Intern. J. Fuzzy Mathematical Archive
Vol. 8, No. 1, 2015, 9-17
ISSN: 2320 –3242 (P), 2320 –3250 (online)
Published on 28 May 2015
www.researchmathsci.org

International Journal of
Fuzzy Mathematical Archive

Totally Regular Property of the Join of two Fuzzy Graphs

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Received 12 May 2015; accepted 21 May 2015

Abstract. In general, the Join of two totally regular fuzzy graphs need not be a totally regular fuzzy graph. In this paper, necessary and sufficient conditions for the Join of two totally regular fuzzy graphs to be totally regular under some restrictions are obtained.

Keywords: Total degree of a vertex, regular fuzzy graph, join

AMS Mathematics Subject Classification (2010): 03E72, 05C72

1. Introduction

A fuzzy subset of a set V is a mapping σ from V to [0,1]. A fuzzy graph G is a pair of functions $G:(\sigma, \mu)$ where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ,satisfying $\mu(uv) \leq \sigma(u) \wedge \sigma(v)$. The underlying crisp graph of $G:(\sigma, \mu)$ is denoted by $G^*:(V, E)$ where $E \subseteq V \times V$. Fuzzy graph theory was introduced by Rosenfeld in 1975. Though it is very young, it has been growing fast and has numerous applications in various fields. During the same time Yeh and Bang have also introduced various concepts in connectedness in fuzzy graphs. Mordeson and Peng introduced the concept of operations on fuzzy graphs.

The operations of union, join, Cartesian product and composition on two fuzzy graphs were defined by Mordeson and Peng [2]. Sunitha and Vijayakumar discussed about the complement of the operations of union, join, Cartesian product and composition on two fuzzy graphs. The Regular property of fuzzy graphs which are obtained from two given fuzzy graphs using the operations *union*, *join*, *Cartesian product and composition* was discussed by Nagoorgani and Radha. In our earlier work [9,10], we have discussed the totally regular property of Cartesian product and some composition of two fuzzy graphs. In this paper we study about the Totally Regular property of the Join of two fuzzy graphs and the number of vertices in the fuzzy graphs G_1 *and* G_2 are denoted by p_1 *and* p_2 *respectively.*

First we go through some preliminaries which can be found in [1-8].

2. Basic definitions

Definition 2.1. [7] The order of a fuzzy graph G is defined by

$$O(G) = \sum_{u \in V} \sigma(u)$$

Definition 2.2. [3] Let $G{:}(\sigma, \mu)$ be a fuzzy graph. The degree of a vertex u in G is defined by

$$d_G(u) = \sum_{u \neq v} \mu(uv) = \sum_{uv \in E} \mu(uv)$$

Definition 2.3. [5] Let $G{:}(\sigma, \mu)$ be a fuzzy graph on G^*. The total degree of a vertex $u \in V$ is defined by

$$td_G(u) = \sum_{u \neq v} \mu(uv) + \sigma(u) = d_G(u) + \sigma(u).$$

If each vertex of G has the same total degree k, then G is said to be a totally regular fuzzy graph of total degree k
or a k-totally regular fuzzy graph.

Definition 2.4. [2] Assume that $V_1 \cap V_2 = \emptyset$ · The join (sum) of $G_1$and G_2 is defined as a fuzzy graph
$G = G_1 + G_2$: $(\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ on $G^* : (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E'$ where E' is the set of all edges joining vertices of V_1 with vertices of V_2 , with

$$\begin{aligned}(\sigma_1 + \sigma_2)(u) &= (\sigma_1 \cup \sigma_2)(u) \qquad for\ all\ u \in V_1 \cup V_2\\ &= \sigma_1(u) \ \ if\ u \in V_1\\ &= \sigma_2(u) \ \ if\ u \in V_2\end{aligned}$$

$$\begin{aligned}(\mu_1 + \mu_2)(uv) &= \begin{cases}(\mu_1 \cup \mu_2)(uv), & if\ uv \in E_1 \cup E_2\\ \sigma_1(u) \wedge \sigma_2(v), & if\ uv \in E'\end{cases}\\ &= \mu_1(uv) \quad if\ u \in V_1 \ \text{(i.e.)}\ if\ uv \in E_1\\ &= \mu_2(uv) \quad if\ u \in V_2 \ \text{(i.e.)}\ if\ uv \in E_2\end{aligned}$$

3. Total degree of a vertex in Join

Here $V_1 \cap V_2 = \emptyset$. Hence $E_1 \cap E_2 = \emptyset$.

