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Secondary κ-Kernel Symmetric Fuzzy Matrices

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Abstract. In this paper, characterizations of secondary κ- kernel symmetric fuzzy matrices are obtained. Relation between s- κ- kernel symmetric, s- kernel symmetric, κ-kernel symmetric and kernel symmetric matrices are discussed. Necessary and sufficient conditions are determined for a matrix to be s- κ- kernel symmetric.

Keywords: Fuzzy matrices, kernel symmetric, s-κ- kernel symmetric

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1. Introduction

All matrices considered in this paper are fuzzy matrices, that is, matrices over a fuzzy algebra $\mathcal{F}$ with support [0, 1] under max-min operations. A fuzzy matrix A is range symmetric if $R(A) = R(A^T)$ and kernel symmetric if $N(A) = N(A^T)$. It is well known that for complex matrix, the concept of range and kernel symmetric are same. However this fails for fuzzy matrices. This motivated us to study on s- κ- kernel symmetric matrices. Lee [1] has initiated the study of secondary symmetric matrices, that is matrices whose entries are symmetric about the secondary diagonal. Cantoni and Paul [2] have studied persymmetric matrices, that is matrices which are symmetric about both the diagonals and their applications to communication theory. Hill and Waters [3] have developed a theory of κ-real and κ-hermitian matrices as a generalization of s-real and s-hermitian matrices. A development of κ- kernel symmetric fuzzy matrices is made by Meenakshi and Jayashree [5] analogous to that of k-real and k-hermitian of a complex matrix [3] .

Throughout let κ-be a fixed product of disjoint transpositions in $S_n = \{1,2,\dots,n\}$ and K be the associated permutation matrix. A matrix A=(a_{ij})∈$\mathcal{F}_n$ is κ-symmetric if $a_{ij} = a_{k(j)k(i)}$ for i, j = 1 to n. Meenakshi and krishnamoorthy[6] have introduced the concept of s-k hermitian matrices as a generalization of secondary hermitian and hermitian matrices. In this paper, we extend the concept of s- κ- kernel symmetric fuzzy matrices as a particular case of the results on complex matrices found in [7].

2. Preliminaries

Throughout let 'V' be the permutation matrix with units in its secondary diagonal and let 'κ' be a fixed product of disjoint transpositions in $S_n = \{1,2,\dots,n\}$ and K be the associated permutation matrix. For $x = (x_1, x_2, ..., x_n)^T \in \mathcal{F}_{n1}$ let us define the function $\Re(x) = (x_{\kappa(1)}, x_{\kappa(2)}, ... x_{\kappa(n)})^T \in \mathcal{F}_{n1}$. Since K is involutory, it can be verified that the associated permutation matrix satisfy the following properties.

(P.2.1) $KK^T = K^T K = I_n, K = K^T, K^2 = I$ and $\Re(x) = Kx$

By the definition of V,

(P.2.2) $V = V^T, VV^T = V^T V = I_n$ and $V^2 = I$

(P.2.3) $N(A) = N(AV)$, $N(A) = N(AK)$

(P.2.4) $(AV)^T = VA^T, (VA)^T = A^T V$

If A^+ exists, then

(P.2.5) $(AV)^+ = VA^+, (VA)^+ = A^+ V$

Definition 2.1. [4] $A \in \mathcal{F}_n$ is kernel symmetric matrix if and only if $N(A) = N(A^T)$.

Lemma 2.1. [[4] P. 119] For $A \in \mathcal{F}_n$ and a permutation matrix P, $N(A) = N(B)$ if and only if $N(PAP^T) = N(PBP^T)$.

Lemma 2.2. [5] A matrix $A \in \mathcal{F}_n$ is κ- kernel symmetric $\Leftrightarrow$ KA is kernel symmetric $\Leftrightarrow$ AK is kernel symmetric.

3. Secondary κ-kernel symmetric fuzzy matrices

Definition 3.1. A matrix $A \in \mathcal{F}_n$ is s-symmetric if and only if $A = VA^T V$.

Definition 3.2. A matrix $A \in \mathcal{F}_n$ is s-kernel symmetric if $N(A) = N(VA^T V)$.

Definition 3.3. A matrix $A \in \mathcal{F}_n$ is s- κ-kernel symmetric if $N(A) = N(KVA^T VK)$.

Lemma 3.1. A matrix $A \in \mathcal{F}_n$ is s-kernel symmetric $\Leftrightarrow$ VA is kernel symmetric $\Leftrightarrow$ AV is kernel symmetric.

Proof.

A is s-kernel symmetric	$\Leftrightarrow N(A) = N(VA^T V)$	[By Definition 3.2]
	$\Leftrightarrow N(AV) = N((AV)^T)$	[By P.2.2]
	$\Leftrightarrow AV$ is kernel symmetric	
	$\Leftrightarrow N(VAVV^T) = N(VVA^T V)$	[By Lemme2.1]
	$\Leftrightarrow N(VA) = N((VA)^T)$	[By P.2.2]
	$\Leftrightarrow VA$ is kernel symmetric.	

