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On Sequential Join of Fuzzy Graphs

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Abstract. Sequential join of fuzzy graphs which is analogous to the concept sequential join operation in crisp graph theory is defined. The degree of an edge in sequential join of fuzzy graphs is obtained. Also, the degree of an edge in fuzzy graphs formed by this operation in terms of the degree of edges in the given fuzzy graphs in some particular cases is found.

Keywords: Sequential join; degree of an edge

AMS Mathematics Subject Classification (2010): 03E72, 05C72

1. Introduction

It was Rosenfeld who considered fuzzy relations on fuzzy sets and developed the theory of fuzzy graphs in 1975 [9]. Later on, Bhattacharya gave some remarks on fuzzy graphs [1]. The operations of union, join, Cartesian product and composition on two fuzzy graphs were defined by Moderson and Peng [3]. The degree of a vertex in fuzzy graphs which are obtained from two given fuzzy graphs using these operations were discussed by Nagoor Gani and Radha [5]. Radha and Kumaravel introduced the concept of degree of an edge and total degree of an edge in fuzzy graphs [7] and studied about the degree of an edge in fuzzy graphs which are obtained from two given fuzzy graphs using the operations of union and join [8].

In this paper, we have introduced the concept of sequential join of fuzzy graphs, which are analogous to the concept sequential join in crisp graph theory. We study about the degree of an edge in fuzzy graphs which are obtained from three or more fuzzy graphs using sequential join operation. The degree of an edge in the sequential join of fuzzy graphs obtained in some particular case.

Let V be a nonempty set. A fuzzy graphs is a pair of functions. $G:(\sigma,\mu)$ where σ is a fuzzy subset of v and μ is a symmetric fuzzy relation on $\sigma:V\to[0,1]$ and $\mu:V\times V\to[0,1]$ such that $\mu(u,v)\le\sigma(u)\wedge\sigma(v)$ for all u,v in V [4]. The underlying crisp graph of $G:(\sigma,\mu)$ is denoted by $G^*:(\mathrm{V},\mathrm{E})$ where $E\subseteq V\times V$. $\mu(u,v)>0$ for $(\mathrm{u},\mathrm{v})\in E$, $\mu(u,v)=0$ for $(u,v)\notin E$.

Throughout this paper we assume that μ is reflexive and need not consider loops. Note that $G_i:(\sigma_i,\mu_i)$ denote fuzzy graphs with underlying crisp graphs

$G^*_i : (V_i, E_i),\ \ 1 \le i \le n$ with $|V_i| = p_i,\ \ 1 \le i \le n$. Also, the underlying set V is assumed to be finite and σ can be chosen in any manner so as to satisfy the definition of a fuzzy graph in all the examples and all these properties are satisfied for all fuzzy graphs except null graphs. We shall denote the edge between two vertices u and v by uv.

In [5], the degree of a vertex u in G is defined by

$$d_G(u) = \sum_{u \ne v} \mu(uv) = \sum_{uv \in E} \mu(uv) \tag{1.1}$$

By Nagoor Gani and Basheer Ahamed in [6], the order of a fuzzy graph G is defined by

$$O(G) = \sum_{u \in V} \sigma(\mathrm{u}) \cdot \tag{1.2}$$

The union of two fuzzy graphs $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ is defined as a fuzzy graph $G = G_1 \cup \mathrm{G}_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ on $G^* : (\mathrm{V}, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ with

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u), & u \in V_1 - V_2 \\ \sigma_2(u), & u \in V_2 - V_1 \\ \sigma_1(u) \vee \sigma_2(u), & u \in V_1 \cap V_2 \end{cases}$$

and

$$(\mu_1 \cup \mu_2)(uv) = \begin{cases} \mu_1(uv), & uv \in E_1 - E_2 \\ \mu_2(uv), & uv \in E_2 - E_1 \\ \mu_1(uv) \vee \mu_2(uv), & uv \in E_1 \cap E_2 \end{cases}.$$