By definition,

$$td_{G_1+G_2}(u) = \sum_{uv \in E_1 \cup E_2} (\mu_1 \cup \mu_2)(uv) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v) + (\sigma_1 \wedge \sigma_2)(u)$$

$$For\ any\ u \in V_1, \quad td_{G_1+G_2}(u) = \sum_{uv \in E_1} \mu_1(uv) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v) + \sigma_1(u)$$

$$= d_{G_1}(u) + \sigma_1(u) + \textstyle\sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v)$$

$$td_{G_1+G_2}(u) = td_{G_1}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v) \qquad (3.1)$$

Similarly,

$$For\ any\ u \in V_2,\ td_{G_1+G_2}(u) = td_{G_2}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v) \qquad (3.2)$$

Lemma 3.1. [8] Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs

1. $\sigma_1 \geq \sigma_2$ $then$ $td_{G_1+G_2}(u) = td_{G_1}(u) + O(G_2),\ if\ u \in V_1$
   $= td_{G_2}(u) + p_1\,\sigma_2(u),\ if\ u \in V_2$
2. $\sigma_2 \geq \sigma_1$ $then$ $td_{G_1+G_2}(u) = td_{G_1}(u) + p_2\,\sigma_1(u),\ if\ u \in V_1$
   $= td_{G_2}(u) + O(G_1),\ if\ u \in V_2$

4. Totally Regular Property of the Join

If G_1 and G_2 are totally regular fuzzy graphs, then $G_1 + G_2$ need not be a totally regular fuzzy graph.

Consider G_1 and G_2 in $Fig. 4.1$. Here $\sigma_2 \geq \sigma_1$.

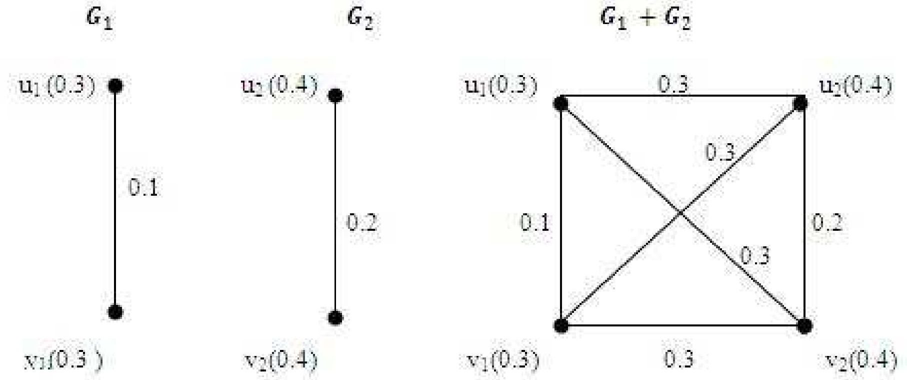


Figure 4.1:

Similarly if $G_1 + G_2$ is totally regular fuzzy graph then *G₁* is totally regular fuzzy graph and *G₂* is not a totally regular fuzzy graph in $Fig. 4.2$.

And if G_1 is a totally regular fuzzy graph and G_2 is not a totally regular fuzzy graph , then $G_1 + G_2$ is a totally regular fuzzy graph in $Fig. 4.3$.

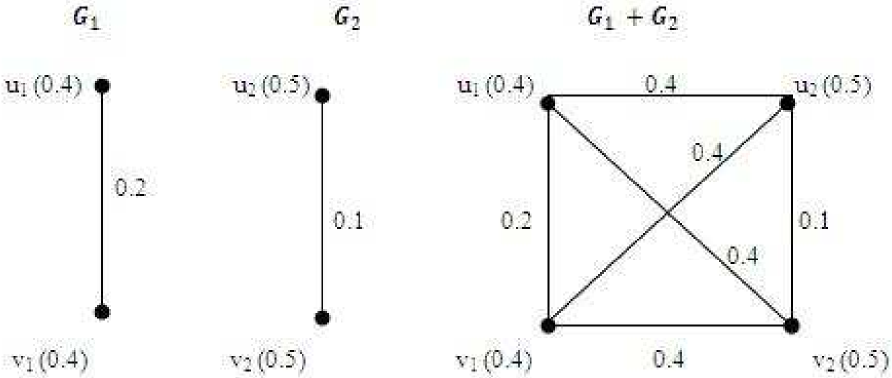


Figure 4.2:

