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# Commutators in Indefinite Inner Product Spaces

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***Abstract.*** Star dagger matrices is extended to indefinite inner product spaces by introducing the commutator of a pair of matrices under the indefinite matrix multiplication and some of its properties are investigated. Sums and products of J-SD matrices to be J-SD are determined. Partial isometry of matrices with respect to the indefinite matrix product are discussed.

***Keywords:*** Indefinite matrix product, indefinite inner product spaces, commutators

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## 1. Introduction

An indefinite inner product in $C^n$ is a conjugate symmetric sesquilinear form $(x, y)$ together with the regularity condition that $(x, y) = 0$ for all $y \in C^n$ only when $x = 0$. Associated with any indefinite inner product is a unique invertible Hermitian matrix J (called a weight) with complex entries such that $(x, y) = < x, Jy >$, where $< , >$ denotes the Euclidean inner product on $C^n$ and vice versa. We also make an additional assumption on $J$, that is $J^2 = I$, to compare our results with the Euclidean case and to present the results with much algebraic ease. There are two different values for dot product of vectors in indefinite inner product spaces. To overcome this difficulty a new matrix product, called indefinite matrix multiplication is introduced and some of its properties are investigated in [6].More precisely, the indefinite matrix product of two complex matrices A and B of sizes $m \times n$ and $n \times l$ respectively.is defined to be the matrix $A \circ B = AJ_nB$. The adjoint of A, denoted as $A^{[*]}$ is defined to be the matrix$J_nA^*J_m$, where $A^*$ is the Hermitianadjoint, $J_n$ and $J_m$ are weights in the appropriate spaces. The aim of this manuscript is to extend star dagger matrices to indefinite inner product spaces by introducing the commutator of a pair of complex matrices under the indefinite matrix multiplication. This class of J-SD matrices includes the class of partial isometries. We recall the definitions and preliminary results in section 2. In section 3, we begin with the definition of J-SD matrices and explore some of its properties. We determine conditions under which indefinite matrix product, usual matrix product and sums of J-SD to be J-SD. In [3], the concept of EP matrices has been extended to indefinite matrix product as J-EP matrices. In this paper, we have obtained equivalence conditions for a matrix to be EP as well as J-EP in terms of its commutators. Inter relations between the class of bi-normal, bi-EP, bi-dagger and J-SD matrices are investigated. Results available in the literature ([2], [3], [5]) are deduced as a special case. Partial isometry is extended to indefinite inner product space.

## 2. Preliminaries

**Definition 2.1.** Let $A \in C^{m\times n}, B \in C^{n\times l}$. Let $J_n$ be an arbitrary but fixed $n \times n$ complex matrix such that $J_n = J_n{}^* = J_n{}^{-1}$. The indefinite matrix product of A and B (relative to $J_n$) is defined as $A \circ B = AJ_nB$.

**Remark 2.1.** When $J_n\ is\ identity$ matrix the product reduces to the usual product of matrices. It can be easily verified that with respect to the indefinite matrix product, $rank(A \circ A^{[*]}) = rank(A^{[*]} \circ A) = rank(A)$, where as this rank property fails under the usual matrix multiplication. Thus the Moore-Penrose inverse of a complex matrix $A$ exits over an indefinite inner product space, with respect to the indefinite matrix product and this is one of the main advantages of the indefinite matrix product.

**Definition 2.2.** For $A \in C^{n\times n}$, $A^{[*]} = JA^*J$ is the adjoint of $A$relative to $J$.

**Definition 2.3.** For $A \in C^{n\times n}$, $A$ is said to be J-invertible if there exists $X \in C^{n\times n}$ such that

$$A \circ X = X \circ A = J.$$

**Definition 2.4. [6]** For $A \in C^{m\times n}$, a matrix $X$ is called the Moore-Penrose if it satisfies the following equations:$A \circ X \circ A = A, X \circ A \circ X = X, (AX)^{[*]} = AX$and $(XA)^{[*]} = XA$. Such an $X$ is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]} = J_nA^{\dagger}J_m$.

**Definition 2.5. [3]** For $A \in C^{n\times n}$, $A$is J-EP if $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$and$A$is EP if $AA^{\dagger} = A^{\dagger}A$.

Thus J-EP is an extension of EP in the indefinite inner product space.In [4], Reverse order law in indefinite inner product space and orderings on matrices are studied in detail.

## 3. Main results

A complex matrix $A \in C^{n\times n}$ is said to be a star dagger (SD) matrix, if the star of $A$ commutes with the dagger of $A$, that is $A^*A^{\dagger} = A^{\dagger}A^*$, where $A^*$ is the Hermitian adjoint and $A^{\dagger}$is the Moore-Penrose inverse of $A$[1]. First we extend this concept to indefinite inner product spaces, by introducing the commutator.