Assume that $V_1 \cap V_2 = \varnothing$. The join of G_1 and G_2 is defined as a fuzzy graph $G = G_1 + G_2 : (\sigma_1 + \sigma_2, \mu_1 + \mu_2)$ on $G^* : (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2 \cup E'$ where E' is the set of all edges joining vertices of V_1 with vertices of V_2, with

$$(\sigma_1 + \sigma_2)(u) = (\sigma_1 \cup \sigma_2)(u) \ \text{ for all } u \in V_1 \cup V_2$$

and

$$(\mu_1 + \mu_2)(uv) = \begin{cases} (\mu_1 \cup \mu_2)(uv), & uv \in E_1 \cup E_2 \\ \sigma_1(u) \vee \sigma_2(v), & uv \in E' \end{cases}.$$

By Radha and Kumaravel [7], the degree of an edge uv is defined

$$d_G(uv) = d_G(u) + d_G(v) - 2\mu(uv) = \sum_{\substack{uw \in E \\ w \ne v}} \mu(uw) + \sum_{\substack{wv \in E \\ w \ne u}} \mu(wv) \tag{1.3}$$

2. Degree of an edge in sequential join

In this section, we give the definition of sequential join operation and calculated degree of an edge of fuzzy graphs that are obtained by this operation.

Sequential join is a graph operation that is introduced by Harary 1181. Graphs obtained by using this operation, represent a communication network construction [2].

Consider for three or more disjoint graphs $G_1, G_2, ..., G_n$ where $G_i = (V_i, E_i)$ and where $V_i \cap V_j = \varnothing$ and $E_i \cap E_j = \varnothing$, $i \neq j$, $1 \leq i, j \leq n$ the sequential join $G = G_1 + G_2 + ... + G_n = (V, E)$ where $V = V_1 \cup V_2 \cup ... \cup V_n$ and where $E = E_1 \cup E_2 \cup ... \cup E_n \cup E'$, is $(G_1 + G_2) \cup (G_2 + G_3) \cup ... (G_{n-1} + G_n)$ [2]. Let σ_i be a fuzzy subset of V_i and let μ_i be a fuzzy subset of E_i, $i = 1, 2, ..., n$. Using definition of join and union, define the fuzzy $(\sigma_1 + \sigma_2) \cup (\sigma_2 + \sigma_3) \cup ... \cup (\sigma_{n-1} + \sigma_n)$ of V and $(\mu_1 + \mu_2) \cup (\mu_2 + \mu_3) \cup ... \cup (\mu_{n-1} + \mu_n)$ of E as follows:

$$(\sigma_1 + \sigma_2) \cup (\sigma_2 + \sigma_3) \cup ... \cup (\sigma_{n-1} + \sigma_n)(u) = (\sigma_1 \cup \sigma_2 \cup ... \cup \sigma_n)(u) \quad \forall u \in V \tag{2.1}$$

$$(\mu_1 + \mu_2) \cup (\mu_2 + \mu_3) \cup ... \cup (\mu_{n-1} + \mu_n)(uv) = \begin{cases} (\mu_1 \cup \mu_2 \cup ... \cup \mu_n)(uv), & uv \in E - E' \\ \sigma_i(u) \wedge \sigma_{i+1}(v) + \sigma_i(u) \wedge \sigma_{i-1}(v), & \\ \qquad uv \in E' \text{ and } u \in V_i & \end{cases} \tag{2.2}$$

Theorem 2.1. Let $G = G_1 + G_2 + ... + G_n$. For any $uv \in E$,

1) If $uv \in E_k$ for $3 \leq k \leq n$ then

$$\begin{aligned} d_G(uv) = d_{G_k}(uv) &+ \sum_{\substack{uw \in E' \\ w \neq v}} \sigma_k(u) \wedge \sigma_{k+1}(w) + \sum_{\substack{uw \in E' \\ w \neq v}} \sigma_k(u) \wedge \sigma_{k-1}(w) \\ &+ \sum_{\substack{wv \in E' \\ w \neq u}} \sigma_{k+1}(w) \wedge \sigma_k(v) + \sum_{\substack{wv \in E' \\ w \neq u}} \sigma_k(v) \wedge \sigma_{k-1}(w) \end{aligned}. \tag{2.3}$$