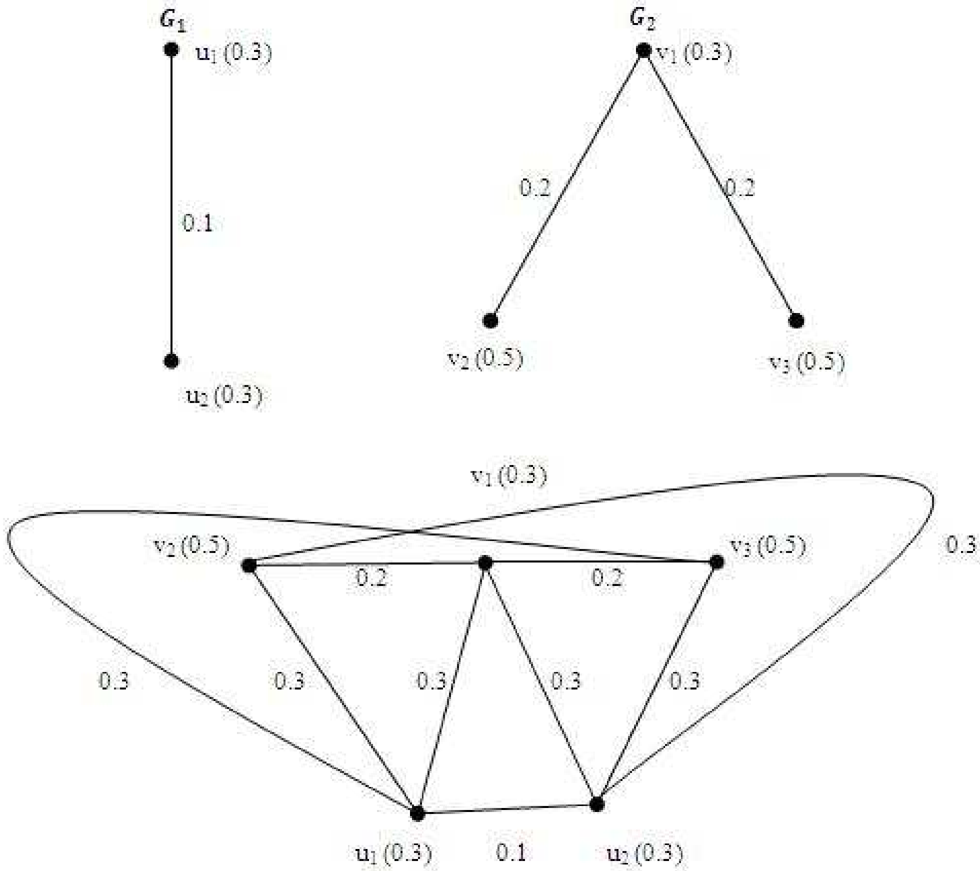


Figure 4.3: $G_1 + G_2$

In the following theorems, we obtain necessary and sufficient conditions for the join of two fuzzy graphs to be totally regular in some particular cases.

Totally Regular Property of the Join of two Fuzzy Graphs

Theorem 4.1. Let G_1 *and* G_2 be totally regular fuzzy graphs of the same degree such that $\sigma_1 \wedge \sigma_2$ is constant function. Then $G_1 + G_2$ is totally regular fuzzy graph if and only if $p_1 = p_2$.

Proof : Let G_1 *and* G_2 be k-totally regular fuzzy graphs.

Let $(\sigma_1(u) \wedge \sigma_2(v)) = c \; for \; all \; u \in V_1 \; and \; v \in V_2$ where c is a constant.

For any $u \in V_1$, from (4.1) ,

$$td_{G_1+G_2}(u) = td_{G_1}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v)$$

$$= td_{G_1}(u) + \sum_{v \in V_2} c = k + cp_2 \tag{4.1}$$

For any $u \in V_2$, from (4.2) ,

$$td_{G_1+G_2}(u) = td_{G_2}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v)$$

$$= td_{G_2}(u) + \sum_{v \in V_1} c = k + cp_1 \tag{4.2}$$

From (4.3) and (4.4) , $G_1 + G_2$ is a totally regular fuzzy graph.

$\Leftrightarrow k + cp_2 = k + cp_1 \Leftrightarrow p_2 = p_1$.

Hence the theorem.

Corollary 4.2. Let $G_1 : (\sigma_1, \mu_1)$ *and* $G_2 : (\sigma_2, \mu_2)$ be two k-totally regular fuzzy graphs such that $\sigma_1 \geq \sigma_2 \; (or \; \sigma_2 \geq \sigma_1) \; and \; \sigma_2 \; (or \; \sigma_1)$ is a constant function. Then $G_1 + G_2$ is a totally regular fuzzy graph if and only if $p_2 = p_1$.

