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Product of k-EP Block Matrices in Minkowski Space

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Abstract. Necessary and sufficient conditions for the product of k-EP matrices of rank r to be k-EP matrix in Minkowski space ***m*** is derived. Also equivalent conditions for the product of two k-EP block matrices to be k-EP are established. As an application we have shown that a block matrix in Minkowski space can be expressed as a product of k-EP matrices in *m*.

Keywords: Minkowski space, Range symmetric matrices

AMS mathematics Subject Classification (2010): 15A57

1. Introduction

Throughout we shall deal with $C^{n\times n}$ the space of complex n-tuples. Let G be the Minkowski metric tensor defined by $Gx = (x_1, -x_2,-x_3,\ldots,-x_n)^T$ for $x = (x_1,x_2,x_3,\ldots,x_n) \in C^n$. Clearly the Minkowski metric matrix

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix} \quad G=G^* \text{ and } G^2=I_2 \tag{1.1}$$

Minkowski inner product on C^n is defined by $(u,v) = [u,Gv]$, where $[.,.]$ denotes the conventional Hilbert space inner product. A space with Minkowski inner product is called a Minkowski space and denoted as ***m***. With respect to the Minkowski inner product, since

$$(Ax,y) = (x,A^{\sim}y),\ A^{\sim} = GA^*G \tag{1.2}$$

is called the Minkowski adjoint of the matrix $A\in C^{n\times n}$ and A^* is the usual Hermitian adjoint.

(P.1) For $A_1,A_2\in C^{n\times n}$, $(A_1+A_2)^{\sim} = A_1^{\sim} + A_2^{\sim}$, $(A_1A_2)^{\sim} = A_2^{\sim} A_1^{\sim}$ and $(A_1^{\sim})^{\sim} = A_1$

Let A^- be a generalized inverse ($AA^-A=A$) and $A^{\dagger}$ is the Moore Penrose of A [9].

A matrix A is called EP_r, if $rk(A) = r$ and $N(A) = N(A^*)$. It is well known that (p.163[1]) A is EP if and only if $AA^{\dagger} = A^{\dagger}A$. The concept of EP matrices over the field of complex number was introduced by Schwerdfeger where rk(A), N(A) and R(A) denote the rank of A, null space of A and range space of A respectively.

Throughout let 'k' be a fixed product of disjoint transpositions in $S_n = \{1,2,\dots.n\}$ and K be the associated permutation matrix satisfying

(P.2) $K^2 = I_n$

(P.3) $K^{\sim} = GK^*G = GKG = K$

A matrix $A = (a_{ij}) \in C^{n\times n}$ is k-hermitian if $a_{ij} = \bar{a}_{k(j),k(i)}$ for i,j = 1,2,…,n. A theory for k-hermitian matrices is developed in [3]. The concept of k-EP matrices is introduced in [6] as a generalization of k-hermitian matrices and as an extension of complex EP matrices ([1]). For $x = (x_1,x_2,\dots,x_n)^T \in C^n$,let $\boldsymbol{k}(x) = (x_{k(1)}, x_{k(2)}, \dots,x_{k(n)})^T \in C^n$.

A matrix $A\in C^{n\times n}$ is said to be k-EP if it satisfies the condition $Ax = 0 \Leftrightarrow A^*\boldsymbol{k}(x) = 0$ (or) equivalently $N(A) = N(A^*K)$. Moreover, A is said to be k-EP_r if A is k-EP and rk (A) = r. Let $k = k_1k_2$ as in Lemma (2.12) of [6], where k_1 is the product of disjoint transpositions on $S_n = \{1,2,\dots.n\}$ leaving (r+1,r+2,…,n) fixed and k_2 is the product of disjoint transpositions leaving (1,2,…,r) fixed. Then the associated permutation matrix of $k = k_1k_2$ is

$$K = \begin{pmatrix} K_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & K_2 \end{pmatrix} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \tag{1.3}$$

where $\begin{pmatrix} K_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}$ and $\begin{pmatrix} I_r & 0 \\ 0 & K_2 \end{pmatrix}$

are the permutation matrices corresponding to the transpositions k_1 and k_2 respectively. For further properties of k-EP matrices one may refer [6]. Let us partition the Minkowski metric tensor G of order n in conformity with that of K in (1.3) as

$G = \begin{pmatrix} G_1 & 0 \\ 0 & -I_{n-r} \end{pmatrix}$ where G_1 is the Minkowski metric tensor of order as that of A.