2) If $uv \in E'$ with $u \in V_k$, $v \in V_{k+1}$ for $3 \leq k \leq n$ then

$$\begin{aligned} d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) &+ \sum_{\substack{uw \in E' \\ w \neq v}} \sigma_k(u) \wedge \sigma_{k+1}(w) + \sum_{\substack{uw \in E' \\ w \neq v}} \sigma_k(u) \wedge \sigma_{k-1}(w) \\ &+ \sum_{\substack{wv \in E' \\ w \neq u}} \sigma_{k+1}(v) \wedge \sigma_k(w) + \sum_{\substack{wv \in E' \\ w \neq u}} \sigma_{k+2}(w) \wedge \sigma_{k+1}(v) \end{aligned} \tag{2.4}$$

Proof. By (1.3), we have

$$d_G(\text{uv}) = \sum_{\substack{uw \in E \\ w \neq v}} \mu(uw) + \sum_{\substack{wv \in E \\ w \neq u}} \mu(wv)$$

$$= \sum_{\substack{uw \in E - E' \\ w \neq v}} \mu(uw) + \sum_{\substack{wv \in E - E' \\ w \neq u}} \mu(wv) + \sum_{\substack{uw \in E' \\ w \neq v}} \mu(uw) + \sum_{\substack{wv \in E' \\ w \neq u}} \mu(wv) \tag{2.5}$$

Assume that $uv\in E_k$ with $u\in E_k$, $v\in E_k$ for $3\le k\le n$. Hence, $w\in V_{k-1}$ *or* $w\in V_{k+1}$ *or* $w\in V_k$. Using (2.2) in (2.5) we get that

$$d_G(uv)=\sum_{\substack{uw\in E_k\\ w\neq v}}\mu(uw)+\sum_{\substack{wv\in E_k\\ w\neq u}}\mu(wv)+\sum_{\substack{uw\in E'\\ w\neq v}}\sigma_k(u)\wedge\sigma_{k+1}(w)$$
$$+\sum_{\substack{uw\in E'\\ w\neq v}}\sigma_k(u)\wedge\sigma_{k-1}(w)+\sum_{\substack{wv\in E'\\ w\neq u}}\sigma_k(w)\wedge\sigma_{k+1}(v)+\sum_{\substack{wv\in E'\\ w\neq u}}\sigma_{k-1}(w)\wedge\sigma_{k+1}(v)$$

Using definition of (1.3), we obtain equation (2.3).

Now, let $uv\in E'$ with $u\in V_k$, $v\in V_{k+1}$ for $3\le k\le n$. So $w\in V_{k+1}$ *or* $w\in V_{k-1}$ if $u\in V_k$ *and* $uw\in E'$. $w\in V_k$ *or* $w\in V_{k+2}$ when $v\in V_{k+1}$ *and* $wv\in E'$. Using (2.2) in equation (2.5) we see that

$$d_G(uv)=\sum_{\substack{u\in V_k\\ w\neq v}}\mu(uw)+\sum_{\substack{v\in V_{k+1}\\ w\neq u}}\mu(wv)+\sum_{\substack{uw\in E'\\ w\neq v}}\sigma_k(u)\wedge\sigma_{k+1}(w)$$
$$+\sum_{\substack{uw\in E'\\ w\neq v}}\sigma_k(u)\wedge\sigma_{k-1}(w)+\sum_{\substack{wv\in E'\\ w\neq u}}\sigma_k(w)\wedge\sigma_{k+1}(v)+\sum_{\substack{wv\in E'\\ w\neq u}}\sigma_{k+2}(w)\wedge\sigma_{k+1}(v)$$

As a conclusion, by (1.1) we obtain equation (2.4). Thus, we complete proof of the theorem. □

In the following theorems, we find the degree of uv in G in terms of those in G_k for $3\le k\le n$ in some particular cases.

Nagoor Gani and Radha in [5] defined the relation $\sigma_1\ge\sigma_2$ means that $\sigma_1(u)\ge\sigma_2(v)$, for every $u\in V_1$ and for every $v\in V_2$, where σ_i is a fuzzy subset of $V_i, i=1,2.$. We also can define the relation $\sigma_k\ge\sigma_j$ means that $\sigma_k(u)\ge\sigma_j(v)$ for every $u\in V_k$ and for every $v\in V_j, k\neq j$ and $1\le k,j\le n$, where σ_i is a fuzzy subset of $V_i, 1\le i\le n.$