Proof: If $\sigma_1 \geq \sigma_2 \; and \; \sigma_2$ is a constant function, then $\sigma_1 \wedge \sigma_2 = \sigma_2$ is a constant function. $\sigma_2 \geq \sigma_1 \; and \; \sigma_1$ is a constant function, then $\sigma_1 \wedge \sigma_2 = \sigma_1$ is a constant function.

Hence the result follows from the above theorem 4.1.

Theorem 4.3. Let G_1 *and* G_2 be two fuzzy graphs such that $p_1 = p_2 \; and \; \sigma_1 \wedge \sigma_2$ is constant function. Then $G_1 + G_2$ is totally regular fuzzy graph if and only if G_1 *and* G_2 are both totally regular fuzzy graphs of the same degree.

Proof: Let $G_1 \; and \; G_2$ be two fuzzy graphs such that $p_1 = p_2 = p$ (say) and $\sigma_1(u) \wedge \sigma_2(v) = c$, for all $u \in V_1 \; and \; v \in V_2$ where c is a constant.

For any $u \in V_1$, from definition (4.1) ,

$$td_{G_1+G_2}(u) = td_{G_1}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v)$$

$$= td_{G_1}(u) + \sum_{v \in V_2} c = td_{G_1}(u) + cp \tag{4.3}$$

For any $w \in V_2$, from definition (4.2)

$$td_{G_1+G_2}(w) = td_{G_2}(w) + \sum_{uv \in E'} \sigma_1(v) \wedge \sigma_2(w)$$

$$= td_{G_2}(w) + \sum_{v \in V_1} c = td_{G_2}(w) + cp \qquad (4.4)$$

From 4.5 and 4.6 ,

$G_1 + G_2$ is a totally regular fuzzy graph.

$\Leftrightarrow td_{G_1}(u) + cp = td_{G_2}(w) + cp$

$\Leftrightarrow \quad td_{G_1}(u) = td_{G_2}(w)$ where $u \in V_1$ and $w \in V_2$ are arbitrary.

Hence $G_1 + G_2$ is totally regular fuzzy graph if and only if G_1 and G_2 are both totally regular fuzzy graphs of the same degree.

Corollary 4.4. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs such that $p_2 = p_1$, $\sigma_1 \geq \sigma_2 (or\ \sigma_2 \geq \sigma_1)$ and $\sigma_2 (\ or\ \sigma_1)$ is a constant function. Then $G_1 + G_2$ is a totally regular fuzzy graph if and only if G_1 and G_2 are both totally regular fuzzy graphs of the same degree .

Proof: Proof is similar to Corollary 4.2.

Theorem 4.5. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two totally regular fuzzy graphs such that $\sigma_1 \geq \sigma_2 (or\ \sigma_2 \geq \sigma_1)$. If $G_1 + G_2$ is a totally regular fuzzy graph, then $\sigma_2 (\ or\ \sigma_1)$ is a constant function.

Proof: Let $\sigma_1 \geq \sigma_2$ and let G_i be a k_i- totally regular fuzzy graph, i =1,2.

For any $u \in V_1$, from definition (4.1) ,

$$td_{G_1+G_2}(u) = td_{G_1}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(u)$$

$$= td_{G_1}(u) + \sum_{v \in V_2} \sigma_2(u) = k_1 + O(G_2) \qquad (4.5)$$

For any $u \in V_2$, from definition (4.2)

$$td_{G_1+G_2}(u) = td_{G_2}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(u)$$

$$= td_{G_2}(u) + \sum_{v \in V_1} \sigma_2(u) = k_2 + p_1 \sigma_2(u) \qquad (4.6)$$

Since $G_1 + G_2$ is totally regular fuzzy graph then

$$k_1 + O(G_2) = k_2 + p_2 \sigma_2(u) \text{, for any } u \in V_2$$

$$\Rightarrow \quad k_1 - k_2 = p_1\sigma_2(u) - O(G_2) \text{, for any } u \in V_2$$

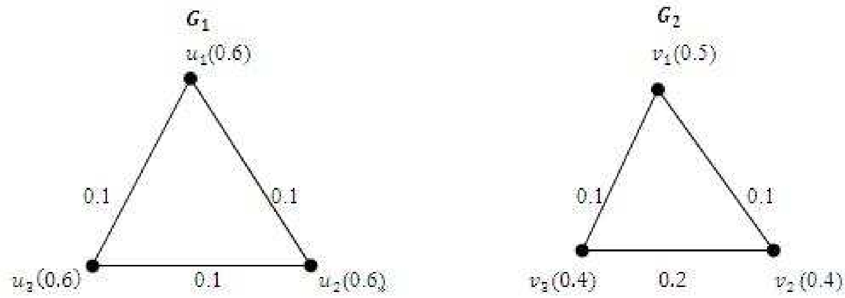
$\therefore$ For any $u, v \in V_2$, $p_1\sigma_2(u) - O(G_2) = k_1 - k_2 = p_1\sigma_2(v) - O(G_2)$

$$\Rightarrow \quad p_1\sigma_2(u) = p_1\sigma_2(v)$$

$$\Rightarrow \quad \sigma_2(u) = \sigma_2(v)$$

Hence σ_2 is a constant function.

