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Analysis of Bulk Queueing System of Variant Threshold for Multiple Vacations, Restricted Admissiblity of Arriving Batches and Setup

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Abstract. In this Paper, a bulk arrival general bulk service queueing system with variant threshold policies for multiple vacations under a restricted admissibility policy of arriving batches and set up time for service is considered. During the server is in non-vacation, the arrivals are admitted with probability 'α 'whereas, with probability 'β ', they are admitted when the server is in vacation. The server starts the service only if at least ‘a’ customers are waiting in the queue, and renders the service according to the general bulk service rule with minimum of ‘a’ customers and maximum of ‘b’ customers. On completion of service, if the queue length is less than 'a' , then the server takes a vacation of type one, repeatedly, until the queue length reaches the threshold value ‘a’. When the server returns from a vacation of type one, if the queue length is at least ‘a’, then the server takes another vacation of type two, repeatedly, until the queue length reaches the threshold value ‘N’ ($N \geq b > a$), and serves a batch of ‘b’ customers. On the other hand, when the server returns from a vacation of type one, if the queue length reaches N, then he servers a batch of 'b' customers. The server requires a setup time U to start the service.

Keywords: Bulk arrival, multiple vacations, restricted admissibility, set up time

AMS Mathematics Subject Classification (2010): 60k25, 60K30, 90B22

1. Introduction

Queueing models with server vacations have been investigated by many authors due to their various applications in production, inventory system, communication systems, banking services, computer systems etc. Very few authors only have studied the comparable work on the bulk queueing models considering variant vacation policy. It is necessary to allow the server to take different types of vacations with different thershold polices to optimize the overall cost. Lee et al (1991) considered a batch arrival queue with different vacations and showed that the waiting time distributions can be obtained by simple iterative procedure. Lee et al (1994) analyzed $M^{[x]}$ /G/1 queueing system with N-policy and multiple vacations, using supplementary variable technique. A batch arrival queue with threshold was discussed by Lee et al (1996). Krishna Reddy and Anitha (1999) studied a M/G(a,b)/1 queue with different vacation polices and obtained Laplace transform of the joint distribution of the queue length and the remaining service time and the remaining vacation time depending on the state of the server. Ke(2003) discussed the optimal control of a M/G/1 queueing system with server startup time and two types of vacations. Madan and Choudhury (2005) discussed a batch arrival queueing system, where the server provides two stages of heterogeneous service with a modified vacation model for aM^x/G/1 queueing systems. Ke (2007) used supplementary varaible technique to study aM^x/G/1 queueing systems with balking under variant vacations.

2. Notations

λ arrival rate
X Group size random variable
g_k The Probability that 'k' customers arrive in a batch
X(z) The Probability generating function.
S(x) Cumulative distribution function of service time
V_1(x) Cumulative distribution function of vacation of type one
V_2(x) Cumulative distribution function of vacation of type two
U(x) Cumulative distribution function of seup time
s(x) The probability density function of S
v_1(x) The probability density function of vacation of type one
v_2(x) The probability density function of vacation of type two
u(x) The probability density function of U
$\tilde{S}(\theta)$ Laplace -Stieltjes transform of S
$\tilde{V}_1(\theta)$ Laplace Stieltjes transform of vacation of type one
$\tilde{V}_2(\theta)$ Laplace Stieltjes transform of vacation of type two
$\tilde{U}(\theta)$ Laplace -Stieltjes transform of U
S^0 (x) Remaining service time
V_1^0(x) Remaining vacation time of type one vacation
V_2^0(x) Remaining vacation time of type two vacation
U^0 (x) Remaining service time of set up time
N_s(t) Number of customers in the service at time t
N_q(t) Number of customers in the queue at time t

The different states of the server at time tare defined as follows

$$C(t) = \begin{cases} 0; & \text{if the server is busy with service} \\ 1; & \text{if the server is on vacation} \\ 2; & \text{if the server is on set up time} \end{cases}$$

z^1(t) = j, if the server is on j th vacation of type one (v_1)
z^2(t) = j, if the server is on j th vacation of type two (v_2)

To obtain the system equations , the following state probabilities are defined;

$$P_{ij}(x,t)dt = P\{N_s(t) = i, N_q(t) = j, x \le S^0(t) \le x + dt,\ C(t) = 0\}, \quad a \le x \le b, j \ge 0,$$

$$Q^1{}_{jn}(x,t)dt = P\{N_q(t) = n, x \le V_1{}^0(t) \le x + dt, C(t) = 1, z^1(t) = j\},\ n\ge0,\ j\ge1,$$

$$Q^2{}_{jn}(x,t)dt = P\{N_q(t) = n, x \le V_2{}^0(t) \le x + dt, C(t) = 1, z^2(t) = j\},\ n\ge a,\ j\ge1,$$

$$U_n(x,t)dt = P\{N_q(t) = n, x \le U^0(t) \le x + dt,\ C(t) = 2\}, n \ge a$$

3. Steady state queue size distribution

The model is then goverened by the following set of difference-differential equations**.**

