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The Neighborhood Graph of a Graph

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Abstract. The neighborhood graph $N(G)$ of a graph $G = (V, E)$ is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G and with two vertices $u, v \in V \cup S$ adjacent if $u \in V$ and v is an open neighborhood set containing u. In this paper, some properties of this new graph are obtained. A characterization is given for graphs G such that $N(G) = G$. Also characterizations are given for graphs (i) whose neighborhood graphs are connected (ii) whose neighborhood graphs are r-regular, (iii) whose neighborhood graphs are eulerian.

Keywords: open neighborhood set, neighborhood graph, eulerian.

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

All graphs considered in this paper are finite, undirected without loops or multiple edges. All definitions and notations not given here may be found in [1].

Let $G = (V, E)$ be a graph with $|V| = p$ vertices and $|E|= q$ edges. For any vertex $u \in V$, the open neighborhood of u is the set $N(u) = \{v \in V : uv \in E\}$. We call $N(u)$ is the open neighborhood set of a vertex u of G. Let $V = \{u_1, u_2, \ldots, u_p\}$ and let $S = \{N(u_1), N(u_2), \ldots, N(u_p)\}$ be the set of all open neighborhood sets of the vertices of G.

A set D of vertices in a graph G is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in G, see [2]. A dominating set D of G is minimal if for any vertex $v \in D$, $D - \{v\}$ is not a dominating set of G.

The dominating graph $D(G)$ of a graph G is the graph with the vertex set $V \cup S_1$ where S_1 is the set of all minimal dominating sets of G and with two vertices u and v in $D(G)$ adjacent if $u \in V$ and v is a minimal dominating set in G containing u. This concept was introduced by Kulli et al in [3]. Several graph valued functions in graph theory were studied, for example, in [4, 5, 6, 7, 8, 9, 10, 11, 12,13, 14, 15, 16] and also several graph valued functions in domination theory were studied, for example, in [17, 18, 19, 20, 21, 22, 23, 24, 25].

The following will be useful in the proof of our result.

Theorem A [1, p.66]. A nontrivial graph is bipartite if and only if all its cycles are even.

In Section 2, we establish some properties of neighborhood graphs.

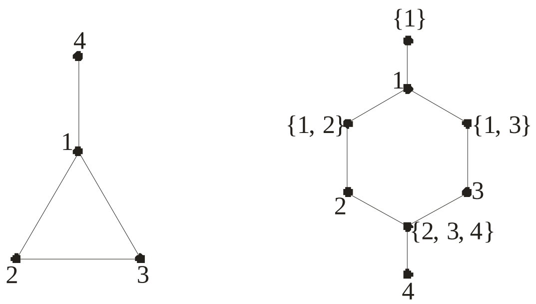
Traversability of some graph valued functions was studied, for example, in [26, 27, 28, 29]. In Section 3, we study traversability of neighborhood graphs.

2. Neighborhood graphs

The concept of the dominating graph inspires us to introduced the neighborhood graph of a graph.

Definition 1. The neighborhood graph $N(G)$ of a graph $G = (V, E)$ is the graph with the vertex set $V \cup S$ where S is the set of all open neighborhood sets of G and with two vertices u and v in $N(G)$ adjacent if $u \in V$ and v is an open neighborhood set containing u.

Example 2. In Figure 1, a graph G and its neighborhood graph $N(G)$ are shown. For the graph G in Figure 1, the open neighborhood sets are $N(1) = \{2, 3, 4\}$, $N(2) = \{1, 3\}$, $N(3) = \{1, 2\}$, $N(4) = \{1\}$.



G *N*(*G*)

Figure 1:

Remark 3. If u is an isolated vertex of a graph, then $N(u)$ is a null set.

Theorem 4. For any graph G, $N(G)$ is bipartite.

Proof: By definition, no two vertices corresponding to vertices in $N(G)$ are adjacent and no two vertices corresponding to open neighborhood sets in $N(G)$ are adjacent. Hence $N(G)$ has no odd cycles. Thus $N(G)$ is bipartite.

Remark 5. If v is a cut vertex of a graph G, then the corresponding vertices of v and $N(v)$ are both cut vertices in $N(G)$.

Theorem 6. If G is a (p, q) graph without isolated vertices, then the neighborhood graph $N(G)$ of G has $2p$ vertices and $2q$ edges.

Proof: Let G be a (p, q) graph without isolated vertices. Then for each vertex v of G, the neighborhood set $N(v)$ exists. Therefore G has p open neighborhood sets. Since the vertex set of $N(G)$ is the union of the set of vertices and the set of open neighborhood sets of G, it implies that $N(G)$ has $2p$ vertices.