In [5] the concept of range symmetric matrix in Minkowski space $\boldsymbol{m}$ is introduced and developed analogous to that of EP matrices.A matrix $A\in C^{n\times n}$ is said to be range symmetric matrix in $\boldsymbol{m} \Leftrightarrow N(A) = N(A^{\sim})$. In our earlier work [7], we have introduced the concept of k-EP matrices in Minkowski space as an extension of EP matrix in $\boldsymbol{m}$. In this paper we have introduced the product of two k-EP matrices in $\boldsymbol{m}$ of rank r to be k-EP in $\boldsymbol{m}$ and k-EP block matrices in $\boldsymbol{m}$.

2. Preliminaries

Lemma 2.1. Let A and B be matrices in $\boldsymbol{m}$. Then $N(A^*) \subseteq N(B^*) \Leftrightarrow N(A^{\sim}) \subseteq N(B^{\sim})$.

Theorem 2.2. [4] For $A,B,C \in C^{m\times n}$, the following are equivalent:

(1) CA^-B is invariant for every $A^- \in C^{n\times m}$
(2) $N(A) \subseteq N(C)$ and $N(A^*) \subseteq N(B^*)$
(3) $C = CA^-A$ and $B = AA^-B$ for every $A^-\in \{1\}$

Definition 2.3. [4] Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ (2.1)

be an n×n matrix. A generalized schur complement of A in M denoted by M/A is defined as $D\text{-}CA^{-}B$, where A^{-} is a generalized inverse of A.

If $CA^{-}B$ is invariant for all choice of g inverse of A, then this reduces to the Schur complement $M/A = D\text{-}CA^{\dagger}B$,where $A^{\dagger}$ the Moore Penrose inverse of A is the unique solution of the equations $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$,$AA^{\dagger}$ and $A^{\dagger}A$ are Hermitian.

Hence forth we are concerned with n×n matrices M partitioned in the form (2.1) with rk(M) = rk(A) = r

It is well known that [2] M of the form (2.1) satisfies $N(A) \subseteq N(C) \Leftrightarrow C = YA = CA^{-}A$ similarly $N(A^*) \subseteq N(B^*) \Leftrightarrow B = AX = AA^{-}B$ and $D = CA^{\dagger}B = YAX$. Hence (2.1) can be written as

$$M = \begin{pmatrix} A & AX \\ YA & YAX \end{pmatrix}$$

Definition 2.4. [7] A matrix $A \in C^{n\times n}$, is said to be k-EP in $\boldsymbol{m}$ if and only if $N(A) = N(A^{\sim}K)$.

Lemma 2.5. [6] For $A \in C^{n\times n}$, the following are equivalent:
(1) A is k-EP
(2) KA is EP
(3) AK is EP
(4) $KA^{\dagger}A = AA^{\dagger}K$

Remark 2.6. In particular, when k(i) = i for each i,j = 1 to n, then K = I and above Theorem reduces to $A^{\dagger}A = AA^{\dagger}$ which implies that A is an EP matrix ([1], p-163)

Lemma 2.7. [7] For $A \in C^{n\times n}$, the following are equivalent:
(1) A is k-EP in $\boldsymbol{m}$
(2) GA is k-EP
(3) AG is k-EP

Theorem 2.8. [8] Let M be of the form (2.1), with rk(M) = rk(A) = r then M is k-EP matrix in $\boldsymbol{m}$ with $k = k_1k_2 \Leftrightarrow$ A is k_1-EP in $\boldsymbol{m}$ and $CA^{\dagger}K_1 = -G_1(A^{\dagger}BK_2)^{\sim}$

3. Product of k-EP matrices in Minkowski space:

In this section we have obtained necessary and sufficient conditions for the product of two k-EP matrices in $\boldsymbol{m}$ of rank r to be k-EP in $\boldsymbol{m}$. Later we have extended the result to k-EP block matrices in $\boldsymbol{m}$.