$-P^{1}_{i,0}(x) = -\lambda P_{i,0}(x) + \lambda(1\text{-}\alpha)P_{i,0}(x)+ \sum_{m=a}^{b} P_{m,i}\,(0)\;\; s(x)\;\; +U_i\;(0)\;s(x),\; a \le i \le b$ (1)

$-P^{1}_{i,j}(x) = -\lambda P_{i,j}(x) + \lambda(1\text{-}\alpha)\;P_{i,j}(x)+ \alpha \sum_{k=1}^{j} P_{i,j-k}\;(x)\;\; \lambda g_k,\; a \le i \le b\;\text{-}1,\; j \ge 1$ (2)

$-P^{1}_{b,j}(x) = -\lambda P_{b,j}(x) + \lambda(1\text{-}\alpha)\;P_{b,j}(x)+ \sum_{m=a}^{b} P_{m,b+j}\;(0)s(x) +\alpha \sum_{k=1}^{j} P_{b,j-k}\;(x)\;\; \lambda g_k,$

$1 \le j \le N - 1 - b$ (3)

$-P^{1}_{b,j}(x) = -\lambda P_{b,j}(x) + \;\lambda(1-\alpha)P_{b,j}(x)+ \sum_{m=a}^{i} P_{m,b+j}\;(0)s\;(x)\; +\alpha \sum_{k=1}^{j} P_{b,j-k}\;(x)\lambda g_k +\; U_{b+j}\;(0)\;s(x),\; j \ge N - b$ (4)

$-Q^{1}_{l,0}(x) = -\lambda Q^{1}_{l,0}(x) + \lambda\;(1\text{-}\beta)\;Q_{l,0}(x) + \sum_{m=a}^{b} P_{m,0}\;(0)\;v_l(x)$ (5)

$-Q^{1}_{l,n}(x) = -\lambda Q^{1}_{l,n}(x) + \lambda\;(1\text{-}\beta)\;Q_{l,n}(x)+ \sum_{m=a}^{b} P_{m,n}\;\;(0)\;v_l(x)\;\; + \beta \sum_{k=l}^{n} Q^{1}_{l,n-k}\;\;(x)\;\lambda g_k,$

$0 \le n \le a\text{-}1$ (6)

$-Q^{1}_{j,n}(x) = -\lambda Q^{1}_{j,n}(x) + \lambda(1-\beta)\;Q_{j,n}(x) + \beta \sum_{k=l}^{n} Q^{1}_{j,n-k}\;(x)\;\lambda g_k\;,\; n \ge a,\; j\ge 1$ (7)

$-Q^{1}_{j,0}(x) = -\lambda Q^{1}_{j,0}(x)\;\; +\lambda(1\text{-}\beta)\;Q^{1}_{j,0}(x)+\;Q^{1}_{j-1,0}(0)v_1(x),\;\; j \ge 2$ (8)

$-Q^{1}_{j,n}(x) = -\lambda Q^{1}_{j,n}(x) + \lambda(1\text{-}\beta)Q^{1}_{j,n}(x)+Q^{1}_{j-1,n}(0)v_1(x) + \beta\sum_{k=1}^{n} Q^{1}_{j,n-k}(x)\;\lambda\;g_k,$

$0 \le n \le a\text{-}1,\; j \ge 2$ (9)

$-Q^{2}_{1,n}(x) = -\lambda Q^{2}_{1,n}(x) + \lambda(1\text{-}\beta)Q^{2}_{j,n}(x) +\beta \sum_{k=1}^{n} Q^{2}_{j,n-k}(x)\;\lambda\;g_k + \sum_{k=1}^{\infty} Q^{1}_{k,n}(0)v_2(x),$

$a \le n \le N\text{-}1$ (10)

$-Q^{2}_{j,n}(x) = -\lambda Q^{2}_{j,n}(x) + \lambda(1\text{-}\beta)Q^{2}_{j,n}(x)\;\; + \beta\sum_{k=1}^{n} Q^{2}_{j,n-k}(x)\;\lambda\;g_k + Q^{2}_{j-1,n}(0)v_2(x)\;,$

$a \le n \le N\text{-}1,\;\; j \ge 2$ (11)

