Intern. J. Fuzzy Mathematical Archive
Vol. 7, No. 2, 2015, 213-224
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 22 January 2015
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Solution and Stability of an ACQ Functional Equation in Generalized 2-Normed Spaces

***John M. Rassias*[1], *M. Arunkumar*[2], *E. Sathya*[3] and *N. Mahesh Kumar*[4]**

[1]Pedagogical Department E.E., Section of Mathematics and Informatics
National and Capodistrian University of Athens, 4, Agamemnonos Str.
Aghia Paraskevi, Athens 15342, Greece
e-mail: jrassias@primedu.uoa.gr, URL:http: www.primedu.uoa.gr/~jrassias/
[1]Corresponding Author

[2,3]Department of Mathematics, Government Arts College, Tiruvannamalai – 606 604
TamilNadu, India. e-mail: [2]annarun2002@yahoo.co.in,
[3]sathya24mathematics@gmail.com
[4]Department of Mathematics, Arunai Engineering College, Tiruvannamalai – 606 604
TamilNadu, India. e-mail: [4]mrnmahesh@yahoo.com

Received 3 November 2014; accepted 15 December 2014

Abstract. In this paper, the authors investigate the solution and generalized Ulam – Hyers stability of a mixed type additive-cubic-quartic functional equation

$$11[g(u+2v+2w)+g(u-2v-2w)]+66g(u)+48g(2v+2w)$$
$$= 44[g(u+v+w)+g(u-v-w)]+12g(3v+3w)+60g(v+w)$$

in generalized 2-normed spaces. Counterexamples for non-stability cases are also discussed.

Keywords: Additive functional equation, cubic functional equations, quartic functional equation, mixed type functional equations. Generalized Ulam –Hyers Stability, Generalized 2-normed spaces

AMS Mathematics Subject Classification (2010): 39B55, 39B52, 39B8

1. Introduction

Ulam [27] is the pioneer for the famous stability problem in functional equations. In 1940, while he was delivering a talk before the Mathematics Club of University of Wisconsin, he proposed a number of unsolved problems. Among those the following question was concerning the stability of homomorphisms:

"Let G be group and H be a metric group with metric $d(.,.)$. Given $\varepsilon > 0$ does there exists a $\delta > 0$ such that if a function $f: G \to H$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then is there exists a homomorphism $a: G \to H$ with $d(f(x), a(x)) \le \varepsilon$ for all $x \in G$.?" The case of approximately additive functions was solved by Hyers [14] under the assumption that G and H are Banach spaces. It was further generalized excellent results were obtained by number of mathematicians [3, 5 – 13, 15 – 26] .

In this paper, the authors investigate general solution and the generalized Ulam - Hyers stability of a mixed type additive-cubic-quartic functional equation

$$\begin{aligned} &11[g(u+2v+2w)+g(u-2v-2w)]+66g(u)+48g(2v+2w) \\ &\quad = 44[g(u+v+w)+g(u-v-w)]+12g(3v+3w)+60g(v+w) \end{aligned} \tag{1.1}$$

in generalized 2-normed spaces. The function $g(x) = ax+bx^3+cx^4$ is the solution for the functional equation (1.1). Counterexamples for non-stability casesare also discussed.

2. Basic definitions on generalized 2-normed space

Definition 2.1. [4] Let X be a linear space. A function $N(.,.): X \times X \to [0,\infty)$ is called a generalized 2-normed space if it satisfies the following properties

[M1] $N(x,y) = N(y,x)$ for all $x, y \in X$,

[M2] $N(\lambda x, y) = |\lambda| N(x,y)$ for all $x, y \in X$ and X $=\varphi$, φ is a real or complex field,

[M3] $N(x+y,z) \le N(x,z)+N(y,z)$ for all $x, y, z \in X$.

The generalized 2-normed space is denoted by (X,N(. , .)).

Definition 2.2. [4] A sequence $\{x_n\}$ in a generalized 2-normed space (X,N(. , .)) is called convergent if there exist $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, y) = 0$ then $\lim_{n\to\infty} N(x_n, y) = N(x,y)$ for all $y \in X$.

Definition 2.3. [4] A sequence $\{x_n\}$ in a generalized 2-normed space (X,N(. , .)) is called Cauchy sequence if there exist two linearly independent elements y and z in X such that $\{N(x_n, y)\}$ and $\{N(x_n, z)\}$ are real Cauchy sequences.

Definition 2.4. [4] A generalized 2-normed space (X,N (. , .)) is called generalized 2-Banach space if every Cauchy sequence is convergent.

3. Solution for the functional equation (1.1)

In this section, the authors discuss the general solution for the functional equation (1.1). Throughout this section, let us consider U and V be real vector spaces.

