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Fuzzy Metric Space: A Study on Level Sets Notion

***M. Shahjalal*[1], *Abeda Sultana*[2], *Jahanara Begum*[3] and *N.K Mitra*[4]**

[1]Department of Basic Science, Primeasia University, Bangladesh
E-mail: shahajalal@gmail.com
[2]Department of Mathematics, Jahangirnagar University, Bangladesh
[3]Times University, Bangladesh
[4]School of Engineering and Computer Science, Fiji National University

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Abstract. In this paper, the Level sets concept of fuzzy numbers is considered to study fuzzy metric space. Some properties of metric space along with contraction mapping are revised in terms of the notion of level sets.

Keywords: Fuzzy fixed point, fuzzy metric space, fuzzy contraction, fuzzy sequence, level sets.

AMS Mathematics Subject Classification (2010): 94D05

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh [11] in 1965, now the expansion of fuzzy sets theory in pure and applied mathematics has been widely recognized. Specially to mention about metric spaces what is extended to the fuzzy metric spaces were introduced by Deng [2], Erceg [4], Kaleva and Seikkala [6], Kramosil and Michalek [8]. Recently Phiangsungnoena, and Kumama [9] worked on fuzzy fixed point theorems for multivalued fuzzy contraction mapping. The concept of fuzzy mapping in Hausdorff metric space was dscussed in [10]. In this paper, the concept of fuzzy metric space is discussed in the knowledge of level sets concept. Fuzzy contraction mapping is applied to study the fixed point in a fuzzy space.

2. Preliminaries

Definition 2.1. Let A be a fuzzy number and the level sets of A will be denoted by $[A]^r = [\underline{a}(r), \overline{a}(r)]$, $r \in]0,1]$ or simply by $[A]^r = [\underline{a}, \overline{a}]$. Level sets of a fuzzy number are bounded interval.

Definition 2.2. Two intervals $[\underline{a}, \overline{a}], [\underline{b}, \overline{b}] \subset [0, \infty[$ are **equal** iff $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$.

Definition 2.3. Let $[\underline{a}, \overline{a}], [\underline{b}, \overline{b}] \subset [0, \infty[$ be two intervals. Interval $[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}]$; if and only if $\underline{a} \le \underline{b}$ and $\overline{a} \le \overline{b}$.

Definition 2.4. Let $[\underline{a},\overline{a}],[\underline{b},\overline{b}],[\underline{c},\overline{c}]\subset[0,\infty[$ be three intervals. The less then relation among three intervals is denoted by

$$[\underline{a},\overline{a}]<[\underline{b},\overline{b}]<[\underline{c},\overline{c}] \text{ iff } \underline{a}<\underline{b}<\underline{c} \text{ and } \overline{a}<\overline{b}<\overline{c}.$$

Definition 2.5. Distance between two intervals: Let $[\underline{a},\overline{a}],[\underline{b},\overline{b}]\subset[0,\infty[$ be two intervals. The distance between the intervals is defined by

$$d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)=\max\left\{|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\right\}.$$

Properties 2.6. Let $X=[0,\infty[$ and $[\underline{a},\overline{a}],[\underline{b},\overline{b}],[\underline{c},\overline{c}]\subset X$ are such that $\underline{a}<\underline{b}<\underline{c}$, and $\overline{a}<\overline{b}<\overline{c}$. Then

- **i)** $d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)\geq 0$
- **ii)** $d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)=0$ iff $[\underline{a},\overline{a}]=[\underline{b},\overline{b}]$
- **iii)** $d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)=d\left([\underline{b},\overline{b}],[\underline{a},\overline{a}]\right)$
- **iv)** $d\left([\underline{a},\overline{a}],[\underline{c},\overline{c}]\right)\leq d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)+d\left([\underline{b},\overline{b}],[\underline{c},\overline{c}]\right)$
- **v)** $d\left([\underline{a},\overline{a}]+p,[\underline{b},\overline{b}]+p\right)=d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)$ for any $p\in\mathbb{R}$
- **vi)** $d\left(\lambda[\underline{a},\overline{a}],\lambda[\underline{b},\overline{b}]\right)=0$ for any $\lambda\in\mathbb{R}^{+}$

Proof : Given that $X=[0,\infty[$ and $[\underline{a},\overline{a}],[\underline{b},\overline{b}],[\underline{c},\overline{c}]\subset X$ are such that $\underline{a}<\underline{b}<\underline{c}$, and $\overline{a}<\overline{b}<\overline{c}$.

i) By $d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)=\max\left\{|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\right\}$ where $\underline{a}<\underline{b}$, and $\overline{a}<\overline{b}$.

$\Rightarrow |\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\neq 0$ and $|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|>0$.