$-Q^{2}_{j,n}\;(x) = -\lambda Q^{2}_{j,n}(x) + \lambda(1\text{-}\beta)Q^{2}_{j,n}(x)\;\; + \beta \sum_{k=1}^{n} Q^{2}_{j,n-k}(x)\;\lambda\;g_k,\; n \ge N,\;\; j \ge 1$ (12)

$-U^{1}_{n}(x) = \text{-}\lambda U_n\;(x)+ \lambda(1\text{-}\alpha)\;U_n\;(x)+\alpha \sum_{k=1}^{n} U_{n-k}\;\;(x)\;\lambda\;g_k\;\; +\sum_{l=1}^{\infty} Q^{1}_{l,n}\;(0)\;s(x)$

$+\; \sum_{l=1}^{\infty} Q^{2}_{l,n}\;(0)\;s(x),\;\; n\ge a$ (13)

Taking LST on both sides of the equation (1) through (13), we have

$\theta\widetilde{P}_{i,0}(\theta) - P_{i,0}(0) = \lambda(1-\alpha)\widetilde{P}_{i,0}(\theta) - \sum_{m=a}^{b} P_{m,i}(0)\tilde{S}(\theta)\;\text{-}\;U_i\;(0)\;\tilde{S}(\theta\;),\;\; a \le i \le b$ (14)

$\theta\widetilde{P}_{i,j}(\theta) - P_{i,j}(0) = \lambda\widetilde{P}_{i,j}(\theta) - \lambda(1-\alpha)\widetilde{P}_{i,j}(\theta) - \alpha\sum_{k=1}^{j} \widetilde{P}_{i,j-k}(\theta)\lambda g_k\;,\; a \le i \le b\text{-}1,\;\; j\ge 1$ (15)

$\theta\widetilde{P}_{b,j}(\theta) - P_{b,j}(0) = \lambda\widetilde{P}_{bj}(\theta) - \lambda(1-\alpha)\widetilde{P}_{b,j}(\theta) - \sum_{m=a}^{b} P_{mb+j}(0)\tilde{S}(\theta) - \alpha\sum_{k=1}^{j} \widetilde{P}_{bj-\,k}(\theta)\lambda g_k\;,$

$1 \le j \le N - b - 1$ (16)

$\theta\widetilde{P}_{b,j}(\theta) - P_{b,j}(0) = \lambda\widetilde{P}_{bj}(\theta)\lambda(1-\alpha)\widetilde{P}_{b,j}(\theta) - \sum_{m=a}^{b} P_{m,b+j}(0)\tilde{S}(\theta) - \alpha\sum_{k=1}^{j} \widetilde{P}_{b,j-\,k}(\theta)\lambda g_k$

$- \sum_{1=1}^{\infty} Q^{1}_{1,b+j}(0)\;\tilde{S}(\theta) - U_{b+j}(0)\;\tilde{S}(\theta),\; j \ge N - b$ (17)

$\theta\widetilde{Q}^{1}_{1,0}(\theta) - Q^{1}_{1,0}(0) = \lambda\widetilde{Q}^{1}_{1,0}(\theta) - \lambda(1-\beta)\widetilde{Q}_{1,0}(\theta) - \sum_{m=a}^{b} P_{m,0}(0)\tilde{v}_1(\theta)$ (18)

$$\theta\widetilde{Q}^{1}_{1,n}(\theta) - Q^{1}_{1,n}(0) = \lambda\widetilde{Q}^{1}_{1,n}(\theta) - \lambda(1-\beta)\widetilde{Q}^{1}_{1,n}(\theta) - \sum_{m=a}^{b} P_{m,n}(0)\tilde{v}_1(\theta)$$

$-\beta \sum_{k=1}^{n} \widetilde{Q}^{1}_{1,n-k}(\theta)\lambda g_k,\; 0\le n\le a-1$ (19)

$\theta\widetilde{Q}^{1}_{j,n}(\theta) - Q^{1}_{j,n}(0) = \lambda\widetilde{Q}^{1}_{j,n}(\theta) - \lambda(1-\beta)\widetilde{Q}^{1}_{l,n}(\theta) - \beta\sum_{k=1}^{n} \widetilde{Q}^{1}_{l,n-k}(\theta)\lambda g_k,\; n\ge a,\qquad j\ge 1$ (20)

$\theta\widetilde{Q}^{1}_{j,0}(\theta) - Q^{1}_{j,0}(0) = \lambda\widetilde{Q}^{1}_{j,0}(\theta) - \lambda(1-\beta)\widetilde{Q}^{1}_{l,n}(\theta) - Q^{1}_{j-1,0}(0)\tilde{v}_l(\theta) j \ge 2$ (21)

$$\theta\widetilde{Q}^{1}_{j,n}(\theta) - Q^{1}_{j,n}(0) = \lambda\widetilde{Q}^{1}_{j,n}(\theta) - \lambda(1-\beta)Q^{1}_{j,n}(\theta) - Q^{1}_{j-1,n}(0)\tilde{v}_l(\theta) - \beta\sum_{k=1}^{n} \widetilde{Q}^{1}_{j,n-k}(\theta)\lambda g_k,$$