In $N(G)$, the corresponding vertex v_i of the vertex v_i of G contributes $\sum d(v_i)$ edges and the corresponding vertex $N(v_i)$ of the open neighborhood set $N(v_i)$ of G contributes $\sum d(N(v_i))$ edges. Clearly $\sum d(v_i) = \sum d(N(v_i)) = 2q$. Thus the number of edges in $N(G)$

$$= \frac{1}{2}\left[\sum d(v_i) + \sum d(N(v_i))\right] = 2q.$$

Theorem 7. If T is a tree with $p\geq 2$ vertices, then $N(T) = 2T$.

Proof: Let T be a tree with $p \geq 2$ vertices. We employ induction on p. One can verify that the result is true for $p=2$ or 3. This completes the first step of the induction.

We now assume the result is true for a tree T with $p=k$ vertices. Then by induction $N(T) = 2T$. It implies that each component of $N(T)$ is the tree T with $p = k$ vertices and for any vertex v, $N_T(v)$ and v are in different components of $N(T)$. Consider a tree T_1 with $k+1$ vertices. Let u be a vertex of T_1 and $u \notin T$. In T_1, the vertex u is adjacent with a vertex v of T. Now $N_{T_1}(v) = N_T(v) \cup \{u\}$ and $N_{T_1}(u) = \{v\}$. In $N(T_1)$, $N_{T_1}(u)$ is adjacent with u in one component of $N(T)$ and $N_{T_1}(u)$ is adjacent with v in another component of $N(T)$, since $N_{T_1}(v)$ and v are in different components of $N(T)$. Then each component of $N(T_1)$ is an acyclic and with $k+1$ vertices. Hence $N(T_1) = 2T_1$ and hence the result is true for a tree with $k+1$ vertices.

Hence the result follows.

The following results follow from Theorem 7.

Corollary 8. For any path P_p with $p\geq 2$ vertices,

$$N(P_p) = 2P_p.$$

Corollary 9. For any star $K_{1,p}$ with $p\geq 1$ vertices,

$$N(K_{1,p}) = 2K_{1,p}.$$

Corollary 10. For a graph mK_2 with $m\geq 1$,

$$N(mK_2) = 2mK_2.$$

Theorem 11. For a cycle C_p with $p\geq 3$ vertices,

$$N(C_p) = 2C_p, \text{ if } p \text{ is even}, \quad = C_{2p}, \text{ if } p \text{ is odd}.$$

Proof: Let $V(C_p) = \{u_1, u_2,\ldots,u_p\}$, $p\geq 3$. Let $N(u_i) = X_i$, $1\leq i\leq p$. Then $V(N(C_p)) = \{u_1, u_2,\ldots,u_p, X_1, X_2,\ldots, X_p\}$.

Consider

$X_1 = N(u_1) = \{u_2, u_p\}$
$X_2 = N(u_2) = \{u_1, u_3\}$
$X_3 = N(u_3) = \{u_2, u_4\}$
: : :

. . .

$X_{p-1} = N(u_{p-1}) = \{u_{p-2}, u_p\}$

$X_p = N(u_p) = \{u_{p-1}, u_1\}$.

In $N(C_p)$, no two corresponding vertices of $u_1, u_2, \ldots, u_p$ are adjacent and no two corresponding vertices of $X_1, X_2, \ldots, X_p$ are adjacent. The adjacencies of the vertices in $N(C_p)$ are

X_1 is adjacent with u_2 and u_p.

X_2 is adjacent with u_3 and u_1.

X_3 is adjacent with u_4 and u_2.

: : :

. . .

X_{p-1} is adjacent with u_{p-2} and u_p

X_p is adjacent with u_{p-1} and u_1.

Case 1. Suppose p is even. Then the adjacency of the vertices of $N(C_p)$ is given below.

$X_1\, u_2\, X_3\, u_4 \ldots X_{p-1}\, u_p\, X_1$

and $X_2\, u_3\, X_4\, u_5 \ldots X_p\, u_{p-1}\, X_2$.

Since p is even, $X_1\, u_2\, X_3\, u_4 \ldots X_{p-1}\, u_p\, X_1$ is a cycle with p vertices and $X_2\, u_3\, X_4\, u_5 \ldots X_p\, u_{p-1} X_2$ is also a cycle with p vertices and they are disjoint. Hence $N(C_p) = 2C_p$.