Theorem 2.2. Let $G=G_1+G_2+...+G_n$. For $\sigma_{k-1}\ge\sigma_k, \sigma_k\ge\sigma_{k+1}$ *and* $\sigma_{k+1}\ge\sigma_{k+2}$ the following equalities holds:

1) If $uv\in E_k$ for $3\le k\le n$ then

$$d_G(uv)=d_{G_k}(uv)+2O(G_{k+1})+p_{k-1}(\sigma_k(u)+\sigma_k(v)).$$

2) If $uv\in E'$ with $u\in V_k$, $v\in V_{k+1}$ for $3\le k\le n$ then

$$d_G(uv)=d_{G_k}(u)+d_{G_{k+1}}(v)+O(G_{k+1})+O(G_{k+2})+p_{k-1}\sigma_k(u)+(p_k-2)\sigma_{k+1}(v)$$

Proof. We have $\sigma_{k-1}\ge\sigma_k, \sigma_k\ge\sigma_{k+1}$ *and* $\sigma_{k+1}\ge\sigma_{k+2}$. Let any $uv\in E_k$ for $3\le k\le n$. From equation (2.3) we have

$$d_G(uv)=d_{G_k}(uv)+\sum_{\substack{w\in V_{k+1}\\ w\neq v}}\sigma_{k+1}(w)+\sum_{\substack{w\in V_{k-1}\\ w\neq v}}\sigma_k(u)+\sum_{\substack{w\in V_{k+1}\\ w\neq u}}\sigma_{k+1}(w)+\sum_{\substack{w\in V_{k-1}\\ w\neq u}}\sigma_k(v).$$

Recall that $|V_i|=p_i, 1\le i\le n.$ By equation (1.2), we have

$$d_G(uv)=d_{G_k}(uv)+O(G_{k+1})+p_{k-1}\sigma_k(u)+O(G_{k+1})+p_{k-1}\sigma_k(v)$$

Therefore we obtain Theorem 2.2 (1).

Using conditions of the Theorem 2.2 in equation (2.4), we get that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w\in V_{k+1}\\ w\neq v}} \sigma_{k+1}(w) + \sum_{\substack{w\in V_{k-1}\\ w\neq v}} \sigma_k(u) + \sum_{\substack{w\in V_k\\ w\neq u}} \sigma_{k+1}(v) + \sum_{\substack{w\in V_{k+2}\\ w\neq u}} \sigma_{k+2}(w)$$

Now, using equation (1.2) and $|V_i| = p_i, 1\le i\le n$, we obtain that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k+1}) - \sigma_{k+1}(v) + p_{k-1}\sigma_k(u) + (p_k - 1)\sigma_{k+1}(v) + O(G_{k+2})$$

□

Theorem 2.3. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_k \ge \sigma_{k-1}, \sigma_k \ge \sigma_{k+1}$ *and* $\sigma_{k+1} \ge \sigma_{k+2}$ the following equalities holds:

1) If $uv\in E_k$ for $3\le k\le n$ then

$$d_G(uv) = d_{G_k}(uv) + 2O(G_{k+1}) + 2O(G_{k-1}).$$

2) If $uv\in E'$ with $u\in V_k$, $v\in V_{k+1}$ for $3\le k\le n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k+1}) + O(G_{k-1}) + O(G_{k+2}) + (p_k - 2)\sigma_{k+1}(v)$$

Proof. We have $\sigma_k \ge \sigma_{k-1}, \sigma_k \ge \sigma_{k+1}$ *and* $\sigma_{k+1} \ge \sigma_{k+2}$. Let any $uv\in E_k$ for $3\le k\le n$. In similar a way, by equation (2.3) we have

$$d_G(uv) = d_{G_k}(uv) + \sum_{\substack{w\in V_{k+1}\\ w\neq v}} \sigma_{k+1}(w) + \sum_{\substack{w\in V_{k-1}\\ w\neq v}} \sigma_{k-1}(w) + \sum_{\substack{w\in V_{k+1}\\ w\neq u}} \sigma_{k+1}(w) + \sum_{\substack{w\in V_{k-1}\\ w\neq u}} \sigma_{k-1}(w).$$

By equation (1.2), we get

$$d_G(uv) = d_{G_k}(uv) + O(G_{k+1}) + O(G_{k-1}) + O(G_{k+1}) + O(G_{k-1}).$$

Therefore we obtain Theorem 2.3 (1).