$0 \le n \le a\text{-}1,\qquad j \ge 2$ (22)

$\theta\widetilde{Q}^{2}_{l,n}(\theta) - Q^{2}_{l,n}(0) = \lambda\widetilde{Q}^{2}_{l,n}(\theta) - \lambda(1-\beta)Q^{2}_{j,n}(\theta) - \sum_{k=1}^{\infty} Q^{1}_{k,n}\;(0)\tilde{v}_2(\theta)$

$-\beta \sum_{k=1}^{n} \widetilde{Q}^{2}_{l,n-k}(\theta)\lambda g_k\;,\; a \le n \le N\;\text{-}\;1$ (23)

$$\theta\widetilde{Q}^{2}_{j,n}(\theta)-Q^{2}_{j,n}(0)=\lambda\widetilde{Q}^{2}_{j,n}(\theta)-\lambda(1-\beta)Q^{2}_{j,n}(\theta)-Q^{2}_{j-1,n}(0)\tilde{v}_{l}(\theta)-\beta\sum_{k=1}^{n}\widetilde{Q}^{2}_{l,n-k}(\theta)\lambda g_{k}$$

$a \le n \le N-1, \quad j \ge 2$ (24)

$\theta\widetilde{Q}^{2}_{j,n}(\theta)-Q^{2}_{j,n}(0)=\lambda\widetilde{Q}^{2}_{j,n}(\theta)-\lambda(1-\beta)Q^{2}_{j,n}(\theta)-\beta\sum_{k=1}^{n}\widetilde{Q}^{2}_{j,n-k}(\theta)\lambda g_{k}$ (25)

$\theta\widetilde{U}_{n}(\theta)-U_{n}(0)=-\lambda\widetilde{U}_{n}(\theta)-\lambda(1-\alpha)\widetilde{U}_{n}(\theta)-\alpha\sum_{k=1}^{n}U_{n-k}(x)\lambda g_{k}-\sum_{l=1}^{\infty}Q^{1}_{l,n}(0)s(\theta)$
$-\sum_{l=1}^{\infty}Q^{2}_{l,n}(0)s(\theta)$, $n \ge a$ (26)

Define the following probability generating functions:

$\tilde{P}_{i}(z,\theta)=\sum_{j=0}^{\infty}\tilde{P}_{i,j}(\theta)z^{j}$, $P_{i}(z,0)=\sum_{j=0}^{\infty}\tilde{P}_{i,j}(0)z^{j}$; $a \le i \le b$

$\widetilde{Q}^{1}_{j}(z,\theta)=\sum_{n=0}^{\infty}\widetilde{Q}^{1}_{j,n}(\theta)z^{n}$, $Q^{1}_{j}(z,0)=\sum_{n=0}^{\infty}Q^{1}_{j,n}(0)z^{n}$; $j \ge 1$ (27)

$\widetilde{Q}^{2}_{j}(z,\theta)=\sum_{n=a}^{\infty}\widetilde{Q}^{2}_{j,n}(\theta)z^{n}$, $Q^{2}_{j}(z,0)=\sum_{i=a}^{\infty}Q^{2}_{j,n}(0)z^{n}$; $j \ge 1$

$\widetilde{U}(z,\theta)=\sum_{n=a}^{\infty}\widetilde{U}(\theta)z^{n}$, $U(z,0)=\sum_{n=a}^{\infty}\widetilde{U}(0)z^{n}$,

The probability generating function P(z) of the number of customers in the queue at an arbitrary time can be obtained using the following equation.

$P(z)=\sum_{i=a}^{b-1}\widetilde{P}_{i}(z,0)+\widetilde{P}_{i}(z,0)+\sum_{j=1}^{\infty}\widetilde{Q}^{1}_{j}(z,0)+\sum_{j=1}^{\infty}\widetilde{Q}^{2}_{j}(z,0)+U(z,0)$ (28)

From equations (18), (19), (20) and (27)

$\left(\theta-\beta(\lambda+\lambda X(z))\right)\widetilde{Q}^{1}_{1}(z,\theta)=Q^{1}_{1}(z,0)-\widetilde{V}_{1}(\theta)\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}(0)z^{n}$ (29)

From equations (21), (22), (20) and (27)

$\left(\theta-\beta(\lambda+\lambda X(z))\right)\widetilde{Q}^{1}_{j}(z,\theta)=Q^{1}_{j}(z,0)-\widetilde{V}_{1}(\theta)\sum_{n=0}^{a-1}Q^{1}_{j-1,n}(0)Z^{n}$, $j \ge 2$ (30)