Lemma 3.1. Let $g: U \to V$ be an even mapping satisfying (1.1), then g is quartic.

Proof: Given $g: U \to V$ is an even mapping satisfying (1.1). Letting w=0 in (1.1), we obtain

$$\begin{aligned} &11[g(u+2v)+g(u-2v)]+66g(u)+48g(2v) \\ &\quad = 44[g(u+v)+g(u-v)]+12g(3v)+60g(v). \end{aligned} \tag{3.1}$$

By Lemma 2.1 of [12], we desired our result.

Lemma 3.2. Let $g: U \to V$ be an odd mapping satisfying (1.1), then g is additive-cubic.

Proof: By data $g: U \to V$ is an odd mapping satisfying (1.1). Letting w=0 in (1.1), we get (3.1). By Lemma 2.2 of [12], we desired our result..

Theorem 3.1. A mapping $g:U \to V$ satisfies (1.1) for all $u,v,w\in U$ if and only if there exist a unique additive mapping $A:U \to V$, a unique cubic mapping $C:U\times U\times U \to V$, and a unique quartic symmetric multi-additive mapping $Q:U\times U\times U\times U \to V$, such that

$$g(u)= A(u)+C(u,u,u)+Q(u,u,u,u) \tag{3.2}$$

for all $u\in U$ and C is symmetric for each fixed one variable and is additive for fixed two variables.

Proof: By hypothesis, $g:U \to V$ satisfies (1.1). Letting w=0 in (1.1), we obtain (3.1). By Lemma 2.3 of [12], our result is demonstrated.

3.1. Stability of the functional equation (1.1)

In this section, the authors investigate generalized Ulam - Hyers stability of the functional equation (1.1) in generalized 2-normed spaces. Let U be generalized 2-normed space and V be generalized 2-Banach space. Define a function $g:U \to V$ by

$$Dg(u,v,w)= 11[g(u+2v+2w)+g(u-2v-2w)]+66g(u)+48g(2v+2w)$$
$$-44[g(u+v+w)+g(u-v-w)]-12g(3v+3w)-60g(v+w).$$

for all $u,v,w\in U$. Also, throughout this paper, we use the following notation $\alpha(u,v,w)=\alpha\left((u,y),(v,y),(w,y)\right)$ and $\|u\|=\|u,y\|$ for all $u,v,w\in U$ and all $y\in U$.

Theorem 3.1.1. Let $j\in\{-1,1\}$. Let $\alpha: X^3 \to [0,\infty)$ be a function such that

$$\sum_{n=0}^{\infty}\frac{\alpha\left(2^{nj}u,2^{nj}v,2^{nj}w\right)}{16^{nj}} \quad converges \ and \ \lim_{n\to\infty}\frac{\alpha\left(2^{nj}u,2^{nj}v,2^{nj}w\right)}{16^{nj}}=0 \tag{3.1.1}$$

for all $u,v,w\in U$. Suppose an even function $g:U \to V$ with $g(0)=0$ satisfies the inequality

$$N(Dg(u,v,w),\ y)\ \le \alpha(u,v,w) \tag{3.1.2}$$

for all $u,v,w\in U$ and $y\in U$. Then there exists a unique quartic function $Q:U \to V$ such that

$$N(g(v)\ -\ Q(v),y)\ \le\ \sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta\left(2^{kj}v\right)}{16^{kj}} \tag{3.1.3}$$

where $\beta(2^{kj}v)=\dfrac{1}{352}\left[12\alpha(2^{kj}v,2^{kj}v,0)+\alpha(0,2^{kj}v,0)\right]$ (3.1.4)

for all $v\in U$. The mapping $Q(v)$ is defined by $\lim_{n\to\infty} N\left(Q(v)\ -\ \dfrac{g(2^{nj}v)}{16^{nj}},\ y\right)=0$ (3.1.5)

for all $v\in U$ and all $y\in U$.

Proof: Assume $j=1$. Setting (u,v,w) by $(v,v,0)$ in (3.1.2), we get

$$N(-g(3v)+4g(2v)+17g(v),\ y)\ \le\alpha(v,v,0) \tag{3.1.6}$$

for all $v\in U$ and all $y\in U$.

From (4.6) and (M2), it is easy to verify that

$$N(12g(3v)-48g(2v)-204g(v),\ y)\ \le 12\alpha(v,v,0) \tag{3.1.7}$$

for all $v\in U$ and all $y\in U$. Replacing $u=w=0$ in (4.2), we get

$$N(70g(2v)-12g(3v)-148f(v),\ y)\ \leq \alpha(0,v,0) \qquad (3.1.8)$$

for all $v\in U$ and all $y\in U$. Combining (3.1.7) and (3.1.8) with the help of (M3), we arrive

$$N\left(\frac{g(2v)}{16}-g(v),\ y\right)\ \leq \frac{1}{352}[12\alpha(v,v,0)+\alpha(0,v,0)] \qquad (3.1.9)$$

for all $v\in U$ and $y\in U$.

