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On Quotient BF-algebras via Interval-valued Fuzzy Ideals

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Abstract. The present paper gives a new construction of a quotient BF-algebra $X/\tilde{\mu}$ by a interval-valued fuzzy ideal $\tilde{\mu}$ in X and establishes that interval-valued fuzzy homomorphism, we show that if $\tilde{\mu}$ is a interval-valued fuzzy ideal of X, then $X/\tilde{\mu}$ is a BF-algebra X if and only if $\tilde{\mu}$ is a interval-valued fuzzy ideal of X and investigate some of its properties.

Keywords: BF-algebras, interval-valued fuzzy set, interval-valued fuzzy homomorphism.

AMS Mathematics Subject Classification (2010): 06F35, 03E72, 03G25

1. Introduction and preliminaries

The concept of fuzzy set was introduced by Zadeh [13]. Recently, Walendzik [12] defined BF-algebras. The notion of interval-valued fuzzy set was first introduced by Zadeh [14] as an extension of fuzzy sets. In [3], Liu and Meng constructed quotient BCI (BCK)-algebra via fuzzy ideals. In this paper we introduce the notion of quotient BF-algebra via interval-valued fuzzy ideals and investigate some interesting properties.
By a BF-algebra we mean an algebra satisfying the axioms:

(1) $x * x = 0$,

(2) $x * 0 = x$,

(3) $0 * (x * y) = y * x$ for all $x, y \in X$

Throughout this paper, X is a BF-algebra.

Example 1.1. Let R be the set of real number and let $A = (R, *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x, \text{if } y = 0 \\ y, \text{if } x = 0 \\ 0, \text{otherwise} \end{cases}$$

Definition 1.2. The subset I of X is said to be an ideal of X, if
(i) $0 \in I$ and (ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$.
We now review some fuzzy logic concepts. A fuzzy set in X is a function $\mu : X \to [0,1]$. For fuzzy sets $\mu \in X$ and $s \in [0,1]$. The sets $U(\mu; t) = \{x \in X : \mu(x) \geq t\}$ is called upper t-level cut of μ.

Definition 1.3. [4] A fuzzy set in a set S is a function μ from S into [0,1].

Definition 1.4. [5] The fuzzy set μ in X is called a fuzzy subalgebra of X, if $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$, for all $x, y \in X$.

Definition 1.5. A fuzzy set μ of X is called a fuzzy ideal of X if
(F1) $\mu(0) \geq \mu(x)$
(F2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ for all $x, y \in X$.
By interval number D we mean an interval $[a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. For interval numbers $D_1 = [a_1^-, b_1^+]$, $D_2 = [a_2^-, b_2^+]$.
We define

- $\min(D_1, D_2) = D_1 \cap D_2 = \min([a_1^-, b_1^+], [a_2^-, b_2^+])$
  $= [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$
- $\max(D_1, D_2) = D_1 \cup D_2 = \max([a_1^-, b_1^+], [a_2^-, b_2^+])$
  $= [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$

and put

- $D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D_1 < D_2 \Leftrightarrow D_1 \leq D_2$ and $D_1 \neq D_2$
- $mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+]$, where $0 \leq m \leq 1$.

It is obvious that $(D[0,1], \leq, \vee, \wedge)$ is a complete lattice with $[0,0]$ as its least element and $[1,1]$ as its greatest element. We now use $D[0,1]$ to denote the set of all closed sub intervals of the interval $[0,1]$.
For interval numbers $D_1 = [a_1^-, b_1^+], D_2 = [a_2^-, b_2^+] \in D[0,1]$ we define

- $D_1 + D_2 = [a_1^- + a_2^- - a_1^-.a_2^-, b_1^+ + b_2^+ - b_1^+.b_2^+]$.

Let L be a given nonempty set. An interval-valued fuzzy set B on L is defined by $B = \{(x, [\mu_B^-(x), \mu_B^+(x)]: x \in L\}$, Where $\mu_B^-(x)$ and $\mu_B^+(x)$ are fuzzy sets of L such that $\mu_B^-(x) \leq \mu_B^+(x)$ for all $x \in L$. Let $\tilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)]$, then $B = \{(x, \tilde{\mu}_B(x)): x \in L\}$ Where $\tilde{\mu}_B: L \to D[0, 1]$

Definition 1.6. The Interval-valued fuzzy set $\tilde{\mu}$ in X is called interval-valued fuzzy subalgebra of X, if $\tilde{\mu}(x * y) \geq \min\{\tilde{\mu}(x), \tilde{\mu}(y)\}$, for all $x, y \in X$.