Theorem 3.1. Let A and B be k-EP matrices in $\boldsymbol{m}$ of rank r and AB be of rank r. Then AB is k-EP matrix in $\boldsymbol{m}$ of rank r if and only if N(A) = N(B).

Proof: AB is k-EP matrix in $\boldsymbol{m}$ of rank r

$\Rightarrow N(AB) = N(AB)^{\sim}K$ (by Definition 2.4)

$\Rightarrow N(B) = N(B^{\sim}A^{\sim})K$, since $rk(B) = rk(AB) = r$ (by P.1)

$\Rightarrow N(B) \subseteq N(A^{\sim}K)$

$\Rightarrow N(B) \subseteq N(A)$ (by Definition 2.4)

$\Rightarrow N(B) = N(A)$, since $rk(A) = rk(B) = r$

Conversely,

Let N(A) = N(B). To prove that AB is k-EP in $\boldsymbol{m}$. Clearly $N(AB) \subseteq N(B)$. Since $rk(AB) = rk(B) = r$, we get

$N(AB) = N(B)$ (3.1)

$N((AB)^{\sim}K) = N(B^{\sim}A^{\sim})K \subseteq N(A^{\sim}K) = N(A)$ (by Definition 2.4)

Now $N(A) = N(A^{\sim}K) \Rightarrow rk(A^{\sim}K) = rk(A) = r$

$N(B) = N(B^{\sim}K) \Rightarrow rk(B^{\sim}K) = rk(B) = r$

$N((AB)^{\sim}K) \subseteq N(A)$

$rk(AB)^{\sim}K = rk(AB)^{\sim} = rk(AB) = r$

$N((AB)^{\sim}K) = N(A)$, since $rk(A) = r$ (3.2)

From (3.1) and (3.2) we get

$N(AB) = N(AB)^{\sim}K$, since $N(A) = N(B)$

Thus AB is k-EP in $\boldsymbol{m}$.

Theorem 3.2. Let A, B and AB be k-EP matrices in $\boldsymbol{m}$ of rank r and BA is of rank r, then BA is k-EP_r matrix in $\boldsymbol{m}$.

Proof: Let A, B and AB be k-EP matrices in $\boldsymbol{m}$ and $rk(AB) = rk(A) = rk(B) = r$

We claim BA is k-EP_r in $\boldsymbol{m}$.

$N(BA) \subseteq N(A)$

$rk(BA) = rk(A) = r$

Therefore $N(BA) = N(A)$ (3.3)

$N(AB) \subseteq N(B)$

$rk(AB) = rk(B) = r$

Therefore $N(AB) = N(B)$ (3.4)

by Theorem (3.1), $N(A) = N(B)$

Therefore $N(BA) = N(AB)$ (3.5)

Also $N(BA)^{\sim}K = N(A^{\sim}B^{\sim})K \subseteq N(B^{\sim}K) = N(B)$

$N(BA)^{\sim}K \subseteq N(B) = N(AB)$

$N(BA)^{\sim}K \subseteq N(AB)$

$rk(BA)^{\sim}K = rk(BA)^{\sim} = rk(BA) = rk(A) = r$

$N(BA)^{\sim}K = N(AB)$ (3.6)

From (3.5) and (3.6) it follows that $N(BA) = N(BA)^{\sim}K$

Therefore BA is k-EP_r matrix in $\boldsymbol{m}$.