**Definition 3.1.** For a pair of square complex matrices $A$ and $B$ of same order, the commutator of $A$ and $B$ with respect to $J$ is defined by $[A, B]_J = A \circ B - B \circ A$.

**Definition 3.2.**: Let $A \in C^{n\times n}$. $A$is said to be J-SD if $[A^{[*]}, A^{[\dagger]}]_J = 0$.

**Remark 3.1.** In particular, for $J = I_n$, $A^{[*]} = A^*$ and $A^{[\dagger]} = A^{\dagger}$. Definition 3.2 of J-SD matrix reduces to a SD matrix.

**Example 3.1.** Let us consider $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and J $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .
Here $AJ \neq JA$.
$A^* = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $A^\dagger = \frac{1}{trace(A^*A)} \cdot A^* = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.
$[A^*, A^\dagger] = 0$. Hence $A$ is SD.
$A^* \circ A^\dagger = A^\dagger J A^* = A^\dagger \circ A^*$. Hence $[A^*, A^\dagger]_J = 0$.
$A^{[*]} \circ A^{[\dagger]} = (JA^*J)J(JA^\dagger J) = JA^*JA^\dagger J = J(A^* \circ A^\dagger)J$.
$A^{[\dagger]} \circ A^{[*]} = (JA^\dagger J)J(JA^*J) = JA^\dagger JA^*J = J(A^\dagger \circ A^*)J$.
Thus $A$ is SD and $A$ is J-SD.

**Example 3.2.** Let us consider $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and J $= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ .

For this $A, A^\dagger = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.
$A^*A^\dagger = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = A^\dagger = A^\dagger A^*$.
$A$ is SD
$A^* \circ A^\dagger = \begin{bmatrix} 1 & 0 \\ 1/2 & 0 \end{bmatrix}$; $A^\dagger \circ A^* = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}$
$A^* \circ A^\dagger \neq A^\dagger \circ A^*$

$J(A^* \circ A^\dagger)J \neq J(A^\dagger \circ A^*)J$
Hence $A^{[*]} \circ A^{[\dagger]} \neq A^{[\dagger]} \circ A^{[*]}$
Thus $A$ is SD but $A$ is not J-SD.

**Theorem 3.1.** For $A \in C^{n\times n}$, $[A^{[*]}, A^{[\dagger]}]_J = 0$ if and only if $[A^*, A^\dagger]_J = 0$
**Proof:** $[A^{[*]}, A^{[\dagger]}]_J = 0 \Leftrightarrow A^{[*]} \circ A^{[\dagger]} = A^{[\dagger]} \circ A^{[*]}$ (By Definition 3.1)
$\Leftrightarrow (JA^*J)J(JA^\dagger J) = (JA^\dagger J)J(JA^*J)$ (By Definition 2.2 and Definition 2.4)
$\Leftrightarrow A^*JA^\dagger = A^\dagger JA^*$
$\Leftrightarrow A^* \circ A^\dagger = A^\dagger \circ A^*$ (By Definition 2.1)
$\Leftrightarrow [A^*, A^\dagger]_J = 0$(By Definition 3.1).

**Theorem 3.2.** For $A \in C^{n\times n}$,
(i) $A$ is J-SD $\Leftrightarrow$ $JA$is SD $\Leftrightarrow$ $AJ$ is SD.
(ii) $JA$ is J-SD $\Leftrightarrow$ $A$ is SD $\Leftrightarrow$ $AJ$ is J-SD.

**Proof:**
**(i)** $A$ is J-SD $\Leftrightarrow [A^{[*]}, A^{[\dagger]}]_J = 0$
$\Leftrightarrow [A^*, A^\dagger]_J = 0$ (By Theorem 3.1)
$\Leftrightarrow A^* \circ A^\dagger = A^\dagger \circ A^*$
$\Leftrightarrow A^*JA^\dagger = A^\dagger JA^*$
$\Leftrightarrow (A^*J)(A^\dagger J) = (A^\dagger J)(A^*J)$

$\Leftrightarrow (JA)^*(JA)^\dagger = (JA)^\dagger(JA)^*$

$\Leftrightarrow JA$is SD

If $A$ is J-SD, then as in the above proof,

$A$ is J-SD $\Leftrightarrow A^*JA^\dagger = A^\dagger JA^*$

$\Leftrightarrow (JA^*)(JA^\dagger) = (JA^\dagger)(JA^*)$

$\Leftrightarrow (AJ)^*(AJ)^\dagger = (AJ)^\dagger(AJ)^*$

$\Leftrightarrow AJ$is SD

Thus equivalence in (i) holds.

(ii)Follows from (i) by replacing $A$ by $JA$ and $AJ$ respectively and using $J^2 = I_n$.