Therefore $d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)>0$.

ii) Let $d\left([\underline{a},\overline{a}],[\underline{b},\overline{b}]\right)=0$

$\Rightarrow \max\left\{|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\right\}=0$

Since both $|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\geq 0$, so $\max\left\{|\underline{a}-\underline{b}|,|\overline{a}-\overline{b}|\right\}=0$ shall occur only when

$$|\underline{a}-\underline{b}|=0 \text{ and } |\overline{a}-\overline{b}|=0,$$

$$\Rightarrow \underline{a}-\underline{b}=0 \text{ and } \overline{a}-\overline{b}=0$$

$$\Rightarrow \underline{a}=\underline{b} \text{ and } \overline{a}=\overline{b},$$

Therefore $[\underline{a},\overline{a}] = [\underline{b},\overline{b}]$.

Again let $[\underline{a},\overline{a}] = [\underline{b},\overline{b}]$ which implies that $\underline{a} = \underline{b}$ and $\overline{a} = \overline{b}$. Therefore

$\underline{a} - \underline{b} = 0$ and $\overline{a} - \overline{b} = 0$.

So $\max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\} = 0$

$\Rightarrow d([\underline{a},\overline{a}],[\underline{b},\overline{b}]) = 0.$

iii)

$$d([\underline{a},\overline{a}],[\underline{b},\overline{b}]) = \max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\} = \max\{|\overline{a}-\overline{b}|, |\underline{a}-\underline{b}|\} = d([\underline{b},\overline{b}],[\underline{a},\overline{a}])$$

iv) We have $\underline{a} < \underline{b} < \underline{c}$, $\overline{a} < \overline{b} < \overline{c}$. So

$$\max\{|\underline{a}-\underline{c}|, |\overline{a}-\overline{c}|\} \le \max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\} + \max\{|\underline{b}-\underline{c}|, |\overline{b}-\overline{c}|\}$$

$$\Rightarrow d([\underline{a},\overline{a}],[\underline{c},\overline{c}]) \le d([\underline{a},\overline{a}],[\underline{b},\overline{b}]) + d([\underline{b},\overline{b}],[\underline{c},\overline{c}]).$$

v) Clearly, $([\underline{a},\overline{a}]+p, [\underline{b},\overline{b}]+p) = ([\underline{a}+p, \overline{a}+p], [\underline{b}+p, \overline{b}+p])$

$$\Rightarrow d([\underline{a},\overline{a}]+p, [\underline{b},\overline{b}]+p) = d([\underline{a}+p,\overline{a}+p],[\underline{b}+p,\overline{b}+p]) = \max\{|\underline{a}+p-\underline{b}-p|, |\overline{a}+p-\overline{b}-p|\}$$

$$\Rightarrow d([\underline{a},\overline{a}]+p, [\underline{b},\overline{b}]+p) = \max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\} = d([\underline{a},\overline{a}],[\underline{b},\overline{b}]).$$

vi) Clearly, for $\lambda > 0, \lambda[\underline{a},\overline{a}] = [\lambda\underline{a}, \lambda\overline{a}]$ and $\lambda[\underline{b},\overline{b}] = [\lambda\underline{b}, \lambda\overline{b}]$

$$\Rightarrow d(\lambda[\underline{a},\overline{a}], \lambda[\underline{b},\overline{b}]) = d([\lambda\underline{a},\lambda\overline{a}],[\lambda\underline{b},\lambda\overline{b}]) = \max\{|\lambda\underline{a}-\lambda\underline{b}|, |\lambda\overline{a}-\lambda\overline{b}|\}$$

$$\Rightarrow d(\lambda[\underline{a},\overline{a}], \lambda[\underline{b},\overline{b}]) = \max\{\lambda|\underline{a}-\underline{b}|, \lambda|\overline{a}-\overline{b}|\} = \lambda\max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\} = \lambda d([\underline{a},\overline{a}],[\underline{b},\overline{b}])$$

$$\Rightarrow d(\lambda[\underline{a},\overline{a}], \lambda[\underline{b},\overline{b}]) = \max\{\lambda|\underline{a}-\underline{b}|, \lambda|\overline{a}-\overline{b}|\} = \lambda\max\{|\underline{a}-\underline{b}|, |\overline{a}-\overline{b}|\}$$

Therefore $d(\lambda[\underline{a},\overline{a}], \lambda[\underline{b},\overline{b}]) = \lambda d([\underline{a},\overline{a}],[\underline{b},\overline{b}])$.

3. Level sets based approach to fuzzy metric space

Definition 3.1. Let *A, B* and *C* be three fuzzy numbers, their corresponding level sets are $[A]^r = [\underline{a},\overline{a}], [B]^r = [\underline{b},\overline{b}]$ and $[C]^r = [\underline{c},\overline{c}]$ for $r \in [0,1]$; respectively.