Lemma 3.3. For complex matrices A and B, $N(A^*K) \subseteq N(B^*K)$ if and only if $N(A^{\sim}K) \subseteq N(B^{\sim}K)$

Proof: Let us assume that $N(A^*K) \subseteq N(B^*K)$ we need to prove $N(A^{\sim}K) \subseteq N(B^{\sim}K)$

Let us choose $x \in N(A^{\sim}K) \Rightarrow A^{\sim}Kx = 0$

$\Rightarrow GA^*GKx = 0$

$\Rightarrow A^*GKx = 0$

$\Rightarrow A^*KKGKx = 0$ (by P.2)

$\Rightarrow A^*Ky = 0$, where $y = KGKx$, and hence $Ky = GKx$

$\Rightarrow y \in N(A^*K) \subseteq N(B^*K)$

$\Rightarrow B^*Ky = 0$

$\Rightarrow B^*GKx = 0$

$\Rightarrow GB^*GKx = 0$

$\Rightarrow B^{\sim}Kx = 0$

$\Rightarrow x \in N(B^{\sim}K)$

Thus $N(A^{\sim}K) \subseteq N(B^{\sim}K)$.

Conversely, let us assume that $N(A^{\sim}K) \subseteq N(B^{\sim}K)$.

We need to prove that $N(A^*K) \subseteq N(B^*K)$

Let us choose $x \in N(A^*K) \Rightarrow A^*Kx = 0$

$\Rightarrow GA^*GGKx = 0$

$\Rightarrow A^{\sim}GKx = 0$

$\Rightarrow A^{\sim} Ky = 0$, where $y = KGKx$

$\Rightarrow y \in N(A^{\sim}K) \subseteq N(B^{\sim}K)$.

$\Rightarrow B^{\sim}Ky = 0$

$\Rightarrow GB^*GKy = 0$

$\Rightarrow GB^*GGKx = 0$

$\Rightarrow B^*Kx = 0$

$\Rightarrow x \in N(B^*K)$

Thus $N(A^*K) \subseteq N(B^*K)$. Hence the result.

Lemma 3.4. Let M be of the form (2.1) be k-EP in $\boldsymbol{m}$ with $k = k_1k_2$ A is k_1-EP in $\boldsymbol{m}$ and there exists an $r\times(n-r)$ matrix X such that

$$M = \begin{pmatrix} A & AX \\ -G_1K_2 \sim X \sim K_1A & -G_1K_2 \sim X \sim K_1AX \end{pmatrix}$$

Proof: Since M is of the form (2.1) by using Lemma 3.3 and Theorem2.2, M satisfy $N(A) \subset N(C)$, $N(A^*) \subseteq N(B^*) \Leftrightarrow N(A^*K_1) \subseteq N(B^*K_1) \Leftrightarrow N(A^{\sim}K_1) \subset N(B^{\sim}K_1)$ and $D = CA^{\dagger}B$. Hence there exist $(n-r) \times r$ matrix Y and $n \times (n-r)$ matrix X such that $C = YA$ and $B=AX$. Since A is k_1-EP in $\boldsymbol{m}$ by using Lemma (2.7) and Remark (2.6), K_1G_1A is EP

$(K_1G_1A)(K_1G_1A)^{\dagger} = (K_1G_1A)^{\dagger}(K_1G_1A)$

$K_1G_1AA^{\dagger}G_1K_1 = A^{\dagger}G_1K_1K_1G_1A_1$, since $G_1 = G_1^{\dagger}$, $K_1 = K_1^{\dagger}$

$= A^{\dagger}A$ (by (1.1) & (P.2)) (3.7)

$CA^{\dagger}K_1 = -G_1K_2^{\sim}(A^{\dagger}B)^{\sim}$ (by Theorem 2.8)

$CA^{\dagger} = -G_1K_2^{\sim}(A^{\dagger}B)^{\sim}K_1$

$= -G_1K_2^{\sim}(A^{\dagger}AX)^{\sim}K_1$ (by $B = AA^{-}B = AX$)

$= -G_1K_2^{\sim}X^{\sim}(A^{\dagger}A)^{\sim}K_1$ (by (P.1))

$= -G_1K_2^{\sim}X^{\sim}G_1(A^{\dagger}A)^*G_1K_1$ (by (1.2))

$CA^{\dagger} = -G_1 K_2^{\sim} X^{\sim} K_1AA^{\dagger}$ (by (3.7))

$C = CA^{\dagger}A = -G_1 K_2^{\sim} X^{\sim} K_1AA^{\dagger}A$ (since $N(A) \subseteq N(C)$)

Therefore $C = -G_1 K_2^{\sim} X^{\sim} K_1A$ and $D = CA^{\dagger}B = -G_1 K_2^{\sim} X^{\sim} K_1AA^{\dagger}AX$

$= -G_1 K_2^{\sim} X^{\sim} K_1AX$.