Hence the theorem.

**Lemma 3.1.** For $A \in C^{n\times n}$, the following are equivalent

(i) $AJ = JA$

(ii) $A^{[*]} = A^*$

(iii) $A^{[\dagger]} = A^\dagger$

**Proof:** $AJ = JA \Leftrightarrow (AJ)^* = (JA)^*$

$\Leftrightarrow JA^* = A^*J$

$\Leftrightarrow A^{[*]} = A^*$

Thus (i) $\Leftrightarrow$(ii) holds.

$AJ = JA \Leftrightarrow (AJ)^\dagger = (JA)^\dagger$

$\Leftrightarrow JA^\dagger = A^\dagger J$

$\Leftrightarrow A^{[\dagger]} = A^\dagger$

Thus (i) $\Leftrightarrow$ (iii) holds.

**Theorem 3.3.** For $A \in C^{n\times n}$, if $AJ = JA$, then $A$ is J-SD $\Leftrightarrow$ $A$ is SD.

**Proof:** Since $AJ = JA$, by Lemma 3.1, $A^{[*]} = A^*$ and $A^{[\dagger]} = A^\dagger$; $A^*J = JA^*$ and $A^\dagger J = JA^\dagger$.

$A$ is J-SD $\Leftrightarrow [A^{[*]}, A^{[\dagger]}]_J = 0$

$\Leftrightarrow [A^*, A^\dagger]_J = 0$ (By Theorem 3.1)

$\Leftrightarrow A^* \circ A^\dagger = A^\dagger \circ A^*$

$\Leftrightarrow A^*JA^\dagger = A^\dagger JA^*$

$\Leftrightarrow A^*A^\dagger = A^\dagger A^*$

$\Leftrightarrow A$is SD

**Theorem 3.4.** For $A \in C^{n\times n}$, if $AJ = JA$, then$[A, A^{[\dagger]}]_J = 0 \Leftrightarrow [A, A^\dagger]_J = 0 \Leftrightarrow [A, A^\dagger] = 0$.

**Proof:** The first equivalence follows from Lemma 3.1, by using $A^{[\dagger]} = A^\dagger$.

Now, $[A, A^\dagger]_J = 0 \Leftrightarrow A \circ A^\dagger = A^\dagger \circ A$

$\Leftrightarrow AJA^\dagger = A^\dagger JA$

$\Leftrightarrow AA^\dagger = A^\dagger A \Leftrightarrow [A, A^\dagger] = 0$

**Remark 3.2.** The concept of J-EP matrix introduced in [3] (refer Definition 2.5) can be reformulated in terms of commutator as $A$ is J-EP $\Leftrightarrow [A, A^{[\dagger]}]_J = 0$. Then Theorem 3.4 reduces to the following:

**Corollary 3.1.** (Theorem 3.7(a) of [3]): For $A \in C^{n\times n}$, if $AJ = JA$, then $A$is J-EP$\Leftrightarrow$ $A$ is EP.

**Theorem 3.5.** Let $A, B \in C^{n\times n}$

(i) If $AJ = JA$, then $(AB)$ is SD $\Leftrightarrow$ $(A \circ B)$ is J- SD.

(ii) If $BJ = JB$, then $(AB)$ is J-SD $\Leftrightarrow$ $(A \circ B)$ is SD.

(iii) If $AJ = JA$and $BJ = JB$, then $(AB)$ is SD $\Leftrightarrow$ $(AB)$is J-SD $\Leftrightarrow$ $(A \circ B)$ is J-SD$\Leftrightarrow$ $(A \circ B)$ is SD.

**Proof:**

(i) If $AJ = JA$, then $AB = J(A \circ B)$ and by Theorem 3.2(i), $AB$ is SD $\Leftrightarrow$ $(A \circ B)$ is J- SD.

(ii) If $BJ = JB$, then $AB = (A \circ B)J$ and by Theorem 3.2(ii), $AB$ is J-SD $\Leftrightarrow$ $(A \circ B)$ is SD.

(iii) If $AJ = JA$and $BJ = JB$, then $J(A \circ B) = (A \circ B)J$. Hence by Theorem 3.3 $(A \circ B)$ is J- SD$\Leftrightarrow$ $(A \circ B)$ is SD.

This combined with (i) and (ii), yields (iii).

Hence the Theorem.

**Remark 3.3.** The condition on $A$, that is $AJ = JA$ is essential can be seen by the following example:

**Example 3.3.**

$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, J= $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ here $AJ \neq JA$

$A^T = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$; $AA^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$; $A^\dagger = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \frac{1}{4}A^T$

$A^{[\dagger]} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \frac{1}{4}A$here$A^{[\dagger]} \neq A^\dagger$.