From (3.1.9), we arrive $N\left(\frac{g(2v)}{16}-g(v),\ y\right)\ \leq \beta(v)$ (3.1.10)

where $\beta(v)=\frac{1}{352}[12\alpha(v,v,0)+\alpha(0,v,0)]$ for all $v\in U$ and all $y\in U$. Replacing v by $2v$ and divided by 16 in (3.1.10), we have

$$N\left(\frac{g(2^2v)}{16^2}-\frac{g(2v)}{16},\ y\right)\ \leq \frac{\beta(2v)}{16} \qquad (3.1.11)$$

for all $v\in U$ and all $y\in U$. Combining (3.1.10) and (3.1.11) and using (M3), we arrive

$$N\left(\frac{g(2^2v)}{16^2}-g(v),\ y\right)\leq N\left(\frac{g(2^2v)}{16^2}-\frac{g(2v)}{16},\ y\right)+N\left(\frac{g(2v)}{16}-g(v)\ ,y\right) \qquad (3.1.12)$$

$$\leq \beta(v)+\frac{\beta(2v)}{16}$$

for all $v\in U$ and all $y\in U$. In general for any positive integer n, we have

$$N\left(\frac{g(2^nv)}{16^n}-g(v),\ y\right)\leq \sum_{k=0}^{n-1}\frac{\beta(2^kv)}{16^k}\leq \sum_{k=0}^{\infty}\frac{\beta(2^kv)}{16^k} \qquad (3.1.13)$$

for all $v\in U$ and all $y\in U$. In order to prove the convergence of the sequence $\left\{\frac{g(2^nv)}{16^n}\right\}$, replace v by 2^mv and divided by 16^m in (4.13), for any m,n>0, we arrive

$$N\left(\frac{g\left(2^{n+m}v\right)}{16^{n+m}}-\frac{g\left(2^mv\right)}{16^m},y\right)=\frac{1}{16^m}N\left(\frac{g\left(2^{n+m}v\right)}{16^n}-g\left(2^mv\right),y\right)\leq\sum_{k=0}^{\infty}\frac{\beta\left(2^{n+m}v\right)}{16^{n+m}}\to 0\ \ as\ \ m\to\infty \qquad (3.1.14)$$

for all $v\in U$ and all $y\in U$. Also

$$N\left(\frac{g\left(2^{n+m}v\right)}{16^{n+m}}-\frac{g\left(2^mv\right)}{16^m},z\right)=\frac{1}{16^m}N\left(\frac{g\left(2^{n+m}v\right)}{16^n}-g\left(2^mv\right),z\right)\leq\sum_{k=0}^{\infty}\frac{\beta\left(2^{n+m}v\right)}{16^{n+m}}\to 0\ \ as\ \ m\to\infty \qquad (3.1.15)$$

for all $v\in U$ and all $z\in U$.

Hence there exist two linearly independent elements y and z in U such that $\left\{N\left(\frac{g(2^nv)}{16^n},y\right)\right\}$ and $\left\{N\left(\frac{g(2^nv)}{16^n},z\right)\right\}$ are real Cauchy sequences. Thus the sequence $\left\{\frac{g(2^nv)}{16^n}\right\}$ is Cauchy sequence. Since V is complete, there exists a mapping $Q:U\to V$ such that $\lim_{n\to\infty}N\left(Q(v)-\frac{g(2^nv)}{16^n},y\right)=0$ for all $v\in U$ and all $y\in U$. Letting $n\to\infty$ in

(3.1.13) we see that (3.1.3) holds for all $v \in U$ and all $y \in U$. To prove Q satisfies (1.1), replacing (u,v,w) by $(2^n u, 2^n v, 2^n w)$ and divided by 16^n in (3.1.2), we get $\frac{1}{16^n} N\left(Dg\left(2^n u,\ 2^n v,\ 2^n w\right),\ y\right) \leq \frac{1}{16^n}\alpha\left(2^n u,\ 2^n v,\ 2^n w\right)$ (3.1.16) for all $u,v,w \in U$ and all $y \in U$. Letting $n \to \infty$ and using the definition of $Q(v)$ we see that Q satisfies (1.1). To prove $Q(v)$ is unique, let $R(v)$ be another quartic mapping satisfying (1.1) and (3.1.3), we arrive

$$N\left(Q(v)-R(v),y\right)=\frac{1}{16^n}N\left(Q\left(2^n v\right)-f\left(2^n v\right),y\right)+N\left(f\left(2^n v\right)-R\left(2^n v\right),y\right)\leq\sum_{k=0}^{\infty}\frac{\beta\left(2^{k+n}v\right)}{16^{k+n}}$$

tends to 0 as n approaches infinity, for all $v \in U$ and all $y \in U$. Hence Q is unique. For $j=-1$, we can prove the similar stability result. This completes the proof of the theorem. The following corollary is an immediate consequence of Theorem 3.1.1 concerning the stability of (1.1).