Thus

$$M = \begin{pmatrix} A & AX \\ -G_1K_2 \sim X \sim K_1A & -G_1K_2 \sim X \sim K_1AX \end{pmatrix}$$

Theorem 3.5.

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $L = \begin{pmatrix} F & U \\ H & K \end{pmatrix}$**,** be k-EP in $\boldsymbol{m}$ with $k = k_1k_2$ both of the form (2.1) and ML be of rank r, then the following are equivalent

(i) ML is k-EP in $\boldsymbol{m}$ with $k= k_1k_2$

(ii) AF is k_1-EP in $\boldsymbol{m}$ $CA^{\dagger} = HF^{\dagger}$

(iii) AF is k_1-EP in $\boldsymbol{m}$ $A^{\dagger}B = F^{\dagger}U$

Proof: Since M and L are of the form (2.1) by Lemma (3.4) there exists $r\times(n-r)$ matrices X and Y such that

$$M = \begin{pmatrix} A & AX \\ -G_1K_2^{\sim} X^{\sim}K_1A & -G_1K_2^{\sim} X^{\sim}K_1AX \end{pmatrix}$$

$$L = \begin{pmatrix} F & FY \\ -G_1K_2^{\sim} Y^{\sim}K_1F & -G_1K_2^{\sim} Y^{\sim}K_1FY \end{pmatrix}$$

$$\text{Now } ML = \begin{pmatrix} A & AX \\ -G_1K_2^{\sim} X^{\sim}K_1A & -G_1K_2^{\sim} X^{\sim}K_1AX \end{pmatrix}\begin{pmatrix} F & FY \\ -G_1K_2^{\sim} Y^{\sim}K_1F & -G_1K_2^{\sim} Y^{\sim}K_1FY \end{pmatrix}$$

$$= \begin{pmatrix} A(I-XG_1K_2^{\sim} Y^{\sim}K_1)F & A(I-XG_1K_2^{\sim} Y^{\sim}K_1)FY \\ -G_1K_2^{\sim} X^{\sim}K_1A(I-XG_1K_2^{\sim} Y^{\sim}K_1)F & -G_1K_2^{\sim} X^{\sim}K_1A(I-XG_1K_2^{\sim} Y^{\sim}K_1)FY \end{pmatrix}$$

$$= \begin{pmatrix} AZF & AZFY \\ -G_1K_2^{\sim} X^{\sim}K_1AZF & -G_1K_2^{\sim} X^{\sim}K_1AZFY \end{pmatrix}$$

where $Z = I-XG_1K_2^{\sim} Y^{\sim}K_1$.

Clearly, $N(AZF) \subset N(-G_1K_2^{\sim} X^{\sim}K_1AZF) = N(G_1K_2^{\sim} X^{\sim}K_1AZF)$,

$N(AZF)^{\sim} \subset N(AZFY)^{\sim}$ and the schur complement of AZF in ML is zero. For

$ML/AZF = -G_1K_2^{\sim} X^{\sim} K_1AZFY + G_1K_2^{\sim} X^{\sim} K_1AZF(AZF)^{\dagger}(AZF)Y$

$= -G_1K_2^{\sim} X^{\sim} K_1AZFY + G_1K_2^{\sim} X^{\sim} K_1AZFY = 0$

Hence $rk(AZF) = rk(ML) = r$. Thus ML is also of the form (2.1). Since M and L are k-EP in $\boldsymbol{m}$. By Theorem (2.8) and Lemma(3.4), A and F are k_1-EP in $\boldsymbol{m}$. Now

$-G_1(K_1(A^{\dagger}AX)K_2)^{\sim} = -G_1K_2^{\sim}(A^{\dagger}AX)^{\sim}GK_1G$ (by (P.1) & (P.3))