For any $uv\in E'$ with $u\in V_k$, $v\in V_{k+1}$ for $3\le k\le n$. Using (2.4) and $\sigma_k \ge \sigma_{k-1}, \sigma_k \ge \sigma_{k+1}$ *and* $\sigma_{k+1} \ge \sigma_{k+2}$, we get that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w\in V_{k+1}\\ w\neq v}} \sigma_{k+1}(w) + \sum_{\substack{w\in V_{k-1}\\ w\neq v}} \sigma_{k-1}(w) + \sum_{\substack{w\in V_k\\ w\neq u}} \sigma_{k+1}(v) + \sum_{\substack{w\in V_{k+2}\\ w\neq u}} \sigma_{k+2}(w)$$

By equation (1.2), we obtain that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k+1}) - \sigma_{k+1}(v) + O(G_{k-1}) + (p_k - 1)\sigma_{k+1}(v) + O(G_{k+2})$$

□

Theorem 2.4. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_{k-1} \ge \sigma_k, \sigma_{k+1} \ge \sigma_k$ *and* $\sigma_{k+1} \ge \sigma_{k+2}$ the following equalities holds:

1) If $uv\in E_k$ for $3\le k\le n$ then

$$d_G(uv) = d_{G_k}(uv) + (p_{k+1} + p_{k-1})(\sigma_k(u) + \sigma_k(v)).$$

2) If $uv\in E'$ with $u\in V_k$, $v\in V_{k+1}$ for $3\le k\le n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_k) + O(G_{k+2}) + (p_{k-1} + p_k - 2)\sigma_k(u)$$

Proof. We have $\sigma_{k-1} \geq \sigma_k, \sigma_{k+1} \geq \sigma_k$ *and* $\sigma_{k+1} \geq \sigma_{k+2}$. Let any $uv \in E_k$ for $3 \leq k \leq n$. From equation (2.3) we obtain that

$$d_G(uv) = d_{G_k}(uv) + \sum_{\substack{w \in V_{k+1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_{k-1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_{k+1} \\ w \neq u}} \sigma_k(v) + \sum_{\substack{w \in V_{k-1} \\ w \neq u}} \sigma_k(v).$$

By $|V_i| = p_i, 1 \leq i \leq n.$, we have

$$d_G(uv) = d_{G_k}(uv) + p_{k+1}\sigma_k(u) + p_{k-1}\sigma_k(u) + p_{k+1}\sigma_k(v) + p_{k-1}\sigma_k(v)$$

Therefore we obtain Theorem 2.4 (1).
For $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \leq k \leq n$. From (2.4), we obtain that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w \in V_{k+1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_{k-1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_k \\ w \neq u}} \sigma_k(w) + \sum_{\substack{w \in V_{k+2} \\ w \neq u}} \sigma_{k+2}(w)$$

Using equation (1.2) and $|V_i| = p_i, 1 \leq i \leq n$, we get that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + (p_{k-1} - 1)\sigma_k(u) + p_{k-1}\sigma_k(u) + O(G_k) - \sigma_k(u) + O(G_{k+2})$$

Thus, the proof completed. □

Theorem 2.5. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_k \geq \sigma_{k-1}, \sigma_{k+1} \geq \sigma_k$ *and* $\sigma_{k+1} \geq \sigma_{k+2}$ the following equalities holds:

1) If $uv \in E_k$ for $3 \leq k \leq n$ then

$$d_G(uv) = d_{G_k}(uv) + 2O(G_{k-1}) + p_{k+1}(\sigma_k(u) + \sigma_k(v)).$$

2) If $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \leq k \leq n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k-1}) + O(G_k) + O(G_{k+2}) + (p_{k+1} - 2)\sigma_k(u)$$

Proof. We have $\sigma_k \geq \sigma_{k-1}, \sigma_{k+1} \geq \sigma_k$ *and* $\sigma_{k+1} \geq \sigma_{k+2}$. Assume that any $uv \in E_k$ for $3 \leq k \leq n$. From equation (2.3) we see that

$$d_G(uv) = d_{G_k}(uv) + \sum_{\substack{w \in V_{k+1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_{k-1} \\ w \neq v}} \sigma_{k-1}(w) + \sum_{\substack{w \in V_{k+1} \\ w \neq u}} \sigma_k(v) + \sum_{\substack{w \in V_{k-1} \\ w \neq u}} \sigma_{k-1}(w).$$