From equations (23), (24), (25) and (27)

$\left(\theta-\beta(\lambda+\lambda X(z))\right)\widetilde{Q}^{2}_{1}(z,\theta)=Q^{2}_{1}(z,0)-\widetilde{V}_{2}(\theta)\sum_{n=a}^{N-1}\sum_{k=1}^{\infty}Q^{1}_{k,n}(0)z^{n}$ (31)

$\left(\theta-\beta(\lambda+\lambda X(z))\right)\widetilde{Q}^{2}_{j}(z,\theta)=Q^{2}_{j}(z,0)-\widetilde{V}_{2}(\theta)\sum_{n=a}^{N-1}Q^{2}_{j-1,n}z^{n}$, $j \ge 2$ (32)

From equations (14),(15)and(27)

$\left(\theta-\alpha(\lambda+\lambda x(z)\right)\widetilde{P}_{i}(z,\theta)=P_{i}(z,0)-\left[\sum_{m=a}^{b}P_{m,i}(0)+U_{i}(0)\right]\tilde{S}(\theta)$, $a \le i \le b-1$ (33)

From equations (16), (17) and (27)

$$Z^{b}[\theta-\alpha(\lambda+\lambda x(z)]\widetilde{P}_{b}(z,\theta)=z^{b}P_{b}(z,0)-\tilde{s}(\theta)\left[\sum_{m=a}^{b}P_{m}(z,0)-\sum_{j=0}^{b-1}P_{m,j}(0)z^{j}\right]$$

$-\tilde{s}(\theta)\left[U(z,0)-\sum_{n=0}^{b-1}U_{n}(0)z^{n}\right]$ (34)

From equations (26) and (27)

$\left[\theta-\alpha\left(\lambda-\lambda X(z)\right)\right]\widetilde{U}(z,\theta)=U(z,0)-\widetilde{U}(\theta)\sum_{l=1}^{\infty}\left(Q_{l}(z,0)-\sum_{n=0}^{a-1}Q_{l,n}(0)z^{n}\right)$ (35)

By substituting $\theta=\beta(\lambda-\lambda x(z))$ in (27)-(30)

$Q^{1}_{l}(z,0)=\widetilde{V}_{l}\beta(\lambda-\lambda x(z))\sum_{n=0}^{a-1}\sum_{m=a}^{b}P_{m,n}(0)z^{n}$ (36)

$Q^{1}_{j}(z,0)=\widetilde{V}_{l}\beta(\lambda-\lambda x(z))\sum_{n=0}^{a-1}Q^{1}_{j-1}(0)z^{n}$, $j \ge 2$ (37)

$Q^{2}_{1}(z,0)=\widetilde{V}_{2}\beta(\lambda-\lambda x(z))\sum_{n=a}^{N-1}\sum_{k=1}^{\infty}Q^{1}_{k,n}(0)z^{n}$, (38)

$Q^{2}_{j}(z,0)=\widetilde{V}_{2}\beta(\lambda-\lambda x(z))\sum_{n=a}^{N-1}Q^{2}_{j-1,n}z^{n}$; $j \ge 2$ (39)

By substituting $\theta=\alpha(\lambda-\lambda x(z))$ in (33)-(35)

$P_{i}(z,0)=\tilde{S}\alpha(\lambda-\lambda x(z))\sum_{m=a}^{b}P_{m,i}(0)+U_{i}(0)$; $a \le i \le b-1$ (40)

$z^{b}P_{b}(z,0)=\tilde{S}\alpha\left(\lambda-\lambda x(z)\right)\{\sum_{m=a}^{b}\left[P_{m}(z,0)-\sum_{j=0}^{b-1}P_{m,j}(0)z^{j}\right]$
$+\left[U(z,0)-\sum_{n=0}^{b-1}U_{n}(0)z^{n}\right]\}$ (41)

$U(z,0)=\widetilde{U}\left(\alpha(\lambda-\lambda x(z)\right)[\sum_{l=1}^{\infty}\left(Q^{1}_{l}(z,0)-\sum_{n=0}^{a-1}Q^{1}_{l,n}(0)z^{n}\right)]$

$+\sum_{l=1}^{\infty}\left(Q_l^2(z,0)-\sum_{n=0}^{a-1}Q_{l,n}^2(0)z^n\right)$ (42)

solving for $P_b(z,0)$,

$P_b(z,0)$=$P_b(z,0)\left[z^b-\tilde{s}\alpha(\lambda-\lambda X(z))\right]=$

$\tilde{s}\alpha(\lambda-\lambda X(z))\ \{\sum_{m=a}^{b-1}\left[\tilde{S}(\lambda-\lambda X(z))\sum_{n=a}^{b}P_{n,m}(0)\right]-\sum_{m=a}^{b}\sum_{j=0}^{b-1}P_{m,j}(0)z^j$

