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A Bulk Arrival Non-Markovian Queueing System with Two types of Second Optional Services and with Second Optional Vacation

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Abstract. We study an $M^X/G/1$ queueing system where the arrival follows a compound Poisson process. The server provides the first service to all customers which is essential. Only some of the customers demand second service which is optional. The second optional service is given in two types. Soon after the system becomes empty the server takes a vacation and after returning from vacation the server may take second vacation which is optional. Using supplementary variable technique we derive the probability generating function for the number of customers in the system. Some performance measures are calculated. Some particular cases are derived.

Keywords: First essential service, Second optional service, Regular vacation, Second optional vacation and Supplementary variable technique.

AMS Mathematics Subject Classification (2010): 60K25, 60K30

1. Introduction

Various forms of M/G/1 queue has been studied by many authors including Cohen [5], Medhi [9] and Choi and Park [2]. Madan [7] and Thillaigovindan et al. [11] studied the queueing model with second optional service. Vacation queueing models have been effectively modeled in various situations such as production, banking service, communication systems and computer networks etc. Numerous authors are interested in studying queueing models with various vacation polices. Some of them are Baba [1], Choudhury [3], Kalyanaraman and Pazhani Bala Murugan [6], Thangaraj and Vanitha [10]. In digital communication system the server may require two types of vacation. Regular vacation is necessary for maintaining the system and if the system is not in proper working condition then the optional vacation may also be necessary to correct the fault in the system. Choudhury [4], Manoharan and Sankara Sasi [8] studied the queueing model with second optional vacation. Here we consider an $M^X/G/1$ queue with two type of second optional services and with second optional vacation.

2. The model

The following assumptions briefly describe the mathematical model of our study.

- Customers arrive at the system in batches of variable size in a compound Poisson process.
- There is a single server who provides the first essential service (FES) to all customers. The service time for FES follows a general distribution. Let $B_1(x)$ and $b_1(x)$ respectively denote the distribution function and the density function of the FES times.
- As soon as the first service is completed the customer can exercise any one of the following three options.
  i. may leave the system with probability $1-p=q$
  ii. may opt for type 1 optional service with probability pp_1
  iii. may opt for type 2 optional service with probability pp_2

  where $p+q=1$ and $p_1+p_2=1$

  The type i (i=1, 2) optional service times are assumed to be negative exponential with mean service times $\frac{1}{\mu_i}, \mu_i > 0;\ i=1,2$. Let $B_i(\nu)$ and $b_i(\nu); i=1,2$ be the distribution and density function of type 1 and type 2 services respectively.
- Further it is assumed that $\mu_i(x)dx$ is the conditional probability of the completion of the i^{th} service given that the elapsed service time is x so that $\mu_i(x)dx = \frac{b_i(x)}{1-B_i(x)}$ and therefore $b_i(x) = \mu_i(x)\, e^{-\int_0^x \mu_i(t)dt}$; i∈ {1,2,3}
- We assume that the FES and SOS are mutually independent of each other. Let $B_i^*(s), E(B_i^k)\ (k \geq 1), i \in \{1,2,3\}$ denote the LST and finite moments of two service times respectively. Thus the total service time required by the server to complete the service cycle which may be called as modified service period is given by $B = \begin{cases} B_1 + B_2 & \text{with probability} \quad p \\ B_1 & \text{with probability} \quad 1-p \end{cases}$
- Whenever the system becomes empty, the server goes for a first regular vacation (FRV) of random length V_1. Let $V_1(x)$ and $v_1(x)$ respectively be the distribution function and density function of the first vacation times.
- At the end of FRV, the server may take the second optional vacation SOV with probability θ. Otherwise he remains in the system with probability $(1-\theta)$ until a new customer arrives. Let $V_2(x)$ and $v_2(x)$ respectively be the distribution function and density function for the SOV times. Further it is assumed that $\nu_i(x)dx$ is the conditional probability of the completion of the i^{th} vacation given that the elapsed vacation time is x so that $\nu_i(x)\,dx = \frac{v_i(x)}{1-V_i(x)}$ and therefore

  $v_i(x) = \nu_i(x)\, e^{-\int_0^x \nu_i(t)dt}$; $i \in \{1,2\}$.