Corollary 3.1.1. Let λ and s be nonnegative real numbers. If an even function $g:U \to V$ satisfies the inequality

$$N(Dg(u,v,w),y) \leq \begin{cases} \lambda \\ \lambda\left\{\|u\|^s+\|v\|^s+\|w\|^s\right\}, & s<4 \quad or \quad s>4 \\ \lambda\left\{\|u\|^s\|v\|^s\|w\|^s+\|u\|^{3s}+\|v\|^{3s}+\|w\|^{3s}\right\}, & s<\frac{4}{3} \quad or \quad s>\frac{4}{3} \end{cases} \tag{3.1.17}$$

for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique quartic function $Q:U \to V$ such that

$$N(g(v)\ -\ Q(v),\ y)\ \leq \begin{cases} \frac{13\lambda}{330}, \\ \frac{25\lambda}{22|16-2^s|}\|v\|^s \\ \frac{25\lambda}{22|16-2^{3s}|}\|v\|^{3s} \end{cases} \tag{3.1.18}$$

for all $v \in U$ and all $y \in U$.

Theorem 3.1.2. Let $j \in \{-1,1\}$. Let $\alpha: X^3 \to [0,\infty)$ be a function such that

$$\sum_{n=0}^{\infty}\frac{\alpha(2^{nj}u,2^{nj}v,2^{nj}w)}{2^{nj}} \quad converges \quad and \quad \lim_{n\to\infty}\frac{\alpha(2^{nj}u,2^{nj}v,2^{nj}w)}{2^{nj}}=0 \tag{3.1.19}$$

for all $u,v,w \in U$. Suppose an odd function $g:U \to V$ be a function with $g(0)=0$ satisfies the inequality $N(Dg(u,v,w),\ y)\ \leq \alpha(u,v,w)$ (3.1.20)

for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique additive function $A:U \to V$ such that

$$N(g(2v)-8g(v)-A(v),y)\ \le \frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(2^{kj}v)}{2^{kj}} \tag{3.1.21}$$

Where $\gamma(2^{kj}v)\ =\ \frac{14}{33}\alpha(0,2^{kj}v,0)+\frac{1}{11}\alpha(2\cdot 2^{kj}v,2^{kj}v,0)$ (3.1.22)

for all $v\in U$ and all $y\in U$. The mapping $A(v)$ is defined by

$$\lim_{n\to\infty} N\left(A(v)-\frac{g(2^{nj}v)-8g(2^{nj}v)}{2^{nj}}\right)=0 \tag{3.1.23}$$

for all $v\in U$ and all $y\in U$.

Proof: Assume j=1. Setting (u,v,w) by $(0,v,0)$ in (3.1.20) and using oddness of g, we get $N(48g(2v)-12g(3v)-60g(v),\ y)\ \le\alpha(0,v,0)$ (3.1.24)

for all $v\in U$ and all $y\in U$. From (3.1.24) and (M2), it is easy verify that

$$N\left(\frac{14}{11\cdot 3}(48g(2v)-12g(3v)-60g(v)),\ y\right)\le\frac{14}{11\cdot 3}\alpha(0,v,0) \tag{3.1.25}$$

for all $v\in U$ and all $y\in U$. Replacing (u,v,w) by $(2v,v,0)$ in (3.1.20), we get

$$N(11g(4v)-56g(3v)+114g(2v)-104g(v),\ y)\le\alpha(2v,v,0) \tag{3.1.26}$$

for all $v\in U$ and all $y\in U$. From (3.1.26) and (M2), it is easy verify that

$$N\left(\frac{1}{11}(11g(4v)-56g(3v)+114g(2v)-104g(v)),\ y\right)\le\frac{1}{11}\alpha(2v,v,0) \tag{3.1.27}$$

for all $v\in U$ and all $y\in U$. Combining (3.1.25) and (3.1.27) with the help of (M3), we arrive

$$N(g(4v)-10g(2v)+16g(v),y)\le\frac{14}{11\cdot 3}\alpha(0,v,0)+\frac{1}{11}\alpha(2v,v,0) \tag{3.1.28}$$

for all $v\in U$ and all $y\in U$. Equation (3.1.28) can be rewritten as

$$N(g(4v)-8g(2v)-2(g(2v)-8g(v)),y)\ \le\ \gamma(v) \tag{3.1.29}$$

where $\gamma(v)=\frac{14}{11\cdot 3}\alpha(0,v,0)+\frac{1}{11}\alpha(2v,v,0)$ for all $v\in U$ and all $y\in U$.