Using $|V_i| = p_i, 1 \leq i \leq n.$ and equation (1.2), we get that

$$d_G(uv) = d_{G_k}(uv) + p_{k+1}\sigma_k(u) + O(G_{k-1}) + p_{k+1}\sigma_k(v) + O(G_{k-1})$$

Therefore we obtain Theorem 2.5 (1).
In a similar way, if any $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \leq k \leq n$ then from (2.4), we get that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w \in V_{k+1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_{k-1} \\ w \neq v}} \sigma_{k-1}(w) + \sum_{\substack{w \in V_k \\ w \neq u}} \sigma_k(w) + \sum_{\substack{w \in V_{k+2} \\ w \neq u}} \sigma_{k+2}(w)$$

$$= d_{G_k}(u) + d_{G_{k+1}}(v) + (p_{k+1} - 1)\sigma_k(u) + O(G_{k-1}) + O(G_k) - \sigma_k(u) + O(G_{k+2})$$

□

Theorem 2.6. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_{k-1} \geq \sigma_k, \sigma_k \geq \sigma_{k+1}$ *and* $\sigma_{k+2} \geq \sigma_{k+1}$ the following equalities holds:

1) If $uv \in E_k$ for $3 \le k \le n$ then

$$d_G(uv) = d_{G_k}(uv) + 2O(G_{k+1}) + p_{k-1}(\sigma_k(u) + \sigma_k(v)).$$

2) If $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \le k \le n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k-1}) + p_{k-1}\sigma_k(u) + (p_k + p_{k+2} - 2)\sigma_{k+1}(v)$$

Proof. We have $\sigma_{k-1} \ge \sigma_k, \sigma_k \ge \sigma_{k+1}$ *and* $\sigma_{k+2} \ge \sigma_{k+1}$. It is the same proof of Theorem 2.6 (1) with Theorem 2.2 (1). Because, they have the same conditions. While $\sigma_{k+1} \le \sigma_{k+2}$ *or* $\sigma_{k+2} \le \sigma_{k+1}$, it doesn't alter this conclusion.

In similar a way, if $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \le k \le n$ then from equation (2.4), we get that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w \in V_{k+1} \\ w \ne v}} \sigma_{k+1}(w) + \sum_{\substack{w \in V_{k-1} \\ w \ne v}} \sigma_k(u) + \sum_{\substack{w \in V_k \\ w \ne u}} \sigma_{k+1}(v) + \sum_{\substack{w \in V_{k+2} \\ w \ne u}} \sigma_{k+1}(v)$$

$$= d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k+1}) - \sigma_{k+1}(v) + p_{k-1}\sigma_k(u) + (p_k - 1)\sigma_{k+1}(v) + p_{k+2}\sigma_{k+1}(v)$$

□

Theorem 2.7. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_k \ge \sigma_{k-1}, \sigma_{k+1} \ge \sigma_k$ *and* $\sigma_{k+2} \ge \sigma_{k+1}$ the following equalities holds:

1) If $uv \in E_k$ for $3 \le k \le n$ then

$$d_G(uv) = d_{G_k}(uv) + 2O(G_{k-1}) + p_{k+1}(\sigma_k(u) + \sigma_k(v)).$$

2) If $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \le k \le n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k-1}) + O(G_k) + (p_{k+1} - 2)\sigma_k(u) + p_{k+2}\sigma_{k+1}(v)$$

Proof. We have $\sigma_k \ge \sigma_{k-1}, \sigma_{k+1} \ge \sigma_k$ *and* $\sigma_{k+1} \ge \sigma_{k+2}$. Proof of Theorem 2.7 (1) is similar to Theorem 2.5 (1). Because, They have the same conditions. While $\sigma_{k+1} \le \sigma_{k+2}$ *or* $\sigma_{k+2} \le \sigma_{k+1}$, it doesn't change the results. So, let's us prove Theorem 2.7 (2). Let any $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \le k \le n$. By (2.4), we have