$= -G_1K_2^{\sim}(A^{\dagger}AX)^{\sim}K_1$ (by (P.3))

$= -G_1K_2^{\sim}X^{\sim}(A^{\dagger}A)^{\sim}K_1$ (by (P.1))

$= -G_1K_2^{\sim}X^{\sim}G_1(A^{\dagger}A)^*G_1K_1$ (by (1.2))

$= -G_1K_2^{\sim}X^{\sim}K_1K_1G_1(A^{\dagger}A)^*G_1K_1$ (by (P.2))

$= -G_1K_2^{\sim}X^{\sim}K_1K_1G_1(A^{\dagger}A)G_1K_1$

$= -G_1K_2^{\sim}X^{\sim}K_1AA^{\dagger}$ (by (3.7))

Similarly it can be proved that $-G_1K_2^{\sim}Y^{\sim}K_1FF^{\dagger} = -G_1(K_1(F^{\dagger}FY)K_2)^{\sim}$. We now claim AZF is k_1-EP in $\boldsymbol{m}$. $N(F) \subset N(AZF)$, and $rk(AZF) = rk(F) = r$, hence it follows $N(F) = N(AZF)$. Also $N(A^{\sim}K_1) \subset N(AZF)^{\sim}K_1$ and $rk((AZF)^{\sim}K_1) = rk(AZF) = rk(A) = r = rk(F)$, $N(A) = N(A^{\sim}K_1) = N(AZF)^{\sim}K_1 = N(F)$.

Thus $N(AZF) = N(AZF)^{\sim}K_1$ and hence AZF is k_1-EP in $\boldsymbol{m}$. By Lemma (2.5) and Lemma (2.7) K_1G_1AZF is EPr and by using (3.7) we have

$$K_1G_1AZF(AZF)^{\dagger}G_1K_1 = (AZF)^{\dagger}AZF \quad (3.8)$$

By using (3.8) for $N(AZF) = N(F)$, $N(AZF)^{\sim}K_1 = N(A^{\sim}K_1) = N(A)$ we get

$(AZF)(AZF)^{\dagger} = FF^{\dagger} = K_1G_1(AZF)^{\dagger}AZF\, G_1K_1$ and

$$(AZF)^{\dagger}AZF = A^{\dagger}A = K_1G_1AA^{\dagger}G_1K_1 \quad (3.9)$$

Since $H = -G_1K_2^{\sim}Y^{\sim}K_1F$ and $C = -G_1K_2^{\sim}X^{\sim}K_1A$.

We have

$HF^{\dagger} = -G_1K_2^{\sim}Y^{\sim}K_1FF^{\dagger}$ (by 3.9)

$= -G_1K_2^{\sim}Y^{\sim}K_1K_1G_1(AZF)^{\dagger}(AZF)G_1K_1$

$= -G_1K_2^{\sim}Y^{\sim}[(AZF)^{\dagger}(AZF)]^{\sim}K_1$ (by P.2)

$= -G_1K_2^{\sim}Y^{\sim}[(AZF)^{\dagger}(AZF)]^{\sim}K_1^{\sim}$ (by P.3)

$= -G_1[K_1((AZF)^{\dagger}(AZF))YK_2]^{\sim}$

Similarly by using (3.9), we have

$CA^{\dagger} = -G_1K_2^{\sim}X^{\sim}K_1AZF(AZF)^{\dagger}$

Therefore

$$CA^{\dagger} = HF^{\dagger} \Leftrightarrow -G_1K_2^{\sim}X^{\sim}K_1AZF(AZF)^{\dagger} = -G_1[K_1((AZF)^{\dagger}(AZF))YK_2]^{\sim} \quad (3.10)$$

Now the proof runs as follows:

ML is k-EP in $\boldsymbol{m} \Leftrightarrow$ AZF is k_1-EP in $\boldsymbol{m}$ and

$G_1K_2^{\sim}X^{\sim}K_1AZF(AZF)^{\dagger} = G_1[K_1((AZF)^{\dagger}(AZF))YK_2]^{\sim}$