- It is also assumed that the vacation times V_1 and V_2 are mutually independent of each other having general distribution function $V_i(x)$, LST V_i^* and finite moments $E(V_i^k)\ (k \ge 1),\ i \in \{1,2\}$. Thus the total vacation time required to complete the vacation cycle, which may be called as modified vacation period is given by

$$V = \begin{cases} \mathrm{V}_1 + \mathrm{V}_2 \text{ with probability } & \theta \\ \mathrm{V}_1 \qquad\quad \text{with probability } & 1-\theta \end{cases}$$

3. Queue size distribution at a random epoch

Here we first set up the steady state equations for the stationary queue size distribution by treating elapsed service time, FES time, SOS time, FRV time and SOV time as supplementary variables. Then we solve these equations and derive the PGF's, assuming that the system is in steady state condition. Let N(t) be the queue size (including one being served, if any), $B_1^{(0)}(t)$ be the elapsed service time at tfor FES, $B_2^{(0)}(t)$ be the elapsed service time at tfor the SOS, $V_1^{(0)}(t)$ be the elapsed vacation time at t for the FRV, $V_2^{(0)}(t)$ be the elapsed vacation time at t for the SOV. For further development of this model let us introduce the random variable Y(t) as follows.

$$\mathrm{Y(t)} = \begin{cases} 0 & \text{if the server is on FRV at time t} \\ 1 & \text{if the server is on SOV at time t} \\ 2 & \text{if the server is busy giving FES at time t} \\ 3 & \text{if the server is busy giving type 1 SOS at time t} \\ 4 & \text{if the server is busy giving type 2 SOS at time t} \end{cases}$$

The supplementary variables $V_1^{(0)}(t), V_2^{(0)}(t);\ B_1^{(0)}(t), B_2^{(0)}(t)$ and $B_3^{(0)}(t)$ are introduced in order to obtain a bivariate Markov process $\{N(t); \partial(t); t \ge 0\}$ where

$$\partial(\mathrm{t}) = \begin{cases} \mathrm{V}_1^{(0)}(\mathrm{t}) & \text{if} \quad \mathrm{Y(t)} = 0 \\ \mathrm{V}_2^{(0)}(\mathrm{t}) & \text{if} \quad \mathrm{Y(t)} = 1 \\ \mathrm{B}_1^{(0)}(\mathrm{t}) & \text{if} \quad \mathrm{Y(t)} = 2 \\ \mathrm{B}_2^{(0)}(\mathrm{t}) & \text{if} \quad \mathrm{Y(t)} = 3 \\ \mathrm{B}_3^{(0)}(\mathrm{t}) & \text{if} \quad \mathrm{Y(t)} = 4 \end{cases}$$

We define the limiting probabilities as follows.

$$Q_{1,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = V_1^{(0)}(t); x < V_1^{(0)}(t) \le x + dx\right\}; \quad n \ge 0; \quad x > 0$$

$$Q_{2,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = V_2^{(0)}(t); x < V_2^{(0)}(t) \le x + dx\right\}; \quad n \ge 0; \quad x > 0$$

$$P_{1,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = B_1^{(0)}(t); x < B_1^{(0)}(t) \le x + dx\right\}; \quad n \ge 0; \quad x > 0$$

$$P_{2,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = B_2^{(0)}(t); x < B_2^{(0)}(t) \le x + dx\right\}; \quad n \ge 0; \quad x > 0$$

$$P_{3,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = B_3^{(0)}(t); x < B_3^{(0)}(t) \le x + dx\right\}; \quad n \ge 0; \quad x > 0$$

Further it is assumed that $B_i^{(0)}(0)=0;\ B_i^{(0)}(\infty)=1$ for $i\in\{1,2,3\}$ and $V_i^{(0)}(0)=0; V_i^{(0)}(\infty)=1$ for $i\in\{1,2\}$ and are continuous at $x=0$. The differential difference equations governing the system are

$$\frac{d}{dx}P_{1,n}(x)+(\lambda+\mu_1(x))P_{1,n}(x)=\lambda\sum_{k=1}^{n}c_kP_{1,n-1}(x)\quad x>0\ n\geq 1 \tag{1}$$

$$\frac{d}{dx}P_{1,0}(x)+(\lambda+\mu_1(x))P_{1,0}(x)=0\quad x>0 \tag{2}$$

$$\frac{d}{dx}P_{2,n}(x)+(\lambda+\mu_2(x))P_{2,n}(x)=\lambda\sum_{k=1}^{n}c_kP_{2,n-1}(x)\quad x>0\ n\geq 1 \tag{3}$$

