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An M/G/1 Feedback Queueing System with Second Optional Service and with Second Optional Vacation

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Abstract. We study an M/G/1 queueing system with Bernoulli feedback and with second optional service and second optional vacation. Customers arrive singly and are served one by one according to FCFS rule. The service time follows general distribution. All arriving customers are provided first essential service, where as only some of them demand second service which is optional. After the completion of first or second service, if the customer is dissatisfied with service he can immediately join the tail of the queue as a feedback customer for receiving another regular service. Otherwise the customer may depart forever from the system. Soon after the system is empty the server takes a vacation and after returning from vacation the server may opt for second vacation based on a Bernoulli rule. Using supplementary variable technique, we derive the probability generating function for the number of customers in the system. Some performance measures are calculated. Some special cases and particular cases are discussed. A numerical study is also presented.

Keywords: M/G/1 queue, first essential service, second optional service, Bernoulli feedback, regular vacation, second optional vacation, steady state solution.

AMS Mathematics Subject Classification (2010): 60K25, 60K30

1. Introduction

Many authors have contributed to the study of Bernoulli Feedback queueing system. For detailed study one may refer to Disney et al. [5], Choudhury and Paul [3]. Some notable authors who contributed to M/G/1 type queue with second optional service are Madan [9], Medhi [11], Krishnakumar et al. [7], Choudhury [2], and Thillaigovindan et al. [13]. Vacation models under various service disciplines have been investigated by Madan [8], Choudhury [1], Kalyanaraman and Murugan [6] and Thangaraj and Vanitha [12]. A study on M/G/1 type queueing system with optional second vacation have been carried out by Choudhury [4], Manoharan and Sasi [10]. In this paper we consider an M/G/1 feedback queueing system with second optional service and second optional vacation. The motivation for this type of model comes from some digital communication system where the server may require two types of vacations (i) regular vacation, for usual overhauling and maintenance of the system (ii) optional vacation may be necessary for correcting a major fault when the system is not in proper working condition. The organization of the paper is as follows. In section 2 the model is described. In section 3 the probability generating function for the number of customers in the system is derived. In section 4 some performance measures are calculated. In section 5 particular cases are discussed. In section 6 special cases are discussed and in section 7 a numerical study is presented. A conclusion is given in the last section.

2. The model

The following assumptions briefly describe the mathematical model of our study. The arrival follows Poisson distribution with mean arrival rate λ (> 0). There is a single server who provides the first essential service (FES) to all arriving customers. The service time for FES follows a general distribution, with distribution function $B_1(x)$ and density function $b_1(x)$.

Immediately after the FES, the customer may opt with probability p for a second service which is optional (SOS), or he may leave the system with probability (1 − p), in which case another customer at the head of the queue (if any) is taken up for his FES. The service time for SOS is also assumed to be generally distributed. Let $B_2(x)$ and $b_2(x)$ respectively be the distribution function and the density function of the SOS times.

Further it is assumed that $\mu_i(x)dx$ is the conditional probability of completion of the i^{th} service given that the elapsed service time is x so that $\mu_i(x)dx = \frac{b_i(x)}{[1 - B_i(x)]}$ and therefore $b_i(x) = \mu_i(x)\exp(-\int_0^x \mu_i(t)dt)$; $i \in \{1,2\}$.

We assume that the FES and SOS are mutually independent of each other. Let $B_i^*(s)$, $E(B_i^k)$ $(k \geq 1)$, $i \in \{1,2,3\}$ denote the LST and finite moments of two service times respectively. Thus the total service time required by the server to complete the service cycle which may be called as modified service period is given by

$$B = \begin{cases} B_1 + B_2 \text{ with probability} & p \\ B_1 \quad\quad \text{with probability} & 1-p \end{cases}$$

After the completion of first or second service, if the customer is dissatisfied with the service received to him, he can immediately join the tail of the queue as a feedback customer for receiving another regular service with probability r. Otherwise he may depart forever from the system with probability (1 – r) .

Whenever the system becomes empty, the server goes for a first regular vacation (FRV) of random length V_1. Let $V_1(x)$ and $v_1(x)$ respectively be the distribution function and density function of the first vacation times.

At the end of FRV, the server may take the second optional vacation SOV with probability θ. Otherwise he remains in the system with probability $(1-\theta)$ until a new customer arrives. Let $V_2(x)$ and $v_2(x)$ respectively be the distribution function and density function for the SOV times.

Further it is assumed that $\nu_i(x)dx$ is the conditional probability of the completion of the i^{th} vacation given that the elapsed vacation time is x so that $\nu_i(x)\,dx = \frac{v_i(x)}{1 - V_i(x)}$ and therefore $v_i(x) = \nu_i(x)\, e^{-\int_0^x \nu_i(t)dt}$; $i \in \{1,2\}$.