$\Leftrightarrow$ AZF is k_1-EP in $\boldsymbol{m}$ and $CA^{\dagger} = HF^{\dagger}$ (by 3.10)

$\Leftrightarrow N(AZF)=N((AZF)^{\sim}K_1)$ and $CA^{\dagger} = HF^{\dagger}$

$\Leftrightarrow N(F)=N(A^{\sim}K_1) = N(A)$ and $CA^{\dagger} = HF^{\dagger}$

$\Leftrightarrow$ AF is k_1-EP in $\boldsymbol{m}$ and $CA^{\dagger} = HF^{\dagger}$ (by Theorem 3.1)

$\Leftrightarrow$ AF is k_1-EP in $\boldsymbol{m}$ and $A^{\dagger}B = F^{\dagger}U$ (by Theorem 2.8)

Hence the theorem.

Theorem 3.6. Let A and B be k-EP in $\boldsymbol{m}$ of rank r. Then $N(A) = N(B)$ if and only if $N(PAP^{\sim}K) = N(PBP^{\sim}K)$ where P is unitary in unitary space.

Proof: Let A and B be k-EP in $\boldsymbol{m}$

Assume N(A) = N(B)
We prove $N(PAP^{\sim}K) = N(PBP^{\sim}K)$
$x\in N(PAP^{\sim}K) \Leftrightarrow PAP^{\sim}Kx = 0$
$\Leftrightarrow AP^{\sim}Kx = 0$
$\Leftrightarrow Ay = 0$ where $y = P^{\sim}Kx$
$\Leftrightarrow y\in N(A) = N(B)$
$\Leftrightarrow By = 0$
$x\in N(PAP^{\sim}K) \Leftrightarrow BP^{\sim}Kx = 0$
$\Leftrightarrow PBP^{\sim}Kx = 0$
$\Leftrightarrow x\in N(PBP^{\sim}K)$
Thus $N(A) = N(B) \Rightarrow N(PAP^{\sim}K) = N(PBP^{\sim}K)$
Conversely we assume that
$N(PAP^{\sim}K) = N(PBP^{\sim}K)$ we claim N(A) = N(B)
$x\in N(A) \Leftrightarrow Ax = 0$
$\Leftrightarrow AP^{\sim}Ky = 0$
$\Leftrightarrow PAP^{\sim}Ky = 0$
$\Leftrightarrow y\in N(PAP^{\sim}K) = N(PBP^{\sim}K)$
$\Leftrightarrow PBP^{\sim}Ky = 0$
$\Leftrightarrow BP^{\sim}Ky = 0$
$\Leftrightarrow Bx = 0$
Thus $N(PAP^{\sim}K) = N(PBP^{\sim}K) \Rightarrow N(A) = N(B)$.

Product Decomposition of k-EP matrices in *m*

Theorem 3.7. Let M be of the form (2.1) be k-EP in ***m*** with $k = k_1k_2$. Then M can be written as a product of k-EP matrices in ***m***.

Proof: Since M is of the form (2.1) and M is k-EP in ***m***, by Lemma (3.4), A is k_1-EP in ***m***

and $$M = \begin{pmatrix} A & AX \\ -G_1K_2^{\sim}X^{\sim}K_1A & -G_1K_2^{\sim}X^{\sim}K_1AX \end{pmatrix}$$

Since M is k-EP in ***m***, by Lemma (2.7), GM is k-EP$_r$, where

$$GM = \begin{pmatrix} G_1 & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & AX \\ -G_1K_2^{\sim}X^{\sim}K_1A & -G_1K_2^{\sim}X^{\sim}K_1AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & G_1AX \\ G_1K_2^{\sim}X^{\sim}K_1A & G_1K_2^{\sim}X^{\sim}K_1AX \end{pmatrix}$$