$$\frac{d}{dx}P_{2,0}(x)+(\lambda+\mu_2(x))P_{2,0}(x)=0\quad x>0 \tag{4}$$

$$\frac{d}{dx}P_{3,n}(x)+(\lambda+\mu_3(x))P_{3,n}(x)=\lambda\sum_{k=1}^{n}c_kP_{3,n-1}(x)\quad x>0\ n\geq 1 \tag{5}$$

$$\frac{d}{dx}P_{3,0}(x)+(\lambda+\mu_3(x))P_{3,0}(x)=0\quad x>0 \tag{6}$$

$$\frac{d}{dx}Q_{1,n}(x)+(\lambda+\nu_1(x))Q_{1,n}(x)=\lambda\sum_{k=1}^{n}c_kQ_{1,n-1}(x)\quad x>0\ n\geq 1 \tag{7}$$

$$\frac{d}{dx}Q_{1,0}(x)+(\lambda+\nu_1(x))Q_{1,0}(x)=0\quad x>0 \tag{8}$$

$$\frac{d}{dx}Q_{2,n}(x)+(\lambda+\nu_2(x))Q_{2,n}(x)=\lambda\sum_{k=1}^{n}c_kQ_{2,n-1}(x)\quad x>0\ n\geq 1 \tag{9}$$

$$\frac{d}{dx}Q_{2,0}(x)+(\lambda+\nu_2(x))Q_{2,0}(x)=0\quad x>0 \tag{10}$$

$$\lambda Q_{1,0}=(1-p)\int_0^\infty P_{1,0}(x)\mu_1(x)dx+\int_0^\infty P_{2,0}(x)\mu_2(x)dx+\int_0^\infty P_{3,0}(x)\mu_3(x)dx$$
$$+(1-\theta)\int_0^\infty Q_{1,0}(x)\nu_1(x)dx+\int_0^\infty Q_{2,0}(x)\nu_2(x)dx \tag{11}$$

where $Q_{1,0}=\int_0^\infty Q_{1,0}(x)dx$

The boundary conditions are

$$Q_{1,0}(0)=\lambda Q_{1,0} \tag{12}$$

$$Q_{1,n}(0)=0\ n\geq 1 \tag{13}$$

$$Q_{2,n}(0)=\theta\int_0^\infty Q_{1,n}(x)\nu_1(x)dx\ n\geq 0 \tag{14}$$

$$P_{1,0}(0)=(1-p)\int_0^\infty P_{1,1}(x)\mu_1(x)dx+\int_0^\infty P_{2,1}(x)\mu_2(x)dx+\int_0^\infty P_{3,1}(x)\mu_3(x)dx$$
$$+(1-\theta)\int_0^\infty Q_{1,1}(x)\nu_1(x)dx+\int_0^\infty Q_{2,1}(x)\nu_2(x)dx \tag{15}$$

$$P_{1,n}(0)=(1-p)\int_0^\infty P_{1,n+1}(x)\mu_1(x)dx+\int_0^\infty P_{2,n+1}(x)\mu_2(x)dx+\int_0^\infty P_{3,n+1}(x)\mu_3(x)dx$$
$$+(1-\theta)\int_0^\infty Q_{1,n+1}(x)\nu_1(x)dx+\int_0^\infty Q_{2,n+1}(x)\nu_2(x)dx \tag{16}$$

$$P_{2,n}(0)=pp_1\int_0^\infty P_{1,n}(x)\mu_1(x)dx\ n\geq 0 \tag{17}$$

$$P_{3,n}(0) = pp_2 \int_0^\infty P_{1,n}(x)\mu_1(x)dx \quad n \geq 0 \tag{18}$$

and the normalizing condition is

$$\sum_{n=1}^{\infty}\sum_{i=1}^{3}\int_0^\infty P_{i,n}(x)dx + \sum_{n=1}^{\infty}\sum_{i=1}^{2}\int_0^\infty Q_{i,n}(x)dx = 1 \tag{19}$$