Care 2. Suppose p is odd. Then the adjacency of the vertices of $N(C_p)$ is $u_1\, X_2\, u_3\, X_4\, u_5 \ldots u_p X_1\, u_2\, X_3\, u_4 \ldots X_p\, u_1$, which is a cycle with $2p$ vertices. Hence $N(C_p) = C_{2p}$.

Therefore $N(C_p)= 2\, C_p$, if p is even,

$= C_{2p}$, if p is odd.

Theorem 12. For any complete bipartite graph $K_{m,n}$, $1 \leq m \leq n$, $N(K_{m,n}) = 2\, K_{m,n}$.

Proof: Let $K_{m,n}$ be complete bipartite $1 \leq m \leq n$. Let $V(K_{m,n}) = V_1 \cup V_2$, where $V_1 = \{u_1, u_2, \ldots, u_m\}$ and $V_2 = \{v_1, v_2, \ldots, v_n\}$. Then $X_i = N(u_i) = \{v_1, v_2, \ldots, v_n\}$, $1 \leq i \leq n$ and $Y_j = N(v_j) = \{u_1, u_2, \ldots, u_m\}$, $1 \leq j \leq m$. In $N(K_{m,n})$, each vertex $N(u_i)$ is adjacent with the vertices $v_1, v_2, \ldots, v_n$ and each vertex $N(v_j)$ is adjacent with the vertices $u_1, u_2, \ldots, u_m$. Thus the graph with the vertex set $\{v_1, v_2, \ldots, v_n, N(u_1), N(u_2), \ldots, N(u_m)\}$ is a complete bipartite graph $K_{m,n}$ and the graph with the vertex set $\{u_1, u_2, \ldots, u_m, N(v_1), N(v_2), \ldots, N(v_n)\}$ is a complete bipartite graph $K_{m,n}$. Since $X_i \cap Y_j = \phi$, $N(K_{m,n}) = 2K_{m,n}$.

Corollary 13. The neighborhood graph $N(G)$ of a bipartite graph G is disconnected.

Theorem 14. $N(G) = G$ if and only if $G = \overline{K}_p$.

Proof: Suppose $G = \overline{K}_p$. It is known that the open neighborhood set of an isolated vertex is a null set. Clearly $N(G) = \overline{K}_p$. Hence $N(G) = G$.

Conversely suppose $N(G) = G$. We now prove that $G = \overline{K}_p$. That is, we prove that each component of G is an isolated vertex. Assume there exists a component in G which has an edge, say $e = uv$. Then $N(u)$ and $N(v)$ are nonempty open neighborhood sets of vertices u and v respectively. Thus the number of vertices of G is less than the number of vertices of $N(G)$. Hence $N(G) \neq G$, which is a contradiction. Thus each component of G is an isolated vertex. Therefore $G = \overline{K}_p$.

Lemma 15. If v is an end vertex of G, then the corresponding vertices of v and $N(v)$ are end vertices in $N(G)$.
Proof: Let v be an end vertex of G. Then v is adjacent with exactly one vertex of G, say u. Then $N(v) = \{u\}$ and $N(u) = \{v, v_1,\ldots, v_k\}$, $k \geq 1$ and v is not in any other open neighborhood set in G. Thus $N(v)$ is adjacent with exactly one vertex u in $N(G)$ and also v is adjacent with exactly only vertex $N(v)$ in $N(G)$. Thus v and $N(v)$ are endvertices in $N(G)$.

Theorem 16. Let G be a connected graph. Then $N(G)$ is complete if and only G is K_1.
Proof: Suppose $G = K_1$. Then $N(G) = K_1$ and hence $N(G)$ is complete.

Conversely suppose $N(G)$ is complete. We now prove that $G = K_1$. Assume $G \neq K_1$. Let G be a connected graph with $p \geq 2$ vertices. For a vertex v of G, deg $v \leq \mathrm{p} - 1 = \Delta(G)$. Therefore v lies in at most Δ (G) open neighborhood sets of vertices of G. Thus deg v in $N(G)$ is at most $p - 1$. Let $N(u)$ be an open neighborhood set of a vertex u of G. Thus deg$N(u)$ in $N(G)$ is at most $p - 1$. Hence the degree of each vertex of $N(G)$ is at most $p - 1$. By Theorem 6, $N(G)$ has $2p$ vertices. Also we have $p - 1 \neq 2p - 1$, $p \geq 2$. Thus $N(G)$ is not complete, which is a contradiction. Hence G has not more than one vertex. Thus $G = K_1$.

Corollary 17. If G is a nontrivial connected graph, then $N(G)$ is not complete.