$+\left[U(z,0)-\sum_{n=0}^{b-1}U_n(0)z^n\right]\}$ (43)

Let $p_i=\sum_{m=a}^{b}P_{m,i}(0), q_i^1=\sum_{l=1}^{\infty}Q^1{}_{l,i}(0)$, and $q_i^2=\sum_{l=1}^{\infty}Q^2{}_{l,i}(0)$, $i\geq 0$

From the equations (36) and (29)

$\tilde{Q}_l^1(z,\theta)==\frac{1}{(\theta-\beta(\lambda+\lambda X(z)))}\{\tilde{V}_1\beta(\lambda-\lambda X(z))-\tilde{V}_1(\theta)\sum_{n=0}^{a-1}p_nz^n\}$, (44)

From equations (37) and (30)

$\tilde{Q}_j^1(z,\theta)==\frac{1}{(\theta-\beta(\lambda+\lambda X(z)))}\{\tilde{V}_2\beta(\lambda-\lambda X(z))-\tilde{V}_1(\theta)\}\sum_{n=0}^{a-1}Q_{j-1}^1(0)z^n, j\geq 2$

From the equations (38) and (31)

$\tilde{Q}_l^2(z,\theta)==\frac{1}{(\theta-\beta(\lambda+\lambda X(z)))}\{\tilde{V}_2\beta(\lambda-\lambda X(z))-\tilde{V}_2(\theta)\}\ \sum_{n=a}^{N-1}Q^1{}_{k,n}z^n$

From equations(36) and (30)

$\tilde{Q}_j^2(z,\theta)==\frac{1}{(\theta-\beta(\lambda+\lambda X(z)))}\{\tilde{V}_2\beta(\lambda-\lambda X(z))-\tilde{V}_2(\theta)\}\sum_{n=a}^{N-1}Q_{j-1}^2(0)z^n, j\geq 2$ (45)

From equations (40) and (34)

$\tilde{P}_i(z,\theta)=\frac{1}{(\theta-\alpha(\lambda+\lambda x(z))}\{\tilde{s}\alpha(\lambda+\lambda X(z))-\tilde{S}(\theta)(P_i+U_i)z^i\}$, $a\leq i\leq b-1$ (46)

From equations (43) and (35)

$$\tilde{U}(z,\theta)=\frac{1}{\theta-\alpha(\lambda-\lambda X(z))}\left[\tilde{U}\alpha(\lambda-\lambda X(z))\right]-\tilde{U}(\theta)\sum_{l=1}^{\infty}\left(Q_l^2(z,0)-\sum_{n=0}^{a-1}Q_{l,n}^2(0)z^n\right)$$

$+(Q_l^2(z,0)-\sum_{n=0}^{a-1}Q_{l,n}^2(0)z^n)$ (47)

From equations (39) and (34)

$\tilde{P}_b(z,\theta)=\frac{(\tilde{S}\alpha(\lambda+\lambda x(z))-\tilde{S}(\theta))f(z)}{(\theta-\alpha(\lambda+\lambda x(z))(z^b-\tilde{s}\alpha(\lambda-\lambda X(z))}$ (48)

where $f(z)=\tilde{S}\alpha(\lambda+\lambda x(z))\{\sum_{m=a}^{b-1}p_m(z,0)-\sum_{j=0}^{b-1}p_jz^j\tilde{U}\alpha(\lambda-\lambda X(z))\ [\tilde{V}_1\beta(\lambda-\lambda X(z))\sum_{n=0}^{a-1}(p_n+q_n^1)z^n-\sum_{j=0}^{N-1}Q_j^1z^j+\tilde{V}_2\beta(\lambda-\lambda X(z))\sum_{j=a}^{N-1}(q_n^1+q_n^2)z^i-\sum_{j=a}^{N-1}Q_j^2z^i\ ]$

Let P(z)=be the PGF of the queue size at an arbitrary time epoch.