For $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $AB = 0$ and $A \circ B = \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}$

**Theorem 3.6.** For $A, B \in C^{n\times n}$, if $AJ = JA$ and $B^* \circ A^\dagger = B^\dagger \circ A^*$, then the following are equivalent:

(i) $[A^\dagger A, BB^\dagger] = 0$

(ii) $[A^* A, BB^*] = 0$

(iii) $A^*A \circ BB^*$ is J-EP

(iv) $(A \circ B)^{[\dagger]} = B^{[\dagger]} \circ A^{[\dagger]}$.

**Proof:** Since $AJ = JA$, $J(A^*A) = (A^*A)J$, by Corollary 3.1, $A^*A o BB^*$ is EP $\Leftrightarrow$ $A^*A \circ BB^*$is J-EP. Therefore $A^*ABB^*$is EP $\Leftrightarrow$ $A^*ABB^*$is J-EP. By $AJ = JA$, the condition $B^* \circ A^\dagger = B^\dagger \circ A^*$reduces to $B^*A^\dagger = B^\dagger A^*$. Then, equivalence of (i), (ii) and (iii) follows from Proposition 2 of [5] and (iii) $\Leftrightarrow$ (iv) is precisely Theorem 3.17 of [3].

Hence the theorem.

**Remark 3.4.** In particular for $A = B$, the condition in Theorem 3.6 $AJ = JA$ and $A^* \circ A^\dagger = A^\dagger \circ A^*$, reduces to $A$ is SD.$[A^\dagger A, AA^\dagger] = 0$,$[A^*A, AA^*] = 0$ and $(A^2)^\dagger = (A^\dagger)^2$ reduces to $A$ is bi- EP, bi-normal and bi-dagger respectively. Thus Theorem 3.6 reduces to Corrollary 1 of [5]. A square complex matrix $A$ is a partial isometry if $A^* = A^\dagger$;which is equivalent to $A^{[*]} = A^{[\dagger]}$. It turns out that the definition of a partial isometry carries over as such to indefinite inner product spaces.

**Corollary 3.3.** If $A$ and $B$ are partial isometries such that $AJ = JA$, then the following are equivalent:

(i) $[A^\dagger A, BB^\dagger] = 0$
(ii) $[A^*A, BB^*] = 0$
(iii) $(A \circ B)^\dagger = B^\dagger \circ A^\dagger$
(iv) $(A \circ B)$is partial isometry.

**Proof:** Since $A$ and $B$ are partial isometries $A^* = A^\dagger$and $B^* = B^\dagger$. In Theorem 3.6, the condition $B^* \circ A^\dagger = B^\dagger \circ A^*$ automatically holds.
Hence, the equivalence of (i), (ii) and (iii) follows from Theorem 3.6.
(iii)⟺(iv): $A \circ B$ is partial isometry⟺ $(A \circ B)^\dagger = (A \circ B)^* = (B^* \circ A^*) = B^\dagger \circ A^\dagger$.
Hence the corollary.

**Theorem 3.7.** Let $A$ and $B$ be star orthogonal J-SD matrices. $A + B$ is J-SD ⟺ $(A^* \circ B^\dagger) + (B^* \circ A^\dagger) = (A^\dagger \circ B^*) + (B^\dagger \circ A^*)$.
**Proof:** Since $A$ and $B$ are star orthogonal matrices, by a result of Erdelyi[2], $(A + B)^+ = A^+ + B^+$.

$$
\begin{aligned}
A + B \text{ is J-SD} &\Leftrightarrow (A + B)^* \circ (A + B)^\dagger = (A + B)^\dagger \circ (A + B)^* \\
&\Leftrightarrow (A^* + B^*) \circ (A^\dagger + B^\dagger) = (A^\dagger + B^\dagger) \circ (A^* + B^*) \\
&\Leftrightarrow (A^* \circ A^\dagger) + (A^* \circ B^\dagger) + (B^* \circ A^\dagger) + (B^* \circ B^\dagger) \\
&= (A^\dagger \circ A^*) + (A^\dagger \circ B^*) + (B^\dagger \circ A^*) + (B^\dagger \circ B^*) \\
&\Leftrightarrow (A^* \circ B^\dagger) + (B^* \circ A^\dagger) = (A^\dagger \circ B^*) + (B^\dagger \circ A^*)
\end{aligned}
$$

Hence the theorem.

**Remark 3.5.** In particular for $J = I_n$,Corollary 3.3 and Theorem 3.7 reduces to the Corollary 2 andPropostion 1 of [5] respectively. Theorem 3.7 can be extended to the sum of a finite number of J-SD matrices that are pairwise star orthogonal.

## 4. Conclusion

Investigation into various partial orderings on complex matrices with respect to the indefinite matrix multiplication is currently being undertaken.

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