Using $g_a(v)=g(2v)-8g(v)$, we obtain

$$N(g_a(2v)\text{-}2g_a(v),\ y)\ \le\gamma(v) \tag{3.1.30}$$

for all $v\in U$ and all $y\in U$. The rest of the proof is similar to that of Theorem 3.1.1.The following corollary is an immediate consequence of Theorem 3.1.2 concerning the stability of (1.1).

Corollary 3.1.2. Let λ and s be nonnegative real numbers. If an odd function $g:U\to V$ satisfies the inequality

$$N(Dg(u,v,w),y)\le\begin{cases}\lambda\\ \lambda\{\|u\|^s+\|v\|^s+\|w\|^s\}, & s<1\quad or\quad s>1\\ \lambda\{\|u\|^s\|v\|^s\|w\|^s+\|u\|^{3s}+\|v\|^{3s}+\|w\|^{3s}\}, & s<\frac{1}{3}\quad or\quad s>\frac{1}{3}\end{cases} \tag{3.1.31}$$

for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique additive function $A: U \to V$ such that

$$N(g(v) - A(v), y) \le \begin{cases} \frac{187\lambda}{363}, \\ \left(\frac{187}{363|2-2^{s}|}+\frac{1}{11|1-2^{1-s}|}\right)\lambda\|v\|^{s} \\ \left(\frac{187}{363|2-2^{3s}|}+\frac{1}{11|1-2^{1-3s}|}\right)\lambda\|v\|^{3s} \end{cases} \tag{3.1.32}$$

for all $v \in U$ and $y \in U$

Theorem 3.1.3. Let $j \in \{-1,1\}$. Let $\alpha : X^3 \to [0,\infty)$ be a function such that

$$\sum_{n=0}^{\infty}\frac{\alpha(2^{nj}u,2^{nj}v,2^{nj}w)}{8^{nj}} \text{ converges and } \lim_{n\to\infty}\frac{\alpha(2^{nj}u,2^{nj}v,2^{nj}w)}{8^{nj}}=0 \tag{3.1.33}$$

for all $u,v,w \in U$. Suppose an odd function $g: U \to V$ be a function with $g(0)=0$ satisfies the inequality $N(Dg(u,v,w), y) \le \alpha(u,v,w)$ (3.1.34)

for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique cubic function $C: U \to V$ such that

$$N(g(2v)-2g(v)-C(v),y) \le \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(2^{kj}v)}{8^{kj}} \tag{3.1.35}$$

where $\gamma(2^{kj}v)$ is defined in (3.1.22) for all $v \in U$. The mapping $C(v)$ is defined by

$$\lim_{n\to\infty} N\left(C(v)-\frac{g(2^{nj}v)-2g(2^{nj}v)}{8^{nj}}\right)=0 \tag{3.1.36}$$

for all $v \in U$ and all $y \in U$.

Proof: It follows from (3.1.28) that

$$N(g(4v)-10g(2v)+16g(v),y) \le \frac{14}{11\cdot 3}\alpha(0,v,0)+\frac{1}{11}\alpha(2v,v,0) \tag{3.1.37}$$

for all $v \in U$ and all $y \in U$. Equation (3.1.37) can be rewritten as

$$N(g(4v)-2g(2v)-8(g(2v)-2g(v)),y) \le \gamma(v) \tag{3.1.38}$$

Where $\gamma(v)=\frac{14}{11\cdot 3}\alpha(0,v,0)+\frac{1}{11}\alpha(2v,v,0)$ for all $v \in U$ and all $y \in U$. Using $g_c(v)=g(2v)-2g(v)$, we obtain

$$N(g_c(2v)-8g_c(v), y) \le \gamma(v) \tag{3.1.39}$$

for all $v \in U$ and all $y \in U$. The rest of the proof is similar to that of Theorem 3.1.1.The following corollary is an immediate consequence of Theorem 3.1.3 concerning the stability of (1.1).

Corollary 3.1.3. Let λ and s be nonnegative real numbers. If an odd function $g: U \to V$ satisfies the inequality

$$N(Dg(u,v,w),y) \le \begin{cases} \lambda \\ \lambda\left\{\|u\|^s + \|v\|^s + \|w\|^s\right\}, & s<3 \quad or \quad s>3 \\ \lambda\left\{\|u\|^s\|v\|^s\|w\|^s + \|u\|^{3s} + \|v\|^{3s} + \|w\|^{3s}\right\}, & s<1 \quad or \quad s>1 \end{cases} \quad (3.1.40)$$

for all $v \in U$ and all $y \in U$. Then there exists a unique cubic function $C: U \to V$ such that

$$N(g(v) - A(v), y) \le \begin{cases} \dfrac{187\lambda}{2541}, \\ \left(\dfrac{187}{363|8-2^s|} + \dfrac{1}{11|1-2^{1-s}|}\right)\lambda\|v\|^s \\ \left(\dfrac{187}{363|8-2^{3s}|} + \dfrac{1}{11|1-2^{1-3s}|}\right)\lambda\|v\|^{3s} \end{cases} \quad (3.1.41)$$

for all $v \in U$ and $y \in U$.