Remark 3.1. In particular when κ(i) = i for i = 1, 2, …, n then the associated permutation matrix K reduces to the identity matrix and Definition (3.3) reduces to $N(A) = N(VA^T V)$ which implies that A is s-kernel symmetric matrices.

Remark 3.2. For $\kappa(i) = n - i + 1$, the corresponding permutation matrix K reduces to V and Definition (3.3) reduces to $N(A) = N(A^T)$ which implies that A is kernel symmetric.

Remark 3.3. We note that s- κ-symmetric matrix is s-κ-kernel symmetric for if A is s-κ-symmetric then $A = KVA^TVK$ Hence $N(A) = N(KVA^TVK)$ which implies that A is s-κ -kernel symmetric. However the converse need not be true. This is illustrated in the following example.

Example 3.1. For κ = (1,2), $A = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.5 \end{bmatrix}$ is symmetric

$$KVA^TVK = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.5 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.6 \\ 0.6 & 1 \end{bmatrix} \neq A$$

Here $A = KA^TK$ therefore A is symmetric, κ -symmetric, s- κ -kernel symmetric but not s- κ -symmetric.

Example 3.2. For κ = (1,2), $V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$A = \begin{bmatrix} 0.4 & 0.5 \\ 0.5 & 0.4 \end{bmatrix}$ is symmetric, s-κ-symmetric and hence therefore s- κ-kernel symmetric.

Example 3.3. For κ = (1,2)(3) $K = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ here

$K \neq V, K \neq I$ and $KV \neq VK$.

Now $A = \begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 1 & 0 \\ 0.5 & 0.3 & 0 \end{bmatrix}$ is s- κ-kernel symmetric but not s- κ -symmetric.

$$KVA^TVK = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0.5 & 1 & 0 \\ 0.5 & 0.3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0.3 & 0 \\ 0 & 0 & 0.4 \\ 0.5 & 0.5 & 0 \end{bmatrix} \neq A$$

Hence A is not s- κ-symmetric. But $N(A) = N(KVA^T VK) = \{0\}$.

Theorem 3.1. For $A \in \mathcal{F}_n$ the following are equivalent

(1) A is s- κ -kernel Symmetric
(2) KVA is kernel symmetric
(3) AKV is kernel symmetric
(4) AVK is kernel symmetric
(5) VKA is kernel symmetric
(6) VA is κ-kernel symmetric
(7) AV is κ -kernel symmetric
(8) AK is s-kernel symmetric
(9) KA is s-kernel symmetric
(10) $N(A^T) = N(KVA)$
(11) $N(A) = N(KVA^T)$

Proof:

(1) ⇔ (4) ⇔ (5) ⇔ (9)

A is s- κ-kernel symmetric ⇔ $N(A) = N(KVA^T VK)$ [By Definition 3.2]
⇔ $N(A) = N(KVA^T)$ [By P.2.3]
⇔ $N(AVK) = N((AVK)^T)$
⇔ AVK is kernel symmetric
⇔ $(VK)(AVK)(VK)^T$ is kernel symmetric [By Lemma 2.1]
⇔ VKA is kernel symmetric
⇔ KA is s-kernel symmetric

Thus (1) ⇔ (4) ⇔ (5) ⇔ (9) hold.

(2) ⇔ (6)

KVA is kernel symmetric ⇔ VA is κ -kernel symmetric

Thus (2) ⇔ (6) hold.

(2) ⇔ (10)

KVA is kernel symmetric ⇔ $N(KVA) = N((KVA)^T)$
⇔ $N(KVA) = N(A^T)$ [By P.2.3]

Thus (2) ⇔ (10) hold.

(4) ⇔ (11)

AVK is kernel symmetric ⇔ $N(AVK) = N((AVK)^T)$
⇔ $N(A) = N(KVA^T)$ [By P.2.3]

Thus (4) ⇔ (11) hold.

(1) ⇔ (4) ⇔ (7)

A is s- κ -kernel symmetric ⇔ $N(A) = N(KVA^T VK)$
⇔ $N(A) = N((AVK)^T)$
⇔ $N(AVK) = N((AVK)^T)$

$\Leftrightarrow AVK$ is kernel symmetric

$\Leftrightarrow AV$ is κ -kernel symmetric. Thus (1) ⇔ (4) ⇔ (7) hold.

(3) ⇔ (8)

AKV is kernel symmetric ⇔ AK is s- κ -kernel symmetric.

Hence the Theorem.

In Particular for $K = I$, the above Theorem reduces to the equivalent condition for a matrix to be secondary kernel symmetric.