Definition 3.2. Let X be a fuzzy number and $[X]^r$ be set of all *r*-cuts of X for $r \in [0,1]$. Then the mapping $d : [X]^r \times [X]^r \to [0,\infty[$ is called a fuzzy metric if for all $[\underline{x},\overline{x}], [\underline{y},\overline{y}], [\underline{z},\overline{z}] \in [X]^r$, the following axioms hold.

(i) $d\left([\underline{x},\overline{x}],[\underline{y},\overline{y}]\right)\geq 0$

(ii) $d\left([\underline{x},\overline{x}],[\underline{y},\overline{y}]\right)=0$ if and only if $[\underline{x},\overline{x}]=[\underline{y},\overline{y}]$

(iii) $d\left([\underline{x},\overline{x}],[\underline{y},\overline{y}]\right)=d\left([\underline{y},\overline{y}],[\underline{x},\overline{x}]\right)$

(iv) $d\left([\underline{x},\overline{x}],[\underline{z},\overline{z}]\right)\leq d\left([\underline{x},\overline{x}],[\underline{y},\overline{y}]\right)+d\left([\underline{y},\overline{y}],[\underline{z},\overline{z}]\right)$

The set $[X]^r$ together with metric d on it is called a metric space. It is denoted by $\left([X]^r, d\right)$.

Definition 3.3. Let $[X]^r=[\underline{x},\overline{x}], \forall r\in[0,1]$ be level sets of a fuzzy number X and $C[a,b]$ be set of all continuous functions on interval $[a,b]$. Then $\forall r\in[0,1]$, we define $f[X]^r=f[\underline{x},\overline{x}]=[f(\underline{x}),f(\overline{x})]$.

Example 3.4. Let $C[a,b]$ be set of all continuous functions on interval $[a,b]$. Again suppose that for level sets $[X]^r=[\underline{x},\overline{x}]\in[a,b]$; C is a metric space under following metrics:

(i) $d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=\sup\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|$

(ii) $d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=\int_a^b\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|d[\underline{x},\overline{x}]$

Solution:

(i) Given $d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=\sup\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|$

It is certain,

$\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|\geq 0\Rightarrow\sup\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|\geq 0\Rightarrow d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)$

Again, Let $d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=0$

$\Rightarrow\sup\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|=0\Rightarrow\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|=0\Rightarrow f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]=0$

Therefore, $f[\underline{x},\overline{x}]=g[\underline{x},\overline{x}]$.

For converse; let $f[\underline{x},\overline{x}]=g[\underline{x},\overline{x}]$ that is $f[\underline{x},\overline{x}]=g[\underline{x},\overline{x}]$.

Then $f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]=0\Rightarrow\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|=0\Rightarrow\sup\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|=0$

Therefore $d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=0$.

To show $d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=d\left(g[\underline{x},\overline{x}],f[\underline{x},\overline{x}]\right)$

We have

$d\left(f[\underline{x},\overline{x}],g[\underline{x},\overline{x}]\right)=\sup\left|f[\underline{x},\overline{x}]-g[\underline{x},\overline{x}]\right|$

$= \sup\left|g[\underline{x},\overline{x}] - f[\underline{x},\overline{x}]\right|$

$= d\left(g[\underline{x},\overline{x}], f[\underline{x},\overline{x}]\right)$

Therefore $d\left(f[\underline{x},\overline{x}], g[\underline{x},\overline{x}]\right) = d\left(g[\underline{x},\overline{x}], f[\underline{x},\overline{x}]\right)$

Again $d\left(f[\underline{x},\overline{x}], g[\underline{x},\overline{x}]\right) = \sup\left|f[\underline{x},\overline{x}] - g[\underline{x},\overline{x}]\right|$

$= \sup\left|f[\underline{x},\overline{x}] - h[\underline{x},\overline{x}] + h[\underline{x},\overline{x}] - g[\underline{x},\overline{x}]\right|$

$\le \sup\left|f[\underline{x},\overline{x}] - h[\underline{x},\overline{x}]\right| + \sup\left|h[\underline{x},\overline{x}] - g[\underline{x},\overline{x}]\right|$

$\le d\left(f[\underline{x},\overline{x}], h[\underline{x},\overline{x}]\right) + d\left(h[\underline{x},\overline{x}], g[\underline{x},\overline{x}]\right)$

Therefore $d\left(f[\underline{x},\overline{x}], g[\underline{x},\overline{x}]\right) \le d\left(f[\underline{x},\overline{x}], h[\underline{x},\overline{x}]\right) + d\left(h[\underline{x},\overline{x}], g[\underline{x},\overline{x}]\right)$.

Thus we see that all axioms of metric space are satisfied for (i), thus the solution is completed. The solution of (ii) is similar.

