Intern. J. Fuzzy Mathematical Archive
Vol. 10, No. 2, 2016, 125-129
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 11 May 2016
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Results on Labeled Path and its Iterated Line Graphs

***M.S.Biradar*[1] and *V.R.Kulli*[2]**

[1]Department of Mathematics, Govt. First Grade College, Basavakalyan-585 327, India
e-mail: biradarmallikarjun@yahoo.co.in

[2]Department of Mathematics, Gulbarga University, Gulbarga-585106, India

Received 1 May 2016; accepted 10 May 2016

Abstract. In this paper, we present results on labeled path and its iterated line graphs.

***Keywords*:** labeled path, line graph, iterated line graph

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

All graphs considered here are finite, undirected and without loops or multiple lines. We use the terminology of [2].

The line graph of G, denoted $L(G)$, is the intersection graph $\Omega(X)$. Thus the points of $L(G)$ are the lines of G, with two points of $L(G)$ are adjacent whenever the corresponding lines of G are adjacent. We write $L^1(G)=L(G)$, $L^2(G)=L(L(G))$, and in general the iterated line graph is $L^n(G)=L(L^{n-1}(G))$. Many other graph valued functions in graph theory were studied, for example, in [4-10].

The following will be useful in the proof and discussion of our results.

Theorem A. [1, p.273] *Let G be a graph with p points and q lines. Then*

(i) *The degree in $L(G)$ of a line vw of G is $\deg v+\deg w-2$;*
(ii) *$L(P_p)\cong P_{p-1}$, for $p\geq 1$.*

Theorem B. [3] *A connected graph with $p\geq 2$ points is a nonempty path if and only if*

$$\sum_{i=1}^{p} d_i^2 = 4p-6.$$

2. Results on labeled graphs

In the following theorem, we deduce an equality satisfying the degree of any point of an iterated line graph of a labeled path.

Theorem 1. *Let P_n be a path with n $(n\geq 2)$ points labeled by $1,2,\ldots,n$ in sequence. Then the degree of the point u_k^m, the k^{th} point of an iterated line graph $L^m(P_n)$, where $1\leq k\leq (n-m)$ and $1\leq m<n$ satisfies the following equalities;*

$$du_k^m = d_k + \binom{m}{1} d_{k+1} + \binom{m}{2} d_{k+2} + ... + \binom{m}{m-1} d_{k+m-1} + \binom{m}{m} d_{k+m} - 2^{m+1} + 2$$

=0 when k=1 and m=n–1

=1 when k=1 or k= n–m

=2 when 1<k<(n–m), where d_i (k≤i≤k+m) is the degree of an i^{th} point of P_n.

Proof: Suppose P_n is a path with n (n≥2) points labeled by 1,2,…,n in sequence such that $d_1 = d_n = 1$ and $d_2 = d_3 = ... = d_{n-1} = 2$. Let u_k^m, $1 \le k \le (n-m)$ be the k^{th} point of an iterated line graph $L^m(P_n), 1 \le m < n$. We prove the result by using mathematical induction on m.

Suppose m=1. Then by Theorem *A*, $L^1(P_n) = L(P_n) = P_{n-1}$. Label these $(n-1)$ points by $u_1^1, u_2^1, ..., u_{n-1}^1$ in order that the points $u_1^1, u_2^1, ..., u_{n-1}^1$ represents the lines (1,2), (2,3), … , (n–1,n) of P_n respectively. By Theorem *A*, the degree in *L(G)* of a line (v,w) of *G* is $d_v + d_w - 2$. Thus,

$$du_1^1 = d_1 + d_2 - 2 = d_1 + \binom{1}{1} d_2 - 2^{1+1} + 2 = 1$$

$$du_2^1 = d_2 + d_3 - 2 = d_2 + \binom{1}{1} d_3 - 2^{1+1} + 2 = 2$$

$$du_{n-2}^1 = d_{n-2} + d_{n-1} - 2 = d_{n-2} + \binom{1}{1} d_{n-1} - 2^{1+1} + 2 = 2$$

$$du_{n-1}^1 = d_{n-1} + d_n - 2 = d_{n-1} + \binom{1}{1} d_n - 2^{1+1} + 2 = 1,$$

Since $d_1 = d_n = 1$ and $d_2 = d_3 = ... = d_{n-1} = 2$.

In general, $du_k^1 = d_k + \binom{1}{1} d_{k+1} - 2^{1+1} + 2$

=1 when k=1 or k=n–1

=2 when 1<k<(n–1).

Hence the result is true for m=1.

Suppose m=2. Then $L^2(P_n) = L(L(P_n)) = L(P_{n-1}) = P_{n-2}$. Label these $(n-2)$ points by $u_1^2, u_2^2, ..., u_{n-2}^2$ in order that the points $u_1^2, u_2^2, ..., u_{n-2}^2$ represents the lines $(u_1^1, u_2^1), (u_2^1, u_3^1), \ldots, (u_{n-2}^1, u_{n-1}^1)$ of *L(P$_n$)* respectively.