Definition 1.7. An interval-valued fuzzy set $\tilde{\mu}$ is called interval-valued fuzzy ideal of BF-algebra X if satisfies the following inequality
(i-v F1) $\tilde{\mu}(0) \geq \tilde{\mu}(x)$
(i-v F2) $\tilde{\mu}(x) \geq \min\{\tilde{\mu}(x * y), \tilde{\mu}(y)\}$, for all $x, y, z \in X$.

Example 1.8. Consider a BF-algebra $X = \{0, a, b, c\}$ with following table

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let A be an interval-valued fuzzy set in X defined by $\tilde{\mu}(0) = \tilde{\mu}(a) = [0.6, 0.7]$ and $\tilde{\mu}(b) = \tilde{\mu}(c) = [0.2, 0.3]$, it is easy to verify that A is an interval-valued fuzzy ideal of X.

Proposition 1.9. Every interval-valued fuzzy ideal $\tilde{\mu}$ of X is order reversing.

Proposition 1.10. Let $\tilde{\mu}$ be a interval-valued fuzzy ideal of X. Then $x * y \leq z$ implies $\tilde{\mu}(x) \geq \min\{\tilde{\mu}(y), \tilde{\mu}(z)\}$ for all $x, y, z \in X$.

2. Quotient BF-algebras induced by interval-valued fuzzy ideals

Let $\tilde{\mu}$ be a interval-valued fuzzy ideal of X. For any $x, y \in X$, define relation $\sim$ on X by $x \sim y$ if and only if $\tilde{\mu}(x * y) = \tilde{\mu}(0)$ and $\tilde{\mu}(y * x) = \tilde{\mu}(0)$.

Lemma 2.1: $\sim$ is an equivalence relation of X.
Proof. (i) For any $x \in X$, we have $\tilde{\mu}(x * x) = \tilde{\mu}(0)$. Hence $x \sim x$.

(ii) For any $x, y \in X$, if $x \sim y$, then $\tilde{\mu}(x * y) = \tilde{\mu}(0)$ and $\tilde{\mu}(y * x) = \tilde{\mu}(0)$.
It means that $y \sim x$.

(iii) For any $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then

$$\tilde{\mu}(x * y) = \tilde{\mu}(y * x) = \tilde{\mu}(y * z) = \tilde{\mu}(z * y) = \tilde{\mu}(0)$$

Since $(x * z) * (x * y) \leq y * z$ and $(z * x) * (z * y) \leq y * x$, by proposition 1.10 we have $\tilde{\mu}(x * z) \geq \min\{\tilde{\mu}(x * y), \tilde{\mu}(y * z)\} = \tilde{\mu}(0)$ and $\tilde{\mu}(z * x) \geq \min\{\tilde{\mu}(z * y), \tilde{\mu}(y * x)\} = \tilde{\mu}(0)$, and so $\tilde{\mu}(x * z) = \tilde{\mu}(0)$ and $\tilde{\mu}(z * x) = \tilde{\mu}(0)$. Hence $x \sim z$. The proof is complete.

Lemma 2.2. $x \sim y$ implies $x * z \sim y * z$ and $z * x \sim z * y$ for all $x, y, z \in X$.

Proof: If $x \sim y$, then $\tilde{\mu}(x * y) = \tilde{\mu}(y * x) = \tilde{\mu}(0)$. Since $(x * z) * (y * z) \leq x * y$ and $(y * z) * (x * z) \leq y * x$, by Proposition 1.9 we have $\tilde{\mu}((x * z) * (y * z)) \geq \tilde{\mu}(x * y) = \tilde{\mu}(0)$ and $\tilde{\mu}((y * z) * (x * z)) \geq \tilde{\mu}(y * x) = \tilde{\mu}(0)$ and so $\tilde{\mu}((x * z) * (y * z)) = \tilde{\mu}(0)$ and $\tilde{\mu}((y * z) * (x * z)) = \tilde{\mu}(0)$. Thus $x * z \sim y * z$. Similarly, we can prove that $z * x \sim z * y$. The proof is finished.