Definition 3.5. Let $X \in I^{\mathbb{R}}$ be the set of all fuzzy numbers on real line. Sequence of fuzzy numbers is denoted by (X_n). The level sets of (X_n) are denoted by $\left([\underline{x}_n, \overline{x}_n]\right)$.

Definition 3.6. Let $X \in I^{\mathbb{R}}$ be a fuzzy number and $\left([\underline{x},\overline{x}], d\right)$ be a fuzzy metric space. A sequence $\left([\underline{x}_n, \overline{x}_n]\right)$ in a fuzzy metric space $\left([\underline{x},\overline{x}], d\right)$ is said to be convergent if there exist $[\underline{x}_0, \overline{x}_0] \subset [\underline{x},\overline{x}]$ such that $\lim_{n\to\infty} d\left([\underline{x}_n, \overline{x}_n], [\underline{x}_0, \overline{x}_0]\right) = [0, 0]$. Here $[\underline{x}_0, \overline{x}_0]$ is called the limit of the sequence $\left([\underline{x}_n, \overline{x}_n]\right)$.

Definition 3.7. [7] $\forall K \in [0, \infty[$ with $K = [\underline{k}, \overline{k}]$ is called **generated interval** if

$$\underline{k} \le \overline{k} \text{ with } \underline{k} \ne \overline{k} .$$

Again $\forall K \in [0, \infty[$ with $K = [\underline{k}, \overline{k}]$ is called **degenerated interval** if

$$\underline{k} \le \overline{k} \text{ with } \underline{k} = \overline{k} .$$

Example 3.7. [2, 3] is a generated interval and [2,2] is a degenerated interval. Observe that [2, 2] = {2}.

Definition 3.8. Let $X \in I^{\mathbb{R}}$ be a fuzzy number and $\left([\underline{x},\overline{x}], d\right)$ be a fuzzy metric space. A sequence $\left([\underline{x}_n, \overline{x}_n]\right)$ in the fuzzy metric space $\left([\underline{x},\overline{x}], d\right)$ is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a positive integer N such that $d\left([\underline{x}_n, \overline{x}_n], [\underline{x}_m, \overline{x}_m]\right) \le [\varepsilon, \varepsilon]$ for all $n, m > N$.

Definition 3.9[5]. A metric space (X,d) is said to be complete if every Cauchy sequence in it converges to an element of it.

Theorem 3.10. The metric space $\left(\left[\underline{x},\overline{x}\right],d\right)$ is complete.

Proof: Let be$\left(\left[\underline{x}_m,\overline{x}_m\right]\right)$ a Cauchy sequence in $\left(\left[\underline{x},\overline{x}\right]\right)$.

Let the sequence$\left(\left[\underline{x}_m,\overline{x}_m\right]\right)=\left(\left[\left(\underline{x}_{1m},\underline{x}_{2m},\ldots\right),\left(\overline{x}_{1m},\overline{x}_{2m},\ldots\right)\right]\right)=\left(\left[\left(\underline{x}_{im}\right),\left(\overline{x}_{im}\right)\right]\right)$.

Since $\left(\left[\underline{x}_m,\overline{x}_m\right]\right)$is Cauchy sequence, for given $\varepsilon>0$ there exists a positive integer N such that

$d\left(\left[\underline{x}_m,\overline{x}_m\right],\left[\underline{x}_n,\overline{x}_n\right]\right)=\max_i\left\{\left|\underline{x}_{im}-\underline{x}_{in}\right|,\left|\overline{x}_{im}-\overline{x}_{in}\right|\right\}<[\varepsilon,\varepsilon]$ for all $n,m>\mathbb{N}$.

For every fixed i, we have

$$\left|\underline{x}_{im}-\underline{x}_{in}\right|<\varepsilon \quad m,n>N \text{ and} \left|\overline{x}_{im}-\overline{x}_{in}\right|<\varepsilon \ \ m,n>N \tag{1}$$

Hence for every fixed i, the sequences $\left(\underline{x}_{im}\right)$ and $\left(\overline{x}_{im}\right)$are Cauchy sequence of numbers. So, they converges, say $\underline{x}_{im}\to\underline{x}_i$ and $\overline{x}_{im}\to\overline{x}_i$as $m\to\infty$. Using these infinitely many limits say that$\underline{x}_{im}\to\underline{x}$ and $\overline{x}_{im}\to\overline{x}$ as $m\to\infty$.

where $\underline{x}=\left(\underline{x}_1,\underline{x}_2,\ldots\right)$ and $\overline{x}=\left(\overline{x}_1,\overline{x}_2,\ldots\right)$.