Theorem 3.1.4. Let $j \in \{-1,1\}$. Let $\alpha : X^3 \to [0,\infty)$ be a function satisfying (3.1.19) and (3.1.33) for all $u,v,w \in U$. Suppose an odd function $g: U \to V$ be a function with $g(0)=0$ satisfies the inequalities (3.1.20) and (3.1.34) for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique additive function $A: U \to V$ and a unique cubic function $C: U \to V$ such that

$$N(g(v) - A(v) - C(v), y) \le \frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(2^{kj}v)}{2^{kj}} + \frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(2^{kj}v)}{8^{kj}}\right\} \quad (3.1.42)$$

for all $v \in U$ and all $y \in U$, where $\gamma(v)$ is defined in (3.1.22) for all $v \in U$.

Proof: The proof follows by Theorems 3.1.2 and 3.1.3. The following corollary is an immediate consequence of Theorem 3.1.4 concerning the stability of (1.1).

Corollary 3.1.4. Let λ and s be nonnegative real numbers. If an odd function $g: U \to V$ satisfies the inequality

$$N(Dg(u,v,w),y) \le \begin{cases} \lambda \\ \lambda\left\{\|u\|^s + \|v\|^s + \|w\|^s\right\}, & s<1 \quad or \quad s>1 \\ \lambda\left\{\|u\|^s\|v\|^s\|w\|^s + \|u\|^{3s} + \|v\|^{3s} + \|w\|^{3s}\right\}, & s<\frac{1}{3} \quad or \quad s>\frac{1}{3} \end{cases} \quad (3.1.43)$$

for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique additive function $A: U \to V$ and a unique cubic function $C: U \to V$ such that
$N(g(v) - A(v) - C(v), y)$

$$\leq \begin{cases} \left(\frac{187}{363}+\frac{187}{2541}\right)\lambda, \\ \left\{\left(\frac{187}{363|2-2^s|}+\frac{1}{11|1-2^{1-s}|}\right)+\left(\frac{187}{363|2-8^s|}+\frac{1}{11|1-8^{1-s}|}\right)\right\}\lambda\|v\|^s \\ \left\{\left(\frac{187}{363|2-2^{3s}|}+\frac{1}{11|1-2^{1-3s}|}\right)+\left(\frac{187}{363|2-8^{3s}|}+\frac{1}{11|1-8^{1-3s}|}\right)\right\}\lambda\|v\|^{3s} \end{cases} \tag{3.1.44}$$

for all $v \in U$ and all $y \in U$.

Theorem 3.1.5. Let $j \in \{-1,1\}$. Let $\alpha : X^3 \to [0,\infty)$ be a function satisfies (3.1.1) and (3.1.19) and (3.1.33) for all $u,v,w \in U$ and all $y \in U$. Suppose a function $g:U \to V$ be a function with $g(0)=0$ satisfies the inequalities (3.1.2), (3.1.20) and (3.1.34) for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique additive function $A:U \to V$, a unique cubic function $C:U \to V$ and a unique quartic function $Q:U \to V$ such that

$$N\left(g(v)-A(v)-C(v)-Q(v),y\right) \leq \frac{1}{2}\left\{\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(2^{kj}v)}{16^{kj}}+\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\beta(-2^{kj}v)}{16^{kj}}+\frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(2^{kj}v)}{2^{kj}}+\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(2^{kj}v)}{8^{kj}}\right\}\right.$$
$$\left.+\frac{1}{6}\left\{\frac{1}{2}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(-2^{kj}v)}{2^{kj}}+\frac{1}{8}\sum_{k=\frac{1-j}{2}}^{\infty}\frac{\gamma(-2^{kj}v)}{8^{kj}}\right\}\right\} \tag{3.1.45}$$

for all $v \in U$ and all $y \in U$. where $\gamma(2^{kj}v)$ and $\beta(2^{kj}v)$ are defined in (3.1.22) and (3.1.4), respectively for all $v \in U$.

Proof: The proof theorem by using oddness, evenness of g and Theorems 3.1.1 and 3.1.4. The following corollary is the immediate consequence of Theorem 3.1.5 concerning the stability of (1.1).

