{1, a, b, c\}$ be a set with the following Cayley table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	c
b	1	1	1	c
c	1	1	1	1

Then $\mathcal{A} = (A; *, 1)$ is a distributive implication groupoid. The fuzzy dot ideal ι of $\mathcal{A}$, defined by $\iota(1) = 0.7$ and $\iota(a) = \iota(b) = \iota(c) = 0.2$, is not a fuzzy implicative dot ideal of $\mathcal{A}$. In fact,

$$\begin{aligned}\iota(1) \cdot \iota\Big(1 * \big((a * c) * a\big)\Big) &= \iota(1) \cdot \iota(c * a) \\ &= \iota(1) \cdot \iota(1) \\ &= 0.7 \cdot 0.7 \\ &= 0.49 \\ &\nleq 0.2 \\ &= \iota(a).\end{aligned}$$

Proposition 6.1. Let μ be a fuzzy subset of A. If μ is a fuzzy implicative dot ideal of $\mathcal{A}$, then

a. $\mu(x) \geq \mu(1) \cdot \mu\big((x * y) * x\big)$,

b. $\mu\big((x * y) * x\big) \geq \mu(x) \cdot \mu(1)$; for all $x, y \in A$.

Proof. Assume that μ is a fuzzy implicative dot ideal of $\mathcal{A}$. Let $x, y \in A$.

a. $\mu(x) \geq \mu(1) \cdot \mu\Big(1 * \big((x * y) * x\big)\Big) = \mu(1) \cdot \mu\big((x * y) * x\big)$.

b. $$\begin{aligned}\mu\big((x * y) * x\big) &\geq \mu(x) \cdot \mu\Big(x * \Big[\Big(\big((x * y) * x\big) * y\Big) * \big((x * y) * x\big)\Big]\Big) \\ &= \mu(x) \cdot \mu\Big(\Big[x * \Big(\big((x * y) * x\big) * y\Big)\Big] * \Big[x * \big((x * y) * x\big)\Big]\Big) \\ &= \mu(x) \cdot \mu\Big(\Big[x * \Big(\big((x * y) * x\big) * y\Big)\Big] * 1\Big) \\ &= \mu(x) \cdot \mu(1).\end{aligned}$$

Corollary 6.1. Let μ be a fuzzy subset of A such that $\mu(1) = 1$. If μ is a fuzzy implicative dot ideal of $\mathcal{A}$, then $\mu\big((x * y) * x\big) = \mu(x)$ for all $x, y \in A$.

Proof. Straightforward.

Corollary 6.2. Let μ be a fuzzy dot ideal of $\mathcal{A}$ such that $\mu(1) = 1$. Then μ is a fuzzy implicative dot ideal of $\mathcal{A}$ if and only if $\mu\big((x * y) * x\big) = \mu(x)$ for all $x, y \in A$.

Proof. If μ is a fuzzy implicative dot ideal of $\mathcal{A}$, then $\mu\big((x * y) * x\big) = \mu(x)$ for all $x, y \in A$, by the corollary 6.1.

Conversely, assume that $\mu\big((x * y) * x\big) = \mu(x)$ for all $x, y \in A$. Since μ is a fuzzy dot ideal of $\mathcal{A}$, we have $\mu(x) = \mu\big((x * y) * x\big) \geq \mu(z) \cdot \mu\Big(z * \big((x * y) * x\big)\Big)$ for all $x, y \in A$. Hence, μ is a fuzzy implicative dot ideal of $\mathcal{A}$.

Proposition 6.2. Let μ be a fuzzy implicative dot ideal of $\mathcal{A}$. Then

$z \leq a * \big((x * y) * x\big)$ implies $\mu(x) \geq \mu(a) \cdot \mu(z) \cdot \mu(1)$; for all $x, y, z, a \in A$.

Proof. Let $x, y, z, a \in A$ such that $z \leq a * \big((x * y) * x\big)$.

$$\begin{aligned}\mu(x) &\geq \mu(a)\cdot\mu\Big(a*\big((x*y)*x\big)\Big)\\ &\geq \mu(a)\cdot\mu(z)\cdot\mu\Big(z*\Big[\Big(\Big(a*\big((x*y)*x\big)\Big)*y\Big)*\Big(a*\big((x*y)*x\big)\Big)\Big]\Big)\\ &= \mu(a)\cdot\mu(z)\cdot\mu\Big(\Big[z*\Big(\Big(a*\big((x*y)*x\big)\Big)*y\Big)\Big]*\Big[z*\Big(a*\big((x*y)*x\big)\Big)\Big]\Big)\\ &= \mu(a)\cdot\mu(z)\cdot\mu\Big(\Big[z*\Big(\Big(a*\big((x*y)*x\big)\Big)*y\Big)\Big]*1\Big)\\ &= \mu(a)\cdot\mu(z)\cdot\mu(1).\end{aligned}$$

Corollary 6.3. Let μ be a fuzzy subset of A such that $\mu(1)=1$. Then μ is a fuzzy implicative dot ideal of $\mathcal{A}$ if and only if

$z \leq a*\big((x*y)*x\big)$ implies $\mu(x)\geq\mu(a)\cdot\mu(z)$; for all $x,y,z,a\in A$.