Now from (1), taking$n\to\infty$, we have

$$\left|\underline{x}_{im}-\underline{x}\right|\le\varepsilon \quad m>N \text{ and} \left|\overline{x}_{im}-\overline{x}\right|\le\varepsilon \ \ m>N \tag{2}$$

Since$\underline{x}_m=\left(\underline{x}_{im}\right)$ and $\overline{x}_m=\left(\overline{x}_{im}\right)$ there is a real number k_m such that

$$\left|\underline{x}_{im}\right|\le k_m \text{ and} \left|\overline{x}_{im}\right|=k_m \text{ for all } i.$$

We can write

$\left|\underline{x}_i\right|=\left|\underline{x}_i-\underline{x}_{im}+\underline{x}_{im}\right|$ and $\left|\overline{x}_i\right|=\left|\overline{x}-\overline{x}_{im}+\overline{x}_{im}\right|$

$\Rightarrow\left|\underline{x}_i\right|\le\left|\underline{x}_i-\underline{x}_{im}\right|+\left|\underline{x}_{im}\right|$ and $\left|\overline{x}_i\right|\le\left|\overline{x}-\overline{x}_{im}\right|+\left|\overline{x}_{im}\right|$

$\Rightarrow\left|\underline{x}_i\right|\le\varepsilon+k_m$ and $\left|\overline{x}_i\right|\le\varepsilon+k_m$ $;m>N$

These inequalities hold for every i and right hand side does not involve i. Hence sequences $\left(\underline{x}_i\right)$ and $\left(\overline{x}_i\right)$ are bounded sequence.

Now from (2) we have:

$\max_i\left\{\left|\underline{x}_{im}-\underline{x}\right|,\left|\overline{x}_{im}-\overline{x}\right|\right\}\le[\varepsilon,\varepsilon]$; $m>N$.

Which implies that and $\left|\overline{x}_{im}-\overline{x}\right|\le\varepsilon \ \ m>N$.

Therefore $d\left(\left[\underline{x}_m,\overline{x}_m\right],\left[\underline{x},\overline{x}\right]\right)\leq[\varepsilon,\varepsilon]\ ;m>N$.

This shows that fuzzy metric space is $\left(\left[\underline{x},\overline{x}\right],d\right)$ complete.

4. Fixed point of fuzzy metric space

In classical topology there are notions of fixed point and contraction mapping. In this section we shall present the notions in fuzzy context.

Definition 4.1[9]. Suppose X is any set and $T:X\rightarrow X$ is a mapping. Then $x\in X$ is called fixed point of T if $Tx=x$.

Definition 4.2. Let $X\in I^{\mathbb{R}}$ be a fuzzy number and $\left[\underline{x},\overline{x}\right]\subset\left[0,\infty\right[$ be the set of all level sets of X. Suppose $T:\left[\underline{x},\overline{x}\right]\rightarrow\left[\underline{x},\overline{x}\right]$ is a mapping. Then for any $i\in\mathbb{N}$, $\left[\underline{x}_i,\overline{x}_i\right]\in\left[\underline{x},\overline{x}\right]$ is called a fixed point of T if $T\left(\left[\underline{x}_i,\overline{x}_i\right]\right)=\left[\underline{x}_i,\overline{x}_i\right]$.

Definition 4.3[9]. Let (X,d) be a metric space. A mapping $T:X\rightarrow X$ is called contraction on X if there is a positive real number $k<1$ such that for all $x,y\in X$, $d(Tx,Ty)=kd(x,y)$.

Definition 4.4. Let $X\in I^{\mathbb{R}}$ be a fuzzy number and $\left[\underline{x},\overline{x}\right]\subset\left[0,\infty\right[$ be the set of all level sets of X. A mapping $T:\left[\underline{x},\overline{x}\right]\rightarrow\left[\underline{x},\overline{x}\right]$ is called contraction on $\left[\underline{x},\overline{x}\right]$ if there is a positive real number $K<1$ such that

$$d\left(T\left[\underline{x}_i,\overline{x}_i\right],T\left[\underline{x}_j,\overline{x}_j\right]\right)=k\,d\left(\left[\underline{x}_i,\overline{x}_i\right],\left[\underline{x}_j,\overline{x}_j\right]\right)$$

for all $i,j\in\mathbb{N}$ and $\left[\underline{x}_i,\overline{x}_i\right],\left[\underline{x}_j,\overline{x}_j\right]\in\left[\underline{x},\overline{x}\right]$.

Theorem 4.5. Let T be a contraction on $\left(\left[\underline{x},\overline{x}\right],d\right)$. Then T has a unique fixed point.

Proof: Let for an arbitrary $i\in\mathbb{N}$; $\left[\underline{x}_i,\overline{x}_i\right]\in\left[\underline{x},\overline{x}\right]$.