Corollary 3.1.5. Let λ and s be nonnegative real numbers. If a function $g:U \to V$ satisfies the inequality

$$N(Dg(u,v,w),y) \leq \begin{cases} \lambda \\ \lambda\left\{\|u\|^s+\|v\|^s+\|w\|^s\right\}, & s<1 \quad or \quad s>1 \\ \lambda\left\{\|u\|^s\|v\|^s\|w\|^s+\|u\|^{3s}+\|v\|^{3s}+\|w\|^{3s}\right\}, & s<\frac{1}{3} \quad or \quad s>\frac{1}{3} \end{cases} \tag{3.1.46}$$

for all $u,v,w \in U$ and all $y \in U$. Then there exists a unique additive function $A:U \to V$ and a unique cubic function $C:U \to V$ and a unique quartic function $Q:U \to V$ such that

$$N(g(v)-A(v)-C(v)-Q(v),\ y)$$
$$\leq \begin{cases} \left(\frac{187}{363}+\frac{187}{2541}+\frac{13}{330}\right)\lambda, \\ \left\{\left(\frac{187}{363|2-2^s|}+\frac{1}{11|1-2^{1-s}|}\right)+\left(\frac{187}{363|2-8^s|}+\frac{1}{11|1-8^{1-s}|}\right)+\frac{25}{22|16-2^s|}\right\}\lambda\,\|v\|^s \\ \left\{\left(\frac{187}{363|2-2^{3s}|}+\frac{1}{11|1-2^{1-3s}|}\right)+\left(\frac{187}{363|2-8^{3s}|}+\frac{1}{11|1-8^{1-3s}|}\right)+\frac{25}{22|16-2^{3s}|}\right\}\lambda\,\|v\|^{3s} \end{cases} \quad (3.1.47)$$

for all $v \in U$ and all $y \in U$.

3.2.Counter Examples

Example 3.2.1. Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a function defined by $\alpha(v)=\begin{cases}\mu v^4, & |v|<1\\ \mu, & otherwise\end{cases}$ where $\mu>0$ is a constant, and define a function $g:\mathbb{R}\to\mathbb{R}$ by $g(v)=\sum_{n=0}^{\infty}\frac{\alpha(2^n v)}{16^n}$ $\quad for\ \ all\ \ v\in\mathbb{R}$. Then g satisfies the functional inequality $N(Dg(u,v,w),y)\leq\frac{296\mu\times16^2}{15}\left\{\|u\|^4+\|v\|^4+\|w\|^4\right\}$ (3.2.1) for all $u,v,w\in\mathbb{R}$ and $y\in\mathbb{R}$. Then there does not exist a quartic mapping $Q:\mathbb{R}\to\mathbb{R}$ and a constant $\delta>0$ such that

$$N\big(g(v)-Q(v),y\big)\leq\delta\|v\|^4 \quad for\ \ all\ \ x\in\mathbb{R}. \qquad (3.2.2)$$

Example 3.2.2. Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a function defined by $\alpha(v)=\begin{cases}\mu v, & |v|<1\\ \mu, & otherwise\end{cases}$ where $\mu>0$ is a constant, and define a function $g:\mathbb{R}\to\mathbb{R}$ by $g(v)=\sum_{n=0}^{\infty}\frac{\alpha(2^n v)}{2^n}$ $\quad for\ \ all\ \ v\in\mathbb{R}$. Then g satisfies the functional inequality $N(Dg(u,v,w),y)\leq 2^2\times296\mu\left\{\|u\|+\|v\|+\|w\|\right\}$ (3.2.3) for all $u,v,w\in\mathbb{R}$ and $y\in\mathbb{R}$. Then there does not exist a additive mapping $A:\mathbb{R}\to\mathbb{R}$ and a constant $\delta>0$ such that

$$N\big(g(v)-Q(v),y\big)\leq\delta\|v\| \quad for\ \ all\ \ x\in\mathbb{R}. \qquad (3.2.4)$$

Example 3.2.3. Let $\alpha:\mathbb{R}\to\mathbb{R}$ be a function defined by $\alpha(v)=\begin{cases}\mu v^3, & |v|<1\\ \mu, & otherwise\end{cases}$ where $\mu>0$ is a constant, and define a function $g:\mathbb{R}\to\mathbb{R}$ by $g(v)=\sum_{n=0}^{\infty}\frac{\alpha(2^n v)}{8^n}$ $\quad for\ \ all\ \ v\in\mathbb{R}$. Then g satisfies the functional inequality

$N\left(Dg\left(u,v,w\right),y\right)\le \frac{296\mu \times 8^{3}}{7}\left\{\|u\|^{3}+\|v\|^{3}+\|w\|^{3}\right\}$ (3.2.5) for all $u,v,w\in \square$ and $y\in \square$. Then there does not exist a cubic mapping $C:\square \to \square$ and a constant $\delta >0$ such that

$$N\left(g\left(v\right)-Q\left(v\right),y\right)\le \delta \|v\|^{3} \quad for \quad all \quad v\in \square . \qquad (3.2.6)$$