Now let us define the following PGF's

$$P_i(x,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(x); \; x \geq 0; \; |z| \leq 1, \quad i \in \{1,2,3\} \tag{20}$$

$$P_i(0,z) = \sum_{n=0}^{\infty} z^n P_{i,n}(0); \; |z| \leq 1, \quad i \in \{1,2,3\} \tag{21}$$

$$Q_i(x,z) = \sum_{n=0}^{\infty} z^n Q_{i,n}(x); \; x \geq 0; \; |z| \leq 1, \quad i \in \{1,2\} \tag{22}$$

$$Q_i(0,z) = \sum_{n=0}^{\infty} z^n Q_{i,n}(0); \; |z| \leq 1, \quad i \in \{1,2\} \tag{23}$$

$$P_i(z) = \int_0^\infty P_i(x,z)dx, \; i \in \{1,2,3\} \tag{24}$$

$$Q_i(z) = \int_0^\infty Q_i(x,z)dx, \; i \in \{1,2\} \tag{25}$$

Multiplying (1) by z^n and summing over n=0 to $\infty$and adding with (2), we get

$$\frac{d}{dx}P_1(x,z) + (\lambda - \lambda X(z) + \mu_1(x))P_1(x,z) = 0 \quad P_1(x,z) = P_1(0,z)[1 - B_1(x)]e^{-\lambda(1-X(z))x} \tag{26}$$

where $X(z) = \sum_{k=1}^{\infty} c_k z^k$

Multiplying (3) by z^n and summing over n=0 to ∞ and adding with (4), we get

$$\frac{d}{dx}P_2(x,z) + (\lambda - \lambda X(z) + \mu_2(x))P_2(x,z) = 0 \quad P_2(x,z) = P_2(0,z)[1 - B_2(x)]e^{-\lambda(1-X(z))x} \tag{27}$$

Multiplying (5) by z^n and summing over n=0 to ∞ and adding with (6), we get

$$\frac{d}{dx}P_3(x,z) + (\lambda - \lambda X(z) + \mu_3(x))P_3(x,z) = 0$$

$$P_3(x,z) = P_3(0,z)[1 - B_3(x)]e^{-\lambda(1-X(z))x} \tag{28}$$

Multiplying (7) by z^n and summing over n=0to ∞ and adding with (8), we get

$$\frac{d}{dx}Q_1(x,z) + (\lambda - \lambda X(z) + \nu_1(x))Q_1(x,z) = 0$$

$$Q_1(x,z) = Q_1(0,z)[1 - V_1(x)]e^{-\lambda(1-X(z))x} \tag{29}$$

Multiplying (9) by z^n and summing over n = 0 to∞ and adding with (10), we get

$$\frac{d}{dx}Q_2(x,z) + (\lambda - \lambda X(z) + \nu_2(x))Q_2(x,z) = 0 \quad Q_2(x,z) = Q_2(0,z)[1 - V_2(x)]e^{-\lambda(1-X(z))x} \tag{30}$$

Multiplying (16) by z^{n+1} and summing over n=0to ∞ and adding with z times (15), we get

$$zP_1(0,z) = (1-p)\int_0^\infty P_1(x,z)\mu_1(x)dx + \int_0^\infty P_2(x,z)\mu_2(x)dx + \int_0^\infty P_3(x,z)\mu_3(x)dx$$

$$+ (1-\theta)\int_0^\infty Q_1(x,z)\nu_1(x)dx + \int_0^\infty Q_2(x,z)\nu_2(x)dx - \lambda Q_{1,0} \tag{31}$$

From equation (17), we get $P_2(0,z) = pp_1 P_1(0,z) B_1^*(\lambda - \lambda X(z))$ (32)

From equation (18), we get

$$P_3(0,z) = pp_2 P_1(0,z) B_1^*(\lambda - \lambda X(z)) \tag{33}$$

From equation (26), we get $\int_0^\infty P_1(x,z)\mu_1(x)dx = P_1(0,z) B_1^*(\lambda - \lambda X(z))$ (34)

From equation (27), we get

$$\int_0^\infty P_2(x,z)\mu_2(x)dx = pp_1 P_1(0,z) B_1^*(\lambda - \lambda X(z)) B_2^*(\lambda - \lambda X(z)) \tag{35}$$

From equation (28), we get

$$\int_0^\infty P_3(x,z)\mu_3(x)dx = pp_2 P_1(0,z) B_1^*(\lambda - \lambda X(z)) B_3^*(\lambda - \lambda X(z)) \tag{36}$$

From equation (29), we get $\int_0^\infty Q_1(x,z)\nu_1(x)dx = Q_1(0,z) V_1^*(\lambda - \lambda X(z))$ (37)

From equation (30), we get

$$\int_0^\infty Q_2(x,z)\nu_2(x)dx = \theta Q_1(0,z) V_1^*(\lambda - \lambda X(z)) V_2^*(\lambda - \lambda X(z)) \tag{38}$$