Theorem 18. Let G be a connected graph. The neighborhood graph $N(G)$ of G is connected if and only if G contains an odd cycle.
Proof: Let G be a connected graph. Suppose $N(G)$ is connected. We now prove that G contains an odd cycle. We consider the following cases.

Case 1. Suppose G contains only even cycles. Then G is bipartite. By Corollary 13, $N(G)$ is disconnected, which is a contradiction.

Case 2. Suppose G has no cycles. Then G is a tree and it is bipartite. By Corollary 13, $N(G)$ is disconnected, a contradiction.

By Case 1 and Case 2, we conclude that G contains an odd cycle.

Conversely suppose a connected graph G contains an odd cycle. We prove that $N(G)$ is connected. We consider the following cases.

Case 1. Suppose G is itself an odd cycle. By Theorem 11, $N(G)$ is a cycle. Then $N(G)$ is connected.

Case 2. Suppose G has an odd cycle. Let $C_k = \{u_1, u_2,\ldots, u_k\}$ where k is odd. By Theorem 11, $N(C_k)$ is a cycle. Let $V(G)= \{u_1, u_2,\ldots, u_k, u_{k+1}, \ldots, u_p\}$. Then there exists at least one edge $u_k\, u_{k+1}$ in G such that $u_k\, u_{k+1} \ldots u_p$ is a path. We see that the adjacency of the vertices in $N(G)$ is as follows.

$u_1\, X_2\, u_3\, X_4\, u_5 \ldots u_k\, X_1\, u_2\, X_3\, u_4 \ldots X_k u_1$,

$X_k\, u_{k+1}\, X_{k+2}\, u_{k+3}$ ... and so on,

and $u_k\, X_{k+1}\, u_{k+2}\, X_{k+3}\, u_{k+4}$ ... and so on.

Therefore any two vertices of $N(G)$ are connected by a path. Thus $N(G)$ is connected.

Case 3. Suppose G has an odd cycle and it is a connected graph which is not considered in Case 1 and Case 2. Then one can see that $N(G)$ is connected.

From the above cases, we conclude that for any connected graph having an odd cycle, $N(G)$ is connected.

We characterize neighborhood graphs which are regular.

Theorem 19. A graph G is r-regular if and only if $N(G)$ is r - regular.

Proof: Suppose G is 0-regular. Then $G= \overline{K}_p$. By Theorem 14, $N(G) = \overline{K}_p$ if and only if $G= \overline{K}_p$. Hence G is 0-regular if and only if $N(G)$ is 0-regular.

Suppose G is r-regular, $r \geq 1$. Let $V(G) = [v_1, v_2, \ldots, v_p\}$. Then deg $v_i = r$ and $N(v_i)$ contains r vertices for $1 \leq i \leq p$. Thus in $N(G)$, deg v_i = deg $N(v_i) = r$. Hence $N(G)$ is r-regular.

Conversely suppose $N(G)$ is r-regular. Then the degree of each vertex of $N(G)$ is r. Thus deg v_i = deg $N(v_i) = r$. Hence the degree of each vertex of G is r. Therefore G is r-regular.

3. Traversability

We need the following result.

Theorem B [1,p.76]. A connected graph G is eulerian if and only if every vertex of G has even degree.

Remark 20. If G is eulerian, then $N(G)$ need not be eulerian. For example, for the eulerian graph C_4, the neighborhood graph $N(C_4)$ is $2C_4$, which is not eulerian.

We characterize neighborhood graphs which are eulerian.

Theorem 21. Let G be a nontrivial connected graph. The neighborhood graph $N(G)$ of G is eulerian if and only if the following conditions hold:

(1) G has an odd cycle, and
(2) G is eulerian.

Proof: Suppose $N(G)$ is eulerian. On the contrary, suppose condition (1) is not satisfied. Then G has only even cycles or no cycles. By Theorem 18, $N(G)$ is not connected. Hence $N(G)$ is not eulerian, which is a contradiction. This proves (1). Now suppose (2) is not satisfied. Then G has a vertex v of odd degree. Therefore v lies on odd number of open neighborhood sets in G. Hence the degree of v in $N(G)$ is odd. Thus by Theorem B $N(G)$ is not eulerian, which is a contradiction. This proves (2).

Conversely suppose the given conditions are satisfied. Suppose G has an odd cycle. Then by Theorem 18, $N(G)$ is connected. Suppose G is eulerian. By Theorem B, the degree of each vertex of G is even. Then the corresponding vertices of G and the corresponding vertices of open neighborhood sets of G in $N(G)$ are even. Thus by Theorem B, $N(G)$ is eulerian.

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