Proof. Assume that μ is a fuzzy implicative dot ideal of $\mathcal{A}$. Then for any $x,y,z,a\in A$ such that $z\leq a*\big((x*y)*x\big)$, we have

$$\mu(x)\geq\mu(a)\cdot\mu(z)\cdot\mu(1)=\mu(a)\cdot\mu(z)\cdot 1=\mu(a)\cdot\mu(z).$$

Conversely, assume that $z\leq a*\big((x*y)*x\big)$ implies $\mu(x)\geq\mu(a)\cdot\mu(z)$; for all $x,y,z,a\in A$. For any $x,y,z\in A$, we have

$$\begin{aligned} z*\Big[\Big(z*\big((x*y)*x\big)\Big)*\big((x*y)*x\big)\Big] &=\\ &= \Big[z*\Big(z*\big((x*y)*x\big)\Big)\Big]*\big[z*\big((x*y)*x\big)\big]\\ &= \Big[(z*z)*\Big(z*\big((x*y)*x\big)\Big)\Big]*\big[z*\big((x*y)*x\big)\big]\\ &= \Big[1*\Big(z*\big((x*y)*x\big)\Big)\Big]*\big[z*\big((x*y)*x\big)\big]\\ &= \big[z*\big((x*y)*x\big)\big]*\big[z*\big((x*y)*x\big)\big]\\ &= 1;\end{aligned}$$

thus, $z\leq\Big(z*\big((x*y)*x\big)\Big)*\big((x*y)*x\big)$;

thus, $\mu(x)\geq\mu(z)\cdot\mu\Big(z*\big((x*y)*x\big)\Big)$.

Hence, μ is a fuzzy implicative dot ideal of $\mathcal{A}$.

7. Conclusion

In the present paper, fuzzy dot subalgebras, fuzzy normal dot subalgebras, fuzzy dot ideals and fuzzy implicative dot ideals of distributive implication groupoids are investigated. Using t-norm T and s-norm S, these notions can further be generalized to T-fuzzy sets, S-fuzzy sets and Intuitionistic (T,S)-fuzzy sets.

Acknowledgment. The authors are greatly appreciated the referees for their valuable comments and suggestions for improving the paper.

REFERENCES

1. M.A.A.Ansari and M.Chandramouleeswaran: Fuzzy dot β-subalgebras of β-algebras, *International Journal of Pure and Applied Mathematics*, 90 (2) (2014) 119-129.
2. M.A.A.Ansari and M.Chandramouleeswaran: Fuzzy dot β-ideals of β-algebras, *International Journal of Pure and Applied Mathematics*, 98 (5) (2015) 19-25.
3. R.K.Bandaru, On ideals of implication groupoids, *Advances in Decisions Sciences*, 2012, Article ID 652814, 9 pages.

4. R.K.Bandaru and K.P.Shum, Implicative ideals and fuzzy implicative ideals of a distributive implication groupoid, *Journal of Mathematical Research with Applications*, 34 (6) (2014) 631-639.
5. R.K.Bandaru, K.P.Shum and N.Rafi, Fuzzy ideals of implication groupoids, *Italian Journal of Pure and Applied Mathematics*, 34 (2015) 277-290.
6. R.K.Bandaru and B.Davvaz, Fuzzy implications groupoids, *J. Fuzzy Math.*, 23(1) (2015) 141-148.
7. I.Chajda and R.Halas, Distributive implication groupoids, *Cent. Eur. J. Math.*, 5(3) (2007) 484-492.
8. F.P.Choudhury, A.B.Chakraborty and S.S.Khare, A note on fuzzy subgroups and fuzzy homomorphism, *J. Math. Anal. Appl.*, 131(2) (1988) 537–553.
9. W.A.Dudek, On ideals in Hilbert algebras, *Mathematicia,* 38 (1999) 31-34.
10. S.M.Hong, Y.B. Jun, S.J. Kim and G.I. Kim, On fuzzy dot subalgebras of BCH-algebras, *International Journal of Mathematics and Mathematical Sciences*, 27 (6) (2001) 357-364.
11. V.Murali, Lattice of fuzzy subalgebras and closure systems in I^X, *Fuzzy Sets and Systems,* 41 (1991) 101-111.
12. P.Jia-yin, Fuzzy dot ideals and fuzzy dot H-ideals of BCH-algebras, *Appl. Math. J. Chinese Univ.*, 23(1) (2008) 101-106.
13. S.V.Tchoffo Foka, M.Tonga and T.Senapati, Fuzzy dot structure of bounded lattices, *Journal of Mathematics and Informatics*, 4 (2016) In press.
14. L.A.Zadeh, Fuzzy sets, *Information and Control,* 8 (1965) 338-353.