From equation (13) and (14), we get $Q_1(0,z) = \lambda Q_{1,0}$ (39)

$$Q_2(0,z) = \theta Q_1(0,z) V_1^*(\lambda - \lambda X(z)) \tag{40}$$

Using (34) to (40) in (31), we get

$$P_1(0,z) = \left[\frac{\lambda\{[(1-\theta) + \theta V_2^*(\lambda - \lambda X(z))] V_1^*(\lambda - \lambda X(z))\}}{D_1(z)}\right] Q_{1,0} \tag{41}$$

where $D_1(z) = \{z - [(1-p) + pp_1 B_2^*(\lambda - \lambda X(z)) + pp_2 B_3^*(\lambda - \lambda X(z))] B_1^*(\lambda - \lambda X(z))\}$

Using (41) in (32), we get

$$P_2(0,z) = \left[\frac{\lambda pp_1\{[(1-\theta) + \theta V_2^*(\lambda - \lambda X(z))] V_1^*(\lambda - \lambda X(z))\} B_1^*(\lambda - \lambda X(z))}{D_1(z)}\right] Q_{1,0} \tag{42}$$

Using (41) in (33), we get

$$P_3(0,z) = \left[\frac{\lambda pp_2\{[(1-\theta) + \theta V_2^*(\lambda - \lambda X(z))] V_1^*(\lambda - \lambda X(z))\} B_1^*(\lambda - \lambda X(z))}{D_1(z)}\right] Q_{1,0} \tag{43}$$

Integrating (26) to (30) between 0 and ∞, we get

$$P_1(z) = \int_0^\infty P_1(x,z)dx = \left[\frac{1 - B_1^*(\lambda - \lambda X(z))}{(\lambda - \lambda X(z))}\right] P_1(0,z) \tag{44}$$

$$P_2(z) = \int_0^\infty P_2(x,z)dx = pp_1\left[\frac{1 - B_2^*(\lambda - \lambda X(z))}{(\lambda - \lambda X(z))}\right] B_1^*(\lambda - \lambda X(z)) P_1(0,z) \tag{45}$$

$$P_3(z) = \int_0^\infty P_3(x,z)dx = pp_2\left[\frac{1 - B_3^*(\lambda - \lambda X(z))}{(\lambda - \lambda X(z))}\right] B_1^*(\lambda - \lambda X(z)) P_1(0,z) \tag{46}$$

$$Q_1(z) = \int_0^\infty Q_1(x,z)dx = \left[\frac{1 - V_1^*(\lambda - \lambda X(z))}{(\lambda - \lambda X(z))}\right] Q_1(0,z) \tag{47}$$

$$Q_2(z) = \int_0^\infty Q_2(x,z)dx = \left[\frac{\theta V_1^*(\lambda - \lambda X(z))[1 - V_2^*(\lambda - \lambda X(z))]}{(\lambda - \lambda X(z))}\right] Q_1(0,z) \tag{48}$$

Using (41) in (44), (45) and (46), we get

$$P_1(0,z) = \left[\frac{\{[(1-\theta)+\theta V_2^*(\lambda-\lambda X(z))]V_1^*(\lambda-\lambda X(z))\}(1-B_1^*(\lambda-\lambda X(z)))}{D_1(z)(1-X(z))}\right]Q_{1.0} \quad (49)$$

$$P_2(0,z) = pp_1\left[\frac{\{[(1-\theta)+\theta V_2^*(\lambda-\lambda X(z))]V_1^*(\lambda-\lambda X(z))\}(1-B_2^*(\lambda-\lambda X(z)))B_1^*(\lambda-\lambda X(z))}{D_1(z)(1-X(z))}\right]Q_{1.0} \quad (50)$$

$$P_3(0,z) = pp_2\left[\frac{\{[(1-\theta)+\theta V_2^*(\lambda-\lambda X(z))]V_1^*(\lambda-\lambda X(z))\}(1-B_3^*(\lambda-\lambda X(z)))B_1^*(\lambda-\lambda X(z))}{D_1(z)(1-X(z))}\right]Q_{1.0} \quad (51)$$

Using (39) in (47) and (48), we get $Q_1(z) = \left[\frac{1-V_1^*(\lambda-\lambda X(z))}{(1-X(z))}\right]Q_{1,0}$ (52)

$$Q_2(z) = \left[\frac{\theta V_1^*(\lambda-\lambda X(z))[1-V_2^*(\lambda-\lambda X(z))]}{(1-X(z))}\right]Q_{1,0} \quad (53)$$