Consider $P = \begin{pmatrix} G_1AA^{\dagger}G_1 & G_1AA^{\dagger}G_1K_1X\,K_2 \\ K_2X^*K_1G_1AA^{\dagger}G_1 & K_2X^*K_1G_1AA^{\dagger}G_1K_1X\,K_2 \end{pmatrix}$, $\quad L = \begin{pmatrix} G_1A & 0 \\ 0 & 0 \end{pmatrix}$,

$$Q = \begin{pmatrix} A^{\dagger}A & A^{\dagger}AX \\ X^* A^{\dagger}A & X^* A^{\dagger}AX \end{pmatrix} \qquad \text{By using (1.1)}$$

$$P^* = \begin{pmatrix} (G_1AA^{\dagger}G_1)^* & (K_2X^*K_1G_1AA^{\dagger}G_1)^* \\ (G_1AA^{\dagger}G_1K_1X\,K_2)^* & (K_2X^*K_1G_1AA^{\dagger}G_1K_1X\,K_2)^* \end{pmatrix}$$

$$= \begin{pmatrix} G_1(AA^{\dagger})^*G_1 & G_1(AA^{\dagger})^*G_1K_1X\,K_2 \\ K_2X^*K_1G_1(AA^{\dagger})^*G_1 & K_2X^*K_1G_1(AA^{\dagger})^*G_1K_1X\,K_2 \end{pmatrix}$$

$$= \begin{pmatrix} G_1AA^{\dagger}G_1 & G_1AA^{\dagger}G_1K_1X\,K_2 \\ K_2X^*K_1G_1AA^{\dagger}G_1 & K_2X^*K_1G_1AA^{\dagger}G_1K_1X\,K_2 \end{pmatrix} = P.$$

Similarly $Q^* = Q$ can be proved. Thus $P = P^*$ and $Q = Q^*$ and therefore P, Q are EP_r. Since A is k-EPr in $\boldsymbol{m}$ by Lemma (2.7) G_1A is EP_r and hence L is EP_r. Now

$$PLQ = \begin{pmatrix} G_1AA^{\dagger}G_1 & G_1AA^{\dagger}G_1K_1X\,K_2 \\ K_2X^*K_1G_1AA^{\dagger}G_1 & K_2X^*K_1G_1AA^{\dagger}G_1K_1X\,K_2 \end{pmatrix}\begin{pmatrix} G_1A & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A^{\dagger}A & A^{\dagger}AX \\ X^* A^{\dagger}A & X^* A^{\dagger}AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1AA^{\dagger}G_1G_1A & 0 \\ K_2X^*K_1G_1AA^{\dagger}G_1G_1A & 0 \end{pmatrix}\begin{pmatrix} A^{\dagger}A & A^{\dagger}AX \\ X^* A^{\dagger}A & X^* A^{\dagger}AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & 0 \\ K_2X^*K_1G_1A & 0 \end{pmatrix}\begin{pmatrix} A^{\dagger}A & A^{\dagger}AX \\ X^* A^{\dagger}A & X^* A^{\dagger}AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1AA^{\dagger}A & G_1A\,A^{\dagger}AX \\ K_2X^*K_1G_1AA^{\dagger}A & K_2X^*K_1G_1AA^{\dagger}AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & G_1AX \\ K_2X^*K_1G_1A & K_2X^*K_1G_1AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & G_1AX \\ G_1(K_1XK_2)^{\sim}A & G_1(K_1XK_2)^{\sim}AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & G_1AX \\ G_1K_2^{\sim}X^{\sim}K_1^{\sim}A & G_1K_2^{\sim}X^{\sim}K_1^{\sim}AX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & G_1AX \\ G_1K_2^{\sim}X^{\sim}G\,K_1GA & G_1K_2^{\sim}X^{\sim}G\,K_1GAX \end{pmatrix}$$

$$= \begin{pmatrix} G_1A & G_1AX \\ G_1K_2{}^{\sim}X^{\sim}K_1A & G_1K_2{}^{\sim}X^{\sim}K_1AX \end{pmatrix}$$

$= GM.$

By using (1.1), M = GPLQ = (GP)(LG)(GQ). Since P, Q, L are k-EP by Lemma (2.7), it follows that GP, LG, GQ are k-EP in $\boldsymbol{m}$. Thus k-EP matrix M in $\boldsymbol{m}$ is expressed as a product of k-EP matrices in $\boldsymbol{m}$.

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