Consider,

$$du_1^2 = du_1^1 + du_2^1 - 2$$

$$= (d_1 + d_2 - 2) + (d_2 + d_3 - 2) - 2$$

(From the above case when m=1)

$$= d_1 + 2d_2 + d_3 - 6$$

$$= d_1 + \binom{2}{1} d_2 + \binom{2}{2} d_3 - 2^{2+1} + 2 = 1.$$

Now, $du_2^2 = du_2^1 + du_3^1 - 2$

$$= (d_2 + d_3 - 2) + (d_3 + d_4 - 2) - 2$$
$$= d_2 + 2d_3 + d_4 - 6$$
$$= d_2 + \binom{2}{1} d_3 + \binom{2}{2} d_4 - 2^{2+1} + 2 = 2$$

Similarly, $du_{n-3}^2 = d_{n-3} + \binom{2}{1} d_{n-2} + \binom{2}{2} d_{n-1} - 2^{2+1} + 2 = 2$

$$du_{n-2}^2 = d_{n-2} + \binom{2}{1} d_{n-1} + \binom{2}{2} d_n - 2^{2+1} + 2 = 1,$$

Since $d_1 = d_n = 1 \ and \ d_2 = d_3 = \cdots = d_{n-1} = 2.$

In general, $du_k^2 = d_k + \binom{2}{1} d_{k+1} + \binom{2}{2} d_{k+2} - 2^{2+1} + 2$

=1 when k=1 or k=n–2

=2 when 1<k<(n–2).

Hence the result is true for m=2.

Assume the result is true for $L^{m-1}(P_n)$, where 1<m<n.

We now prove the result is true for $L^m(P_n)$. By Theorem *A*, $L^m(P_n) = L(L^{m-1}(P_n))$ and $L^m(P_n)$ has $(n-m)$ points, where $m<n$. Label these $(n-m)$ points by $u_1^m, u_2^m, ..., u_{n-m}^m$ in order that the points $u_1^m, u_2^m, ..., u_{n-m}^m$ represents the lines (u_1^{m-1}, u_2^{m-1}), (u_2^{m-1}, u_3^{m-1}),…, $(u_{n-m}^{m-1}, u_{n-m+1}^{m-1})$ of $L^{m-1}(P_n)$ respectively. By inductive hypothesis, we have

$$du_k^{m-1} = d_k + \binom{m-1}{1} d_{k+1} + \binom{m-1}{2} d_{k+2} + ... + \binom{m-1}{m-1} d_{k+m-1} - 2^m + 2$$

≠0, since $m<n$ that is m–1≠ n–1

=1 when k=1 or k= n–(m–1)

=2 when 1<k<(n– (m–1)).

Consider,

$$du_1^m = du_1^{m-1} + du_2^{m-1} - 2$$
$$= d_1 + \binom{m-1}{1} d_2 + \binom{m-1}{2} d_3 + ... + \binom{m-1}{m-1} d_m - 2^m + 2$$
$$+ \ d_2 + \binom{m-1}{1} d_3 + \binom{m-1}{2} d_4 + ... + \binom{m-1}{m-2} d_m + \binom{m-1}{m-1} d_{m+1} - 2^m + 2 - 2$$

$$= d_1 + \left[\binom{m-1}{0} + \binom{m-1}{1}\right] d_2 + \left[\binom{m-1}{1} + \binom{m-1}{2}\right] d_3 + \ldots + \left[\binom{m-1}{m-2} + \binom{m-1}{m-1}\right] d_m$$

$$+ \binom{m-1}{m-1} d_{m+1} - 2 \cdot 2^m + 4 - 2 .$$

We know that $\binom{m}{r-1} + \binom{m}{r} = \binom{m+1}{r}$ and $\binom{m}{m} = \binom{m+1}{m+1} = 1$.

Therefore, $du_1^m = d_1 + \binom{m}{1} d_2 + \binom{m}{2} d_3 + \ldots + \binom{m}{m-1} d_m + \binom{m}{m} d_{m+1} - 2^{m+1} + 2$.

Similarly, $du_{n-m}^m = d_{n-m} + \binom{m}{1} d_{n-m+1} + \binom{m}{2} d_{n-m+2} + \ldots + \binom{m}{m} d_{m+n-m} - 2^{m+1} + 2$.

In general,

$$du_k^m = d_k + \binom{m}{1} d_{k+1} + \binom{m}{2} d_{k+2} + \ldots + \binom{m}{m-1} d_{k+m-1} + \binom{m}{m} d_{k+m} - 2^{m+1} + 2$$

=0 when k=1 and m=n–1

=1 when k=1 or k= n–m

=2 when 1<k<(n–m). This completes the proof of the theorem.