Lemma 2.3. $x \sim y$ and $u \sim v$ imply $x * u \sim y * v$ for all $x, y, u, v \in X$.

Proof: If $x \sim y$ and $u \sim v$ by Lemma 2.2 $x * u \sim y * u$ and $x * v \sim y * v$. Using the transitivity of $\sim$, we get $x * u \sim y * v$. This completes the proof.

Summarizing the above lemmas we have following.

Theorem 2.4. $\sim$ is a congruence relation on X.

Definition 2.5. We write $\tilde{\mu}_x = \{y \in X / x * y\}$ the equivalence class containing x and $X/\tilde{\mu} = \{\tilde{\mu}_x / x \in X\}$ is the set of all equivalence classes of X. A binary operation $*$ on $X/\tilde{\mu}$ is defined by $\tilde{\mu}_x * \tilde{\mu}_y = \tilde{\mu}_{x*y}$.

Theorem 2.6. If $\tilde{\mu}$ is interval-valued fuzzy ideal of a BF-algebra X, then quotient algebra $(X/\tilde{\mu}; *, \tilde{\mu}_0)$ is BF-algebra.

Proof: For any $\tilde{\mu}_x, \tilde{\mu}_y \in X/\tilde{\mu}$, we have

(i) $\tilde{\mu}_x * \tilde{\mu}_x = \tilde{\mu}_{x*x} = \tilde{\mu}_0$,

(ii) $\tilde{\mu}_x * \tilde{\mu}_0 = \tilde{\mu}_{x*0} = \tilde{\mu}_x$,

$\tilde{\mu}_0 * (\tilde{\mu}_x * \tilde{\mu}_y) = \tilde{\mu}_{0*(x*y)} = \tilde{\mu}_{y*x}, \forall x, y \in X.$,

$(\tilde{\mu}_0 * \tilde{\mu}_x = \tilde{\mu}_{0*x} = \tilde{\mu}_0)$.

If $\tilde{\mu}_x * \tilde{\mu}_y = \tilde{\mu}_0$ and $\tilde{\mu}_y * \tilde{\mu}_x = \tilde{\mu}_0$, then $\tilde{\mu}_{x*y} = \tilde{\mu}_0$ and $\tilde{\mu}_{y*x} = \tilde{\mu}_0$ and so $\tilde{\mu}(x*y) = \tilde{\mu}(0) = \tilde{\mu}(y*x)$. Hence $x \sim y$. Then $\tilde{\mu}_x = \tilde{\mu}_y$. Thus we prove the theorem.

A mapping $f: X \to Y$ of BF-algebras is called a homomorphism if $f(x*y) = f(x)*f(y)$ for all $x, y \in X$, f is called an epimorphism, if it is a surjective homomorphism, f is called a monomorphisam if it is an injective homomorphism.

Proposition 2.6. Let $f: X \to Y$ be an epimorphism and $\tilde{v}$ an interval-valued fuzzy ideal of Y. Then $\tilde{v} \circ f$ is a Interval-valued fuzzy ideal of X.

Theorem 2.7. (Fundamental theorem of interval-valued fuzzy homomorphism) Let X and Y be BF-algebra, $f: X \to Y$ an epimorphism and $\tilde{v}$ an interval-valued fuzzy ideal of Y. Then $X/_{\tilde{v} \circ f} \cong Y/_{\tilde{v}}$.

Proof: By using theorem 2.5 and proposition 2.6, $X/_{\tilde{v} \circ f}$ and $Y/_{\tilde{v}}$ are BF-algebras.

Define $\eta: X/_{\tilde{v} \circ f} \to Y/_{\tilde{v}}$ by $\eta((\tilde{v} \circ f)_x) = \tilde{v}_{f(x)}$

(i) $(\tilde{v} \circ f)_x = (\tilde{v} \circ f)_y \Rightarrow (\tilde{v} \circ f)(x*y) = (\tilde{v} \circ f)(y*x) = (\tilde{v} \circ f)(0)$
$\Rightarrow \tilde{v}(f(x)*f(y)) = \tilde{v}(f(y)*f(x)) = \tilde{v}(0')$, where $0'$ is the zero element of Y $\Rightarrow \tilde{v}_{f(x)} = \tilde{v}_{f(y)}$. Hence η is well-defind.