Example 4.6. The converse of theorem 4.5 need not be true , for example , consider fig. 4.2. G_1 *and* G_2 are two totally regular fuzzy graphs such that $\sigma_2 \geq \sigma_1$ *and* σ_1 *is a constant function* . But $G_1 + G_2$ is not a totally regular fuzzy graph.



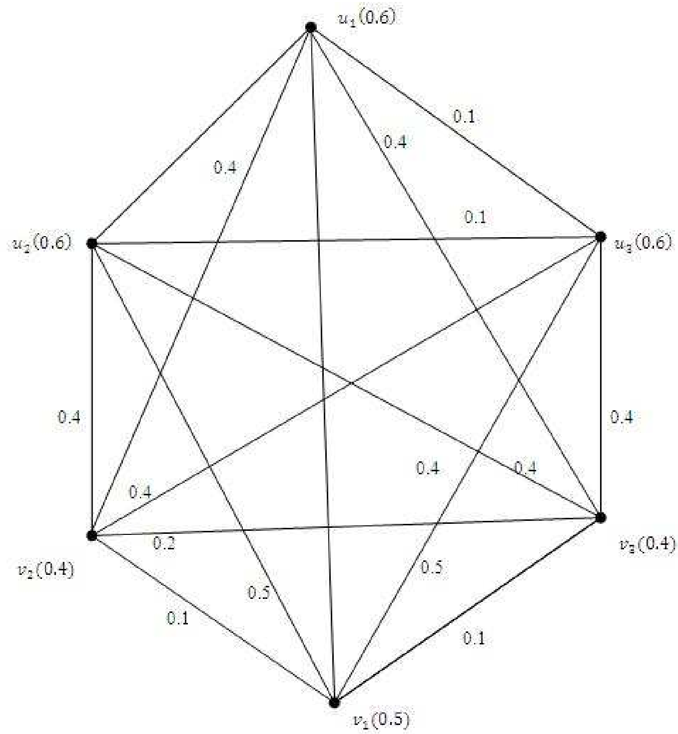


Figure 4.4: $G_1 + G_2$

Theorem 4.7. Let $G_1 : (\sigma_1 , \mu_1)$ *and* $G_2 : (\sigma_2 , \mu_2)$ be two fuzzy graphs of degrees k_1 *and* k_2 such that $\sigma_1 \wedge \sigma_2$ is constant function. Then $G_1 + G_2$ is totally regular fuzzy graph if and only if $k_1 - k_2 = c(p_1 - p_2)$ where c is the constant value of $\sigma_1 \wedge \sigma_2$.

Proof: From (4.3) , for any $u \in V_1$

$$td_{G_1+G_2}(u) = td_{G_1}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v) = td_{G_1}(u) + \sum_{v \in V_2} c$$
$$= k_1 + cp_2$$

From (4.4) , for any $u \in V_2$

$$td_{G_1+G_2}(u) = td_{G_2}(u) + \sum_{uv \in E'} \sigma_1(u) \wedge \sigma_2(v)$$
$$= td_{G_2}(u) + \sum_{v \in V_1} c = k_2 + cp_1$$

Hence $G_1 + G_2$ is totally regular fuzzy graph $\Leftrightarrow k_1 + cp_2 = k_2 + cp_1 \Leftrightarrow k_1 - k_2 = cp_1 - cp_2$

$$\Leftrightarrow k_1 - k_2 = c\,(p_1 - p_2).$$

Corollary 4.8. Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two totally regular fuzzy graphs such that $\sigma_1 \geq \sigma_2 (or\ \sigma_2 \geq \sigma_1)$ and σ_2 $(or\ \sigma_1)$ is a constant function. Then $G_1 + G_2$ is a totally regular fuzzy graph if and only if $k_1 - k_2 = c\,(p_1 - p_2)$ where c is the constant value of σ_2 $(or\ \sigma_1)$.

5. Conclusion

In this paper, we have showed that the Join of two totally regular fuzzy graphs need not be a totally regular fuzzy graph. We have obtained necessary and sufficient condition for the Join of two fuzzy graphs to be totally regular in some particular cases.

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