Corollary 2. *Let P_n be a path with n (n≥2) points labeled by 1,2,…,n in sequence. Then the degree of an isolated point u of the (n–1)th iterated line graph $L^{n-1}(P_n)$ satisfies the following condition;*

$du = d_1 + \binom{n-1}{1} d_2 + \binom{n-2}{2} d_3 + \ldots + \binom{n-1}{n-2} d_{n-1} + \binom{n-1}{n-1} d_n - 2^n + 2 = 0$, *where d_i ($1 \le i \le n$) is the degree of an i^{th} point of P_n.*

Illustration 3. The above Corollary is illustrated by taking a path with 5 points. The points of the path and its iterated line graphs are labeled as shown in Figure 1.

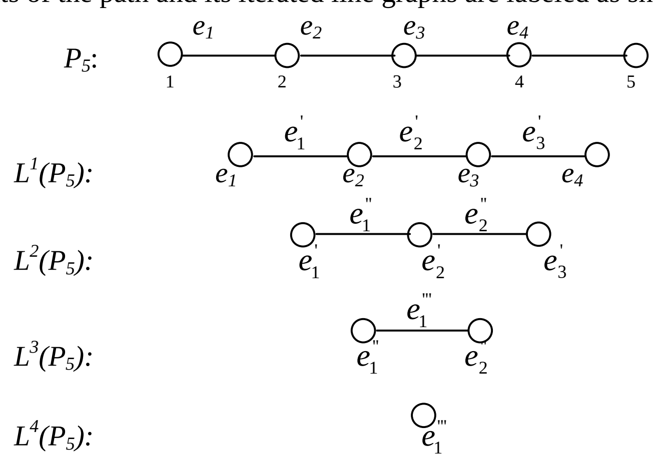

Figure 1:

By Theorem *A*, the degree in *L(G)* of a line *vw* of *G* is d_v+d_w-2. Now consider,

$$de_1''' = de_1'' + de_2'' - 2$$
$$=(de_1' + de_2' - 2) +(de_2' + de_3' - 2)-2$$
$$= de_1' + 2de_2' + de_3' - 6$$
$$= (de_1 + de_2 - 2) + 2(de_2 + de_3 - 2) + (de_3 + de_4 - 2) - 6$$
$$= de_1 + 3de_2 + 3de_3 + de_4 - 14$$
$$= (d_1 + d_2 - 2) + 3(d_2 + d_3 - 2) + 3(d_3 + d_4 - 2) + (d_4 + d_5 - 2) - 14$$
$$= d_1 + 4d_2 + 6d_3 + 4d_4 + d_5 - 30 .$$

Since $d_1 = d_5 = 1$ and $d_2 = d_3 = d_4 = 2$, we have

$$de_1''' = d_1 + \binom{4}{1} d_2 + \binom{4}{2} d_3 + ... + \binom{4}{3} d_4 + \binom{4}{4} d_5 - 2^5 + 2 = 0 .$$

The Theorem B and Theorem 1 lead to the following result:

Theorem 4. *Let P_n be a path with n ($n\geq 2$) points labeled by 1, 2 ,…, n in sequence and u_k^m be the k^{th} point of $L^m(P_n)$, where $1\leq k\leq (n-m)$, $1\leq m<n$ and $m\neq n-1$. Then* $\sum_{i=1}^{n-m} du_k^m = 4(n-m) - 6.$

REFERENCES

1. L.W.Beineke and R.J.Wilson, Selected topics in graph theory, *Academic Press Inc. (London) Lt*d. (1978).
2. V.R. Kulli, *College Graph Theory,* Vishwa International Publications, Gulbarga, India (2012).
3. V.R.Kulli, A characterization of paths, *The Mathematics Education*, 9 (1975) 1-2.
4. V.R.Kulli and M.S.Biradar, The middle blict graph of a graph, *International Research Journal of Pure Algebra,* 5(7) (2015) 111-117.
5. V.R.Kulli and M.S.Biradar, The point block graph of n'a graph, *Journal of Computer and Mathematical Sciences,* 5 (5) (2014) 476-481.
6. V.R.Kulli and M.S.Biradar, Planarity of the point block graph of a graph, *Ultra Scientist,* 18, 609-611 (2006).
7. V.R.Kulli and M.S.Biradar, The point block graphs and crossing numbers, *Acta Ciencia Indica,* 33(2) (2007) 637-640.
8. V.R.Kulli and M.S.Biradar, The blict graph and blitact graph of a graph, *Journal of Discrete Mathematical Sciences & Cryptography,* 4(2-3) (2001) 151-162.
9. V.R.Kulli and M.S.Biradar, The line splitting graph of a graph, *Acta Ciencia Indica,* XXVIII M (3) (2002) 435.
10. V.R.Kulli and M.S.Biradar, On Eulerian blict graphs and blitact graphs, *Journal of Computer and Mathematical Sciences,* 6(12) (2015) 712-717.