We define iterative sequence $\left(\left[\underline{x}_n,\overline{x}_n\right]\right)\in\left[\underline{x},\overline{x}\right]$ by

$$\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]=T\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_2,\overline{x}_2\right]=T\left[\underline{x}_1,\overline{x}_1\right],\left[\underline{x}_3,\overline{x}_3\right]=T\left[\underline{x}_2,\overline{x}_2\right],\ldots,\left[\underline{x}_n,\overline{x}_n\right]=T\left[\underline{x}_{n-1},\overline{x}_{n-1}\right],$$

then

$$\left[\underline{x}_2,\overline{x}_2\right]=TT\left[\underline{x}_0,\overline{x}_0\right]=T^2\left[\underline{x}_0,\overline{x}_0\right],$$

$$\left[\underline{x}_3,\overline{x}_3\right]=TT^2\left[\underline{x}_0,\overline{x}_0\right]=T^3\left[\underline{x}_0,\overline{x}_0\right],\ldots,\left[\underline{x}_n,\overline{x}_n\right]=T^n\left[\underline{x}_0,\overline{x}_0\right]$$

We shall show that sequence $\left(\left[\underline{x}_n,\overline{x}_n\right]\right)$ is a Cauchy sequence.

If $n > m$, then

$$d\left(\left[\underline{x}_{m+1},\overline{x}_{m+1}\right],\left[\underline{x}_{m},\overline{x}_{m}\right]\right)=d\left(T\left[\underline{x}_{m},\overline{x}_{m}\right],T\left[\underline{x}_{m-1},\overline{x}_{m-1}\right]\right)$$

$$\Rightarrow d\left(\left[\underline{x}_{m+1},\overline{x}_{m+1}\right],\left[\underline{x}_{m},\overline{x}_{m}\right]\right)\le k\,d\left(\left[\underline{x}_{m},\overline{x}_{m}\right],\left[\underline{x}_{m-1},\overline{x}_{m-1}\right]\right)$$

$$\Rightarrow d\left(\left[\underline{x}_{m+1},\overline{x}_{m+1}\right],\left[\underline{x}_{m},\overline{x}_{m}\right]\right)\le k^2\,d\left(\left[\underline{x}_{m-1},\overline{x}_{m-1}\right],\left[\underline{x}_{m-2},\overline{x}_{m-2}\right]\right)$$

$$\cdots\cdots\cdots$$

$$\Rightarrow d\left(\left[\underline{x}_{m+1},\overline{x}_{m+1}\right],\left[\underline{x}_{m},\overline{x}_{m}\right]\right)\le k^m\,d\left(\left[\underline{x}_{1},\overline{x}_{1}\right],\left[\underline{x}_{0},\overline{x}_{0}\right]\right)$$

By triangle inequality, we obtain for $n > m$

$$\Rightarrow d\left(\left[\underline{x}_{m},\overline{x}_{m}\right],\left[\underline{x}_{n},\overline{x}_{n}\right]\right)\le d\left(\left[\underline{x}_{m},\overline{x}_{m}\right],\left[\underline{x}_{m+1},\overline{x}_{m+1}\right]\right)+d\left(\left[\underline{x}_{m+1},\overline{x}_{m+1}\right],\left[\underline{x}_{m+2},\overline{x}_{m+2}\right]\right)$$
$$+\ldots+d\left(\left[\underline{x}_{n-1},\overline{x}_{n-1}\right],\left[\underline{x}_{n},\overline{x}_{n}\right]\right)$$

$$\le k^m\,d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{1},\overline{x}_{1}\right]\right)+k^{m+1}\,d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{1},\overline{x}_{1}\right]\right)+\ldots+k^{n-1}\,d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{1},\overline{x}_{1}\right]\right)$$

$$\le k^m\left(1+k+\ldots+k^{n-m-1}\right)d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{1},\overline{x}_{1}\right]\right)=k^m\,\frac{1-k^{n-m}}{1-k}\,d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{1},\overline{x}_{1}\right]\right)$$

Since $0<k<1$, so the number $1-k^{n-m}<1$.

Therefore $d\left(\left[\underline{x}_{m},\overline{x}_{m}\right],\left[\underline{x}_{n},\overline{x}_{n}\right]\right)\le \dfrac{k^m}{1-k}\,d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{1},\overline{x}_{1}\right]\right)$.

Since the space $\left[\underline{x},\overline{x}\right]$ is complete, there exists a $\left[\underline{x}_0,\overline{x}_0\right]\subset\left[\underline{x},\overline{x}\right]$ such that $\left[\underline{x}_n,\overline{x}_n\right]\to\left[\underline{x}_0,\overline{x}_0\right]$.

Now we show that this $\left[\underline{x}_0,\overline{x}_0\right]\subset\left[\underline{x},\overline{x}\right]$ is fixed under the mapping *T*.