Corollary 3.1. For $A \in \mathcal{F}_n$ the following are equivalent

(1) A is s-kernel symmetric

(2) VA is kernel symmetric

(3) AV is kernel symmetric

(4) $N(A^T) = N(VA)$

(5) $N(A) = N(VA^T)$

Lemma 3.2. Let $A \in \mathcal{F}_n$, if A^+ exists ⇔ $(KA)^+$ exists ⇔ $(VKA)^+$ exists.

Proof:

A^+ exists ⇔ $(KA)^+$ exists [follows from Lemma 3.4 in [8]]

$\Leftrightarrow KA = (KA)(KA)^T(KA)$

$\Leftrightarrow VKA = (VKA)(KA)^T VV(KA)$

$\Leftrightarrow VKA = (VKA)(VKA)^T(VKA)$

$\Leftrightarrow (VKA)^T \in (VKA)\{1\}$

$\Leftrightarrow (VKA)^+$ exists.

Lemma 3.2. Let $A \in \mathcal{F}_n$, if A^+ exists ⇔ $(KA)^+$ exists ⇔ $(VKA)^+$ exists.

Proof:

A^+ exists ⇔ $(KA)^+$ exists [follows from Lemma 3.4 in [8]]

$\Leftrightarrow KA = (KA)(KA)^T(KA)$

$\Leftrightarrow VKA = (VKA)(KA)^T VV(KA)$

$\Leftrightarrow VKA = (VKA)(VKA)^T(VKA)$

$\Leftrightarrow (VKA)^T \in (VKA)\{1\}$

$\Leftrightarrow (VKA)^+$ exists.

Remark 3.4. For $A \in \mathcal{F}_n$, A^+ exists ⇔ $(KVA)^+$ exists.

Theorem 3.2. Let $A \in \mathcal{F}_n$. Then any two of the following conditions imply the other one.

(1) A is κ -kernel symmetric

(2) A is s- κ -kernel symmetric

(3) $N(A^T) = N((KAV)^T)$

Proof:

(1) and (2) ⇒ (3)

A is s- κ -kernel symmetric $\Rightarrow N(A) = N((AVK)^T)$ [By Theorem 3.1]

$\Rightarrow N(KAK) = N(VA^TK)$ [By Lemma 2.1]

A is κ -kernel symmetric $\Rightarrow N(A) = N(KA^TK)$

$\Rightarrow N(KAK) = N(A^T)$ [By Lemma 2.1]

Hence (1) and (2) $\Rightarrow N(A^T) = N((KAV)^T)$

Thus (3) hold.

(1) and (3) $\Rightarrow$ (2)

A is κ -kernel symmetric $\Rightarrow N(KAK) = N(A^T)$

Hence (1) and (3) $\Rightarrow N(KAK) = N((KAV)^T)$

$\Rightarrow N(A) = N(KVA^T)$ [By Lemma 2.1]

$\Rightarrow A$ is s- κ -kernel symmetric

[By Theorem 3.1]

Thus (2) hold.

(2) and (3) $\Rightarrow$ (1)

A is s- κ-kernel symmetric $\Rightarrow N(A) = N(KVA^T)$

$\Rightarrow N(KAK) = N(VA^TK)$ [By Lemma 2.1]

Hence (2) and (3) $\Rightarrow N(KAK) = N(A^T)$

$\Rightarrow N(A) = N(KA^TK)$

$\Rightarrow A$ is κ -kernel Symmetric

Thus (1) hold. Hence the theorem.

REFERENCES

1. A.Lee, Secondary Symmetric, Secondary Skew Symmetric, Secondary Orthogonal Matrices, *Period Math, Hungary*, 7 (1976) 63-76.
2. C.Antonio and B.Paul, Properties of the eigen vectors of persymmetric matrices with applications to communication theory, *IEEE Trans. Comm.*, 24 (1976) 804 –809.
3. R.D. Hill and S.R.Waters, On k-Real and k-Hermitian matrices, *Linear Algebra and its Applications*, 169 (1992) 17-29.
4. AR.Meenakshi, *Fuzzy Matrix: Theory and Applications*, MJP Publishers, Chennai, 2008.
5. AR.Meenakshi and D.Jaya Shree, On k-kernel symmetric matrices, *International Journal of Mathematics and Mathematical Sciences*, 2009, Article ID 926217, 8 Pages.
6. AR.Meenakshi and S.Krishanmoorthy, On Secondary k-Hermitian matrices, *Journal of Modern Science*, 1 (2009) 70-78.
7. AR. Meenakshi, S.Krishnamoorthy and G.Ramesh, On s-k-EP matrices, *Journal of Intelligent System Research*, 2(2008) 93-100.
8. AR.Meenakshi and D.Jaya Shree, On K-range symmetric matrices, *Proceedings of the National conference on Algebra and Graph Theory*, MS University, (2009), 58-67.