From the fact that $P_1(1)+P_2(1)+P_3(1)+Q_1(1)+Q_2(1)=1$, we arrived

$$Q_{1,0} = \frac{1-\rho}{\lambda[E(V_1)+\theta E(V_2)]}, \text{ where } \rho = \lambda X'(z)[E(B_1)+pp_1E(B_2)+pp_2E(B_3)] \quad (54)$$

and $B_1^*(0) = -E(B_1),\ B_2^*(0) = -E(B_2),\ B_3^*(0) = -E(B_3)$ are the mean of service times of FES, type1 SOS time and type2 SOS time respectively, $V_1^*(0) = -E(V_1)$ and $V_2^*(0) = -E(V_2)$ are the mean of vacation times of FRV and SOV respectively. Therefore $P(z) = \frac{N(Z)}{D(Z)}$ (55)

where $N(z) = \{[(1-\theta)+\theta V_2^*(\lambda-\lambda X(z))]V_1^*(\lambda-\lambda X(z))-1\}(1-z)Q_{1,0}$

$$D(z) = \{z-[(1-p)+pp_1B_2^*(\lambda-\lambda X(z))+pp_2B_3^*(\lambda-\lambda X(z))]B_1^*(\lambda-\lambda X(z))\}(1-X(z))$$

3.1. Performance measures

Let L_q and L denote the steady state average queue size and system size respectively.

Then $L_q = \frac{d}{dz}P(z)\Big|_{z=1} = \frac{d}{dz}\frac{N(z)}{D(z)}\Big|_{z=1}$ Using the L'Hospital rule twice we obtain

$$L_q = \frac{D'(1)N''(1)-N'(1)D''(1)}{2(D'(1))^2} \quad (56)$$

where $N'(1) = -2\lambda X'(z)(E(V_1)+\theta E(V_2))Q_{1,0}$

$$N''(1) = -3\{\lambda X''(z)\,(E(V_1)+\theta E(V_2))+\lambda^2(X'(z))^2[E(V_1^2)+\theta E(V_2^2)+2\theta E(V_1)E(V_2)]\}Q_{1,0}$$

$$D'(1) = 2X'(z)(\lambda X'(z)(E(B_1)+pp_1E(B_2)+pp_2E(B_3))-1)$$

$$D''(1) = 3\{2\lambda X'(z)X''(z)(E(B_1)+pp_1E(B_2)+pp_2E(B_3))-X''(z)$$
$$+\lambda^2X'(z)^3\{E(B_1^2)+pp_1E(B_2^2)+pp_2E(B_3^2)+2pE(B_1)(p_1E(B_2)+p_2E(B_3))\}\}$$

where $E(B_1^2),\ E(B_2^2),\ E(B_3^2),\ E(V_1^2),\ E(V_2^2)$ second moment of FES, type 1 SOS, type 2 SOS, FRV and SOV time respectively. Now, we can obtain $L = L_q + \rho$, where L_q and ρ have been found in (56) and (54) respectively. Then using Little's formulae, we obtain W_q, the average waiting time in the queue and W, the average waiting time in the system, as $W_q = \frac{L_q}{\lambda}$ and $W = \frac{L}{\lambda}$ respectively.

3.2. Particular cases

Case 1: If there is no one opting for type 2 second optional service then by setting $p_2 = 0$, then (55) takes the form

$$P(z) = \frac{\left\{\left[(1-\theta)+\theta V_2^*(\lambda-\lambda X(z))\right]V_1^*(\lambda-\lambda X(z))-1\right\}(1-z)}{\left\{z-\left[(1-p)+pB_2^*(\lambda-\lambda X(z))\right]B_1^*(\lambda-\lambda X(z))\right\}(1-X(z))}Q_{1,0} \tag{57}$$

which coincides the model studied by Manoharan [8].

Case 2: If there is no one opting for second optional service (both type 1 and type 2) and if there is a single arrival then by setting$p_1 = 0$, $p_2 = 0$ and $X(z) = z$ (55) takes the form

$$P(z) = \frac{\left\{\left[(1-\theta)+\theta V_2^*(\lambda-\lambda z)\right]V_1^*(\lambda-\lambda z)-1\right\}}{\left[z-B_1^*(\lambda-\lambda z)\right]}Q_{1,0} \tag{58}$$

which coincides with the PGF of Choudhury [4] irrespective of the notations used.

6. Conclusion

The analysis carried out in“Abulk arrival non-markovian queueing system with two types of second optional services and with second optional vacation” is to obtain the probability generating function for the number of customers in the system and also to obtain waiting time of a customer in the system.

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