$P(z)=\sum_{i=a}^{b-1}\tilde{P}_i(z,0)+\tilde{P}_i(z,0)+\sum_{j=1}^{\infty}\tilde{Q}_j^1(z,0)+\sum_{j=1}^{\infty}\tilde{Q}_j^2(z,0)+U(z,0)$

Using equations (40), (41), (42), (43), (44), (45), (46) in (27)

$$P(Z)=\frac{\begin{bmatrix}\beta(\tilde{S}(\alpha(\lambda-\lambda(x)-1))\sum_{i=a}^{b-1}(p_i+u_i)(z^b-z^i)\\ \beta\tilde{U}(\tilde{V}_1(\lambda-\lambda X(z)-1)(\sum_{n=0}^{a-1}(p_n+q_n^1)z^n+\\ \beta\tilde{U}(\tilde{V}_2(\lambda-\lambda X(z))-1)\sum_{n=a}^{N-1}(q_n^1+q_n^2)z^n\end{bmatrix}}{\alpha\beta(-\lambda+\lambda x(z))(z^b-\tilde{s}(\alpha(\lambda-\lambda X(z))}\quad (49)$$

The probability generating function P(z) has to satisfy P(1)=1. Applying L 'Hospital's rule and evaluating $\lim_{z\to 1}P(z)$ and equating the expression to 1, b- λE(X)E(S) > 0 is obtained.Define 'ρ' as $\frac{\alpha\lambda E(X)E(S)}{b}$. Thus ρ<1 is the condition to be satisfied for the existence of steady state for model under consideration.

3.1. Computational aspects of unknown probabilities

Equation (49) gives the probability generating function P(z) of the number of customers in the queue at an arbitrary time epoch, which involves 2N + 1- a unknown probabilities namely, $q_0^1, q_1^1, q_2^1, q_3^1, \ldots\ldots q_{N-1}^1, q_a^2, q_{a+1}^2, q_{a+2}^2, q_{a+3}^2, \ldots\ldots q_{N-1.}^2$ $q_{i;i=0,1,2,\ldots N-1}^1$ and $q_{i;i=a,a+1,a+2,\ldots N-1}^2$ are expressed in terms of $p_{i;i=0,1,2,\ldots a-1.}$ Equation (49) has N+b unknowns $p_0, p_1, p_2, p_3 \ldots p_{b-1}, q_0, q_1, \ldots\ldots q_{N-1}$. The following theorms are proven to express q_i in terms p_i in such a way that the numerators has only b constants. By Rouche's theorm of complex variables, it can be proved that $(z^b - 1)\tilde{s}(\alpha(\lambda - \lambda x(z)))$ has b-1 zeros inside and one of the unit circle $|z|$=1. Since P(z) is analytic within and on the unit circle, the numerator must vanish at these points, which gives b equations with b unknowns.

3.1. Expected queue length

The expected queue length E(Q) (i.e. mean number of customers waiting in the queue) at an arbitrary time epoch, is differentiating P(Z) at z=1 and is given by $\lim_{z\to 1} P(z)$ = E(Q),

$$E(Q)=\frac{1}{2\alpha\beta\lambda E(X)[b-S1]2}\begin{bmatrix} \sum_{i=a}^{b-1}(p_i + u_i)\beta\,(b(b-1) - i(i-1)\,f1 \\ + \sum_{i=a}^{b-1}(p_i + u_i)\beta(b-i)f2 \\ + \sum_{n=0}^{a-1}\beta\big(p_n + q_n^1\big)\,f3 + \sum_{n=0}^{b-1}\alpha\big(p_n + q_n^1\big)nf4 \\ + \sum_{n=a}^{N-1}\beta\big(q_n^1 + q_n^2\big)f5 + \sum_{n=a}^{N-1}\alpha\big(q_n^1 + q_n^2\big)nf6 \end{bmatrix}$$

where $S1=\alpha\lambda E(X)E(S)$; $S2 = \alpha\lambda E''E(S)+ \alpha^2\lambda^2 E^2(X)E(S^2)$; $V1=\beta\lambda E(X)E(V_1)$; $V2=\beta\lambda E''(X)E(V_1)+\lambda^2\beta^2E^2(X)E({V_1}^2)$; $V3=\beta\lambda E(X)E(V_2)$; $V4=\beta\lambda E''E(V_2)+ \beta^2\lambda^2E^2(X)E({V_2}^2)$

$T1=\lambda\alpha E(X)(b(b-1) - 2\alpha\lambda E''(1)E(S) - \alpha^2\lambda^2E^2(X)E(S^2))+b(b-1)$;

f1=[b-s1]S1; f2=[b-s1]S2-αE(S)T1;

f3=[b(b-1)(U2 V1+ 2U1V2 +V2U1)+b(U1 V2+ 2U2V1 +V1U2](b-S1)(U2 V1+ 2U1V2 +V2U1)T1;

f4=2b(U2 V1+ 2U1V2 +V2U1) [b-s1];

f5=[b(b-1)V3+bV4]$(b - S1) - b(U1\,V2 + 2U2V1 + V1U2)$ T1 and f6= 2bV3[b-S1]

3.2. Particular cases

In this section, some of the existing models are deduced as a particular case of the proposed model.