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w \in V_{k+1} \\ w \ne v}} \sigma_k(u) + \sum_{\substack{w \in V_{k-1} \\ w \ne v}} \sigma_{k-1}(w) + \sum_{\substack{w \in V_k \\ w \ne u}} \sigma_k(w) + \sum_{\substack{w \in V_{k+2} \\ w \ne u}} \sigma_{k+1}(v)$$

Using equation (1.2) and $|V_i| = p_i, 1 \le i \le n$, we obtain

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + (p_{k+1} - 1)\sigma_k(u) + O(G_{k-1}) + O(G_k) - \sigma_k(u) + p_{k+2}\sigma_{k+1}(v)$$

□

Theorem 2.8. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_{k-1} \ge \sigma_k, \sigma_{k+1} \ge \sigma_k$ *and* $\sigma_{k+2} \ge \sigma_{k+1}$ the following equalities holds:

1) If $uv \in E_k$ for $3 \le k \le n$ then

$$d_G(uv) = d_{G_k}(uv) + (p_{k+1} + p_{k-1})(\sigma_k(u) + \sigma_k(v)).$$

2) If $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \le k \le n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_k) + (p_{k-1} + p_k - 2)\sigma_k(u) + p_{k+2}\sigma_{k+1}(v)$$

Proof. We have $\sigma_{k-1} \geq \sigma_k, \sigma_{k+1} \geq \sigma_k$ *and* $\sigma_{k+2} \geq \sigma_{k+1}$. It is the same proof of Theorem 2.8 (1) with Theorem 2.4 (1). Thus, let's us prove Theorem 2.8 (2).
Let $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \leq k \leq n$. In similar a way, we have that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w \in V_{k+1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_{k-1} \\ w \neq v}} \sigma_k(u) + \sum_{\substack{w \in V_k \\ w \neq u}} \sigma_k(w) + \sum_{\substack{w \in V_{k+2} \\ w \neq u}} \sigma_{k+1}(v)$$

$$= d_{G_k}(u) + d_{G_{k+1}}(v) + (p_{k-1} - 1)\sigma_k(u) + p_{k-1}\sigma_k(u) + O(G_k) - \sigma_k(u) + p_{k+2}\sigma_{k+1}(v)$$

Thus, the proof completed. □

Theorem 2.9. Let $G = G_1 + G_2 + ... + G_n$. For $\sigma_k \geq \sigma_{k-1}, \sigma_k \geq \sigma_{k+1}$ *and* $\sigma_{k+2} \geq \sigma_{k+1}$ the following equalities holds:

1) If $uv \in E_k$ for $3 \leq k \leq n$ then

$$d_G(uv) = d_{G_k}(uv) + 2O(G_{k+1}) + 2O(G_{k-1}).$$

2) If $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \leq k \leq n$ then

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k+1}) + O(G_{k-1}) + (p_k + p_{k+2} - 2)\sigma_{k+1}(v)$$

Proof. We have $\sigma_k \geq \sigma_{k-1}, \sigma_k \geq \sigma_{k+1}$ *and* $\sigma_{k+2} \geq \sigma_{k+1}$. It is the same proof of Theorem 2.9 (1) with Theorem 2.3 (1). Then, if $uv \in E'$ with $u \in V_k, \quad v \in V_{k+1}$ for $3 \leq k \leq n$ then we obtain that

$$d_G(uv) = d_{G_k}(u) + d_{G_{k+1}}(v) + \sum_{\substack{w \in V_{k+1} \\ w \neq v}} \sigma_{k+1}(w) + \sum_{\substack{w \in V_{k-1} \\ w \neq v}} \sigma_{k-1}(w) + \sum_{\substack{w \in V_k \\ w \neq u}} \sigma_{k+1}(v) + \sum_{\substack{w \in V_{k+2} \\ w \neq u}} \sigma_{k+1}(v)$$

$$= d_{G_k}(u) + d_{G_{k+1}}(v) + O(G_{k+1}) - \sigma_{k+1}(v) + O(G_{k-1}) + (p_k - 1)\sigma_{k+1}(v) + p_{k+2}\sigma_{k+1}(v)$$

Proof of the theorem is completed. □

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