By definition and triangle inequality we have:

$$d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],T\left[\underline{x}_{0},\overline{x}_{0}\right]\right)\le d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{n},\overline{x}_{n}\right]\right)+d\left(\left[\underline{x}_{n},\overline{x}_{n}\right],T\left[\underline{x}_{0},\overline{x}_{0}\right]\right)$$

$$\Rightarrow d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],T\left[\underline{x}_{0},\overline{x}_{0}\right]\right)\le d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{n},\overline{x}_{n}\right]\right)+k\,d\left(\left[\underline{x}_{n-1},\overline{x}_{n-1}\right],\left[\underline{x}_{0},\overline{x}_{0}\right]\right)$$

We know that $d(x,y)=0$, iff $x=y$.

Since $\left[\underline{x}_n,\overline{x}_n\right]\to\left[\underline{x}_0,\overline{x}_0\right]$, so

$d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],\left[\underline{x}_{n},\overline{x}_{n}\right]\right)\to 0$ and $d\left(\left[\underline{x}_{n-1},\overline{x}_{n-1}\right],\left[\underline{x}_{0},\overline{x}_{0}\right]\right)\to 0$ Which implies that $d\left(\left[\underline{x}_{0},\overline{x}_{0}\right],T\left[\underline{x}_{0},\overline{x}_{0}\right]\right)=0$, and hence $T\left[\underline{x}_{0},\overline{x}_{0}\right]=\left[\underline{x}_{0},\overline{x}_{0}\right]$. This shows that $\left[\underline{x}_{0},\overline{x}_{0}\right]$ fixed of *T*.

We shall now show that $\left[\underline{x}_{0},\overline{x}_{0}\right]$ is unique of *T*. Suppose that $\left[\underline{x}_{1},\overline{x}_{1}\right]$ is another fixed of *T*. Then $T\left[\underline{x}_{1},\overline{x}_{1}\right]=\left[\underline{x}_{1},\overline{x}_{1}\right]$.

Therefore $d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)=d\left(T\left[\underline{x}_0,\overline{x}_0\right],T\left[\underline{x}_1,\overline{x}_1\right]\right)\le k\,d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)$

Since $k<1$, this implies that $d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)$.

Hence $\left[\underline{x}_0,\overline{x}_0\right]=\left[\underline{x}_1,\overline{x}_1\right]\Rightarrow \underline{x}_0=\underline{x}_1;\overline{x}_0=\overline{x}_1$.

Thus the proof is complete.

4.1. Properties of contrition *T*:

(i) Limit $\left[\underline{x},\overline{x}\right]$ is a fixed of T

(ii) Fixed of *T* is unique

Proof 4.1.

(i) Since *T* is continuous we have

$$T\left[\underline{x},\overline{x}\right]=T\left(\lim_{n\to\infty}\left[\underline{x}_n,\overline{x}_n\right]\right)=\lim_{n\to\infty}\left(T\left[\underline{x}_n,\overline{x}_n\right]\right)=\lim_{n\to\infty}\left(\left[T\underline{x}_n,T\overline{x}_n\right]\right)=\lim_{n\to\infty}\left[\underline{x}_{n+1},\overline{x}_{n+1}\right]=\left[\underline{x},\overline{x}\right]$$

Since the limit of $\left(x_{n+1}\right)$ is same as limit of $\left(x_n\right)$. Thus Limit $\left[\underline{x},\overline{x}\right]$ is a fixed of *T*.

(ii) Fixed of T is unique.

Let $\left[\underline{x}_0,\overline{x}_0\right]$ is unique of *T*. Suppose that $\left[\underline{x}_1,\overline{x}_1\right]$ is another fixed of *T*. Then $T\left[\underline{x}_1,\overline{x}_1\right]=\left[\underline{x}_1,\overline{x}_1\right]$.

Therefore

$$d\left(T\left[\underline{x}_0,\overline{x}_0\right],T\left[\underline{x}_1,\overline{x}_1\right]\right)=d\left(\left[T\underline{x}_0,T\,\overline{x}_0\right],\left[T\underline{x}_1,T\,\overline{x}_1\right]\right)=d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)$$

Since *T* is contrition mapping for $0<k<1$.

$$d\left(T\left[\underline{x}_0,\overline{x}_0\right],T\left[\underline{x}_1,\overline{x}_1\right]\right)\le k\,d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)$$

Therefore we have $d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)\le k\,d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)$, this is possible only when

$$d\left(\left[\underline{x}_0,\overline{x}_0\right],\left[\underline{x}_1,\overline{x}_1\right]\right)=0\Rightarrow \max\left\{\left|\underline{x}_0-\underline{x}_1\right|,\left|\overline{x}_0-\overline{x}_1\right|\right\}=0\Rightarrow\left|\underline{x}_0-\underline{x}_1\right|=0,\left|\overline{x}_0-\overline{x}_1\right|=0$$

Therefore $\underline{x}_0=\underline{x}_1,\overline{x}_0=\overline{x}_1$. Hence the $\left[\underline{x}_0,\overline{x}_0\right]$ is unique.