Case (i): Considering single service, (i.e. a=b=1), and if there is no vacation of type two $(\tilde{V}_2\beta(\lambda - \lambda X(Z)) = 1)$, then the Equation (49) reduces to

$P(z)=\frac{1}{(-\lambda+\lambda X(z))(z-S\alpha(\lambda-\lambda X(z)))}\,[\tilde{V}_1\beta(\lambda - \lambda X(z) - 1)(p_0 + \sum_{n=0}^{N-1} q_n^1 z^n]$ which coincides with the queue size distribution of a $M^x/G/1$ queueing system with N-policy and multiple vacations.

Case (ii): If there is no vacation of type two (i.e., $\tilde{V}_2\beta(\lambda - \lambda(z)) = \tilde{V}_1\beta(\lambda - \lambda X(z))$),

$P(z)=\frac{1}{(-\lambda+\lambda X(z))(z^b-S\alpha(\lambda-\lambda X(z)))}\,\{\,(\tilde{S}\alpha\big(-\lambda + \lambda X(z)\big) - 1)\sum_{i=a}^{b-1}\beta\big(z^b - z^i\big)p_i + \tilde{V}_1\beta\big(\lambda - \lambda X(z)\big) - 1\,(\sum_{n=0}^{a-1}p_n z^n + \sum_{n=0}^{N-1}q_n^1 z^n)\}$which gives the queue size distribution of a $M^x/G(a,b)/1$ queueing system with multiple vacations and N policy.

Case (iii): If all arrivals are allowed to join the system, i.e. $\alpha=1$, $\beta=1$, then (49) becomes $P(z)=\frac{\{\sum_{i=a}^{b-1}(\tilde{S}(\lambda-\lambda X(z))-1)(z^b-z^i)p_i+(\tilde{V}_2(\lambda-\lambda x(z)-1),\ \tilde{V}_1(\lambda-\lambda X(z))-1\sum_{n=0}^{a-1}(Z^b-1)C_n z^n\}}{[z^b-\tilde{S}(\lambda-\lambda X(z)][\lambda X(z)-\lambda]}$ which coincides with the result $M^x/G(a,b)/1$ and multiple without setup time and N- Policy of Krishna Reddy et al (1998).

6. Conclusion

A bulk arrival general bulk service queueing with variant threshold policies for secondary jobs is analyzed. The probability generating function for queue size at an arbitrary epoch is derived. Various performance measures are also obtained. Some particular cases are also discussed.

REFERENCES

1. M.F.Neuts, The M/G/1 queue with limited number of admission or a limited period during each service time, *Stochastic Models,* 1 (1985) 361-391.
2. H.W.Lee, M/G/1 queue with exceptional first vacation, *Comput, and Oper. Res.,* 15 (5) (1998) 441-445.
3. G.Choudhury, Analysis of the $M^X/G/1$ queueing system with vacation times, *Sankhya-Series B*, 64 (1) (2002) 37-49.
4. K.C.Madan and W.Abu-Davyeh, Steady state analysis of a single server bulk queue with vacation times and restricted admissibility of arriving batches, *Revista Investigation Operational*, 24 (2002) 113-123.
5. G.Choudhury, A batch arrival queue with a vacation time under single vacation policy, *Computers and Operations Research*, 14 (2002) 1941-1955.
6. K.C.Madan and G.Choudhury, A two stage arrival queueing system with a modified Bernoulli schedule vacation under N-policy, *Mathematical and Computer Modelling,* 42 (2005) 71-85.
7. B.T.Doshi, Queueing systems with vacations-a survey, *Queueing Syst.,* 1 (1986) 26-66.
8. R.Arumuganathan and S. Jeyakumar, Analysis of a bulk queue with multiple vacations and closedown times, *Int.J. Inform. Manage. Sci.,* 15 (1) (2004) 45-60.
9. H.W. Lee, Steady state probabilities for the server vacation model with group arrivals and under control operation policy, *J. Korean OR/MS Soc.,* 16 (1991) 36-48.
10. H.W.Lee, S.S.Lee, J.O.Park and K.C.Chae, Analysis of the $M^x/G/1$ queue with N-policy and multiple vacations, *J. Appl. Prob.,* 31 (1994) 476-496.
11. S.S.Lee, H.W.Lee, S.H.Yoon and K.C.Chae, Batch arrival queue with N-policy and single vacation, *Comput. Oper. Res.,* 22(2) (1995) 173-189.
12. G.V.Krishna Reddy, R.Nadarajan and R.Arumuganathan, Analysis of a bulk queue with N-policy multiple vacations and set up times, *Comput. Ops. Res.,* 25(11) (1998) 957-967.