(ii) $\eta((\tilde{v} \circ f)_x * (\tilde{v} \circ f)_y) = \eta((\tilde{v} \circ f)_{x*y}) = \tilde{v}_{f(x*y)} = \tilde{v}_{f(x)*f(y)} = \tilde{v}_{f(x)} * \tilde{v}_{f(y)}$.
$= \eta((\tilde{v} \circ f)_x * (\tilde{v} \circ f)_y)$. Hence η is a homomorphism.

(iii) η is a monomorphism: $\tilde{v}_{f(x)} = \tilde{v}_{f(y)} \Rightarrow \tilde{v}(f(x)*f(y)) = \tilde{v}(f(y)*f(x)) = \tilde{v}(0')$
$\Rightarrow (\tilde{v} \circ f)(x*y) = (\tilde{v} \circ f)(y*x) = (\tilde{v} \circ f)(0) \Rightarrow (\tilde{v} \circ f)_x = (\tilde{v} \circ f)_y$.

Hence $X/_{\tilde{v} \circ f} \cong Y/_{\tilde{v}}$ and the proof is complete.

Reviewing that the characteristic function of a subset I in X, we mean

$$\tilde{\chi}_I(x) = \begin{cases} [1,1], \text{if } x \in I \\ [0,0], \text{otherwise.} \end{cases}$$

In particular, $\tilde{\chi}_{\{0\}}$ is called the zero interval-valued fuzzy ideal of X. If f is an epimorhism, it is easy to verify that $\tilde{\chi}_{\{0\}} \circ f = \tilde{\chi}_{\ker f}$ and $Y/_{\tilde{\chi}_{\{0\}}} \cong Y$.

Applying theorem 2.7, we have the following

Corollary 2.8. Let X and Y be BF-algebras and $f : X \to Y$ an epimorphism. Then $X / \tilde{\chi}_{\ker f} \cong Y$.

REFERENCES

1. A.Borumand Soeid and M.A.Rezvani, On fuzzy BF-algebras, *International Mathematical Forum*, 4 (2009) 13-25.
2. Y.Imai, and K.Iseki, On axiom systems of propositional Calculi, XIV Proc. *Japan Academy*, 42 (1966) 19-22.
3. Y.L.Liu and J.Meng, Construction of quotient BCI (BCK)-algebras via a fuzzy ideal, *J. Appl. Math. & Comuting*, 10 (1-2) (2002) 51-62.
4. B**.**Satyanarayana, D.Ramesh, M.K.Vijaya Kumar and R.Durga Prasad, On fuzzy ideals in BF-algebras, *International J. of .Sci. & Engg. Appls.*, 4 (2010) 263-274.
5. B.Satyanarayana, M.V.Vijaya Kumar, D.Ramesh and R.Durga Prasad, Interval-valued intuitionistic fuzzy BF-subalgebras, *Acta Cienceia Indica*, *XXXVIII M*, 4 (2012) 637-644.
6. T.Senapati, M.Bhowmik, M.Pal and B.Davvaz, Fuzzy translations of fuzzy H-ideals in BCK/BCI-algbras, *J. Indones. Math. Soc.*, 21(1) (2015) 45-48.
7. T.Senapati, M.Bhowmik and M.Pal, Interval-valued intuitionistic fuzzy BG-subalgebras, *J. Uncert. Syst*., 8(1) (2014), 22-30.
8. T.Senapati, M.Bhowmik and M.Pal, Interval-valued intuitionistic fuzzy closed ideals BG-algebras and their products, *Int. J. Fuzzy Logic Syst*., 2(2) (2012) 27-44.
9. T.Senapati, M.Bhowmik and M.Pal, Fuzzy dot structure of BG-algebras, *Fuzzy Information and Engineering*, 6(3) (2014) 315-329.
10. T.Senapati, Bipolar fuzzy structure of BG-subalgebras, *J. Fuzzy Math*., 23(1) (2015), 209-220.
11. T.Senapati, M.Bhowmik and M.Pal, Triangular norm based fuzzy BG-algebras, *Afrika Math*., 27(1-2) (2016) 187-199.
12. A. Walendziak, On BF-algebras, *Math. Slovaca*, 57(2) (2007), 11.
13. L.A.Zadeh, Fuzzy sets, *Information Control*, 8 (1965) 338-353.
14. L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning, *Information Sciences*, 8 (1975) 199-249.