5. Example

Example 5.1.

Let $\forall r\in[0,1]$; $\left[x\right]^r,\left[y\right]^r,\left[z\right]^r\in[a,b]\subset[0,\infty[$ and $\left([a,b],d\right)$ be a fuzzy metric space; and a function $f:[a,b]\to[a,b]$ be continuous and differentiable. Then the solution of the equation $f'(x)\le k<1$ is unique

Solution 5.1. Let $[x]^r,[y]^{r},[z]^r\in[a,\,b]$ and $[x]^r<[z]^r<[y]^r$.

Then we can write by (Lagrange's mean value theorem)

$$\frac{f[\underline{x},\overline{x}]-f\left[\underline{y},\overline{y}\right]}{[\underline{x},\overline{x}]-\left[\underline{y},\overline{y}\right]}=f'\left([\underline{z},\overline{z}]\right)$$

$$=> \frac{[f(\underline{x}),f(\overline{x})]-\left[f(\underline{y}),f(\overline{y})\right]}{[\underline{x},\overline{x}]-\left[\underline{y},\overline{y}\right]}=[f'(\underline{z}),f'(\overline{z})]$$

$$=> [f(\underline{x}),f(\overline{x})]-\left[f(\underline{y}),f(\overline{y})\right]=\left([\underline{x},\overline{x}]-\left[\underline{y},\overline{y}\right]\right)[f'(\underline{z}),f'(\overline{z})]$$

$$=> \left[f(\underline{x})-f(\underline{y}),f(\overline{x})-f(\overline{y})\right]=\left[\left(\underline{x}-\underline{y}\right)f'(\underline{z}),\left(\overline{x}-\overline{y}\right)f'(\overline{z})\right]$$

Implies $\left\|\left[f(\underline{x})-f(\underline{y}),f(\overline{x})-f(\overline{y})\right]\right\|=\left\|\left[\left(\underline{x}-\underline{y}\right)f'(\underline{z}),\left(\overline{x}-\overline{y}\right)f'(\overline{z})\right]\right\|$

$$=> \left[\left|f(\underline{x})-f(\underline{y})\right|,\left|f(\overline{x})-f(\overline{y})\right|\right]=\left[\left|\left(\underline{x}-\underline{y}\right)f'(\underline{z})\right|,\left|\left(\overline{x}-\overline{y}\right)f'(\overline{z})\right|\right]$$

$$=> \left[\left|f(\underline{x})-f(\underline{y})\right|,\left|f(\overline{x})-f(\overline{y})\right|\right]=\left[\left|\left(\underline{x}-\underline{y}\right)\right|\left|f'(\underline{z})\right|,\left|\left(\overline{x}-\overline{y}\right)\right|\left|f'(\overline{z})\right|\right]$$

As f is contraction mapping onto itself.

then $\left[\left|\underline{x}-\underline{y}\right|,\left|\overline{x}-\overline{y}\right|\right]=\left[\left|\underline{x}-\underline{y}\right|\left|f'(\underline{z})\right|,\left|\overline{x}-\overline{y}\right|\left|f'(\overline{z})\right|\right]$

$=> \left[\left|\underline{x}-\underline{y}\right|,\left|\overline{x}-\overline{y}\right|\right]\leq\left[\left|\underline{x}-\underline{y}\right|k,\left|\overline{x}-\overline{y}\right|k\right]$; as here we see $k\geq 1$.

$$=> \left[\left|\underline{x}-\underline{y}\right|,\left|\overline{x}-\overline{y}\right|\right]\leq[k,k]\left[\left|\underline{x}-\underline{y}\right|,\left|\overline{x}-\overline{y}\right|\right]=[k]\left[\left|\underline{x}-\underline{y}\right|,\left|\overline{x}-\overline{y}\right|\right]$$

Since it is given $0<k<1$. So the above relation is possible only when

$$\left[\left|\underline{x}-\underline{y}\right|,\left|\overline{x}-\overline{y}\right|\right]=[0,0]=\{0\}$$

$$=> \left[\underline{x}-\underline{y},\overline{x}-\overline{y}\right]=[0,0]$$

which implies that $\underline{x}=\underline{y}$ and $\overline{x}=\overline{y}$

Therefore the solution of the given equation is unique.

6. Conclusion

In the present work, properties of metric space i.e. limit, sequence and completeness are discussed in Level sets notion. The contraction of fuzzy metric space is defined, and further the existences and uniqueness are shown with an example.

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