REFERENCES

1. M.Acikgoz, A review on 2-normed structures, *Int. Journal of Math. Analysis*, 1(4) (2007) 187-191.
2. J.Aczel and J.Dhombres, Functional Equations in Several Variables, Cambridge University, Press, Cambridge, 1989.
3. T. Aoki, On the stability of the linear transformation in Banach spaces, *J. Math. Soc. Japan*, 2 (1951) 64-66.
4. M. Arunkumar, Solution and stability of modified additive and quadratic functional equation in generalized 2-normed spaces, *International Journal Mathematical Sciences and Engineering Applications*, 7(I) (2013) 383-391.
5. I.S. Chang and Y.S.Jung, Stability of functional equations deriving from cubic and quadratic functions, *J. Math. Anal. Appl.*, 283 (2003) 491-500.
6. S.Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific Publishing Company, New Jersey, 2002.
7. S.Czerwik, Stability of Functional Equations of Ulam-Hyers Rassias Type, Hadronic Press, Plam Harbor, Florida, 2003.
8. M.Eshaghi Gordji, A.Ebadian and S.Zolfaghri, Stability of a functional equation deriving from cubic and quartic functions, *Abstract and Applied Analysis*, 2008, Article ID 801904, 17 pages.
9. M.Eshaghi Gordji, S.Kaboli and S.Zolfaghri, Stability of a mixed type quadratic, cubic and quartic functional equations, arxiv: 0812.2939v1 Math FA, 15 Dec 2008.
10. M. Eshaghi Gordji and H.Khodaie, Solution and stability of generalized mixed type cubic,quadratic and additive functional equation in quasi-Banach spaces, arxiv: 0812. 2939v1 Math FA, 15 Dec 2008.
11. M.Eshaghi Gordji and M.B.Savadkouhi, Stability of cubic, quartic functional equations in non-Archimedian spaces, *Acta Appl. Math.*, DOI 10.1007/s10440009-9512-7, 2009.
12. M.Eshaghi Gordji, S.Kaboli, C.Park and S.Zolfaghri, Stability of an additive-cubic-quartic functional equations, Abstract and Applied Analysis (accepted).
13. P.Gavruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, 184 (1994) 431-436.
14. D.H.Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci.*, 27 (1941) 222 -224.
15. D.H.Hyers, G.Isac and T.M.Rassias, Stability of Functional Equations in Several Variables, Birkhauser Basel, 1998.
16. K.W.Jun, H.M. Kim, On the Hyers-Ulam-Rassias stability of a generalized quadratic and additive type functional equation, *Bull. Korean Math. Soc.*, 42(1) (2005) 133-148.

17. K.W.Jun and H.M.Kim, On the stability of an n-dimensional quadratic and additive type functional equation, *Math. Ineq. Appl*, 9(1) (2006) 153-165.
18. H.M.Kim, On the stability for a mixed type quartic and quadratic functional equation, *J. Math. Anal. Appl.*, 324 (2006) 358-372.
19. A.Najati and M.B.Moghimi, On the stability of a quadratic and additive functional equation, *J. Math. Anal. Appl.*, 337 (2008) 399-415.
20. J.M.Rassias, On approximately of approximately linear mappings by linear mappings, *J. Funct. Anal.*, 46 (1982) 126-130.
21. J.M.Rassias, On approximately of approximately linear mappings by linear mappings, *Bull. Sc. Math.*, 108 (1984) 445-446.
22. T.M.Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, 72 (1978) 297-300.
23. T.M.Rassias, Functional Equations, Inequalities and Applications, Kluwer Acedamic Publishers, Dordrecht, Bostan, London, 2003.
24. K.Ravi, M.Arunkumar and J.M.Rassias, Ulam stability for the orthogonally general Euler-Lagrange type functional equation, *Int. J. Math. Stat.*, 3 (2008) 36-46.
25. K.Ravi, M.Arunkumar, B.V.Senthil Kumar and J.M.Rassias, Solution and Ulam stability of mixed type cubic and additive functional equation, Functional Ulam Notions (F.U.N) Nova Science Publishers, Chapter 13, (2010) 149 – 175.
26. M.A.Sibaha, B.Bouikhalene and E.Elqorachi, Ulam-Gavruta-Rassias stability for a linear functional equation, *Internat. J. Appl. Math. Stat.*, 7 (2007) 157-168.
27. S.M.Ulam, Problems in Modern Mathematics, Rend. Chap.VI, Wiley, New York, 1960.