It is also assumed that the vacation times V_1 and V_2 are mutually independent of each other having LSTs $V_i^*(s)$ and finite moments $E(V_i^k)$, $(k \geq 1)$, $i \in \{1,2\}$. Thus the total vacation time required to complete the vacation cycle, which may be called as modified vacation period is given by

$$V = \begin{cases} V_1 + V_2 & \text{with probability} \quad \theta \\ V_1 & \text{with probability} \quad 1-\theta \end{cases}$$

3. Queue size distribution at a random epoch

Here we first set up the steady state equations for the stationary queue size distribution by treating elapsed service time, FES time, SOS time, FRV time and SOV time as supplementary variables. Then we solve these equations and derive the PGF's. Let N(t) be the queue size (including one being served, if any), $B_1^{(0)}(t)$ be the elapsed service time for FES, $B_2^{(0)}(t)$ be the elapsed service time for the SOS, $V_1^{(0)}(t)$ be the elapsed vacation time for the FRV, $V_2^{(0)}(t)$ be the elapsed vacation time for the SOV at time t respectively. For further development of this model let us introduce the random variable Y (t) as follows.

$$Y(t) = \begin{cases} 0 & \text{if the server is on FRV at time } t \\ 1 & \text{if the server is on SOV at time } t \\ 2 & \text{if the server is busy giving FES at time } t \\ 3 & \text{if the server is busy giving SOS at time } t \end{cases}$$

The supplementary variables $V_1^{(0)}(t), V_2^{(0)}(t);\ B_1^{(0)}(t)$, $B_2^{(0)}(t)$ are introduced in order to obtain a bivariate Markov process $\{N(t); \partial(t);\ t \geq 0\}$ where

$$\partial(t) = \begin{cases} V_1^{(0)}(t) & \text{if} \quad Y(t) = 0 \\ V_2^{(0)}(t) & \text{if} \quad Y(t) = 1 \\ B_1^{(0)}(t) & \text{if} \quad Y(t) = 2 \\ B_2^{(0)}(t) & \text{if} \quad Y(t) = 3 \end{cases}$$

We define the limiting probabilities as follows.

$$Q_{1,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = V_1^{(0)}(t); x < V_1^{(0)}(t) \leq x + dx\right\}; \quad n \geq 0; \quad x > 0$$

$$Q_{2,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t) = n; \partial(t) = V_2^{(0)}(t); x < V_2^{(0)}(t) \leq x + dx\right\}; \quad n \geq 0; \quad x > 0$$

$$P_{1,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t)=n; \partial(t)=B_1^{(0)}(t); x < B_1^{(0)}(t) \le x+dx\right\}; \quad n \ge 0; \quad x > 0$$

$$P_{2,n}(x)dx = \lim_{t\to\infty} \Pr\left\{N(t)=n; \partial(t)=B_2^{(0)}(t); x < B_2^{(0)}(t) \le x+dx\right\}; \quad n \ge 0; \quad x > 0$$

Further it is assumed that $B_i^{(0)}(0)=0; B_i^{(0)}(\infty)=1$ for $i \in \{1,2\}$ and $V_i^{(0)}(0)=0; V_i^{(0)}(\infty)=1$ for $i \in \{1,2\}$ and are continuous at $x = 0$.

By assuming that the system is in steady state condition the differential difference equations governing the system are obtained as

$$\frac{d}{dx}P_{1,n}(x) + (\lambda+\mu_1(x))P_{1,n}(x) = \lambda P_{1,n-1}(x), \qquad x>0, \qquad n \ge 1 \tag{1}$$

$$\frac{d}{dx}P_{1,0}(x) + (\lambda+\mu_1(x))P_{1,0}(x) = 0, \qquad x>0 \tag{2}$$

$$\frac{d}{dx}P_{2,n}(x) + (\lambda+\mu_2(x))P_{2,n}(x) = \lambda P_{2,n-1}(x), \qquad x>0, \qquad n \ge 1 \tag{3}$$

$$\frac{d}{dx}P_{2,0}(x) + (\lambda+\mu_2(x))P_{2,0}(x) = 0, \qquad x>0 \tag{4}$$

$$\frac{d}{dx}Q_{1,n}(x) + (\lambda+\nu_1(x))Q_{1,n}(x) = \lambda Q_{1,n-1}(x), \qquad x>0, \qquad n \ge 1 \tag{5}$$

$$\frac{d}{dx}Q_{1,0}(x) + (\lambda+\nu_1(x))Q_{1,0}(x) = 0, \qquad x>0 \tag{6}$$

$$\frac{d}{dx}Q_{2,n}(x) + (\lambda+\nu_2(x))Q_{2,n}(x) = \lambda Q_{2,n-1}(x), \qquad x>0, \qquad n \ge 1 \tag{7}$$

$$\frac{d}{dx}Q_{2,0}(x) + (\lambda+\nu_2(x))Q_{2,0}(x) = 0, \qquad x>0, \tag{8}$$

$$\begin{aligned}\lambda Q_{1,0} &= (1-p)(1-r)\int_0^\infty P_{1,0}(x)\,\mu_1(x)dx + (1-r)\int_0^\infty P_{2,0}(x)\,\mu_2(x)dx \\ &\quad + (1-\theta)\int_0^\infty Q_{1,0}(x)\nu_1(x)dx + \int_0^\infty Q_{2,0}(x)\nu_2(x)dx\end{aligned} \tag{9}$$

where

$$Q_{1,0} = \int_0^\infty Q_{1,0}(x)dx$$

The boundary conditions are

$$Q_{1,0}(0) = \lambda Q_{1,0} \tag{10}$$

$$Q_{1,n}(0) = 0, \qquad n \ge 1 \tag{11}$$

$$Q_{2,n}(0) = \theta\int_0^\infty Q_{1,n}(x)\nu_1(x)dx, \qquad n \ge 0 \tag{12}$$

$$P_{1,0}(0) = r(1-p)\int_0^\infty P_{1,0}(x)\mu_1(x)dx + \int_0^\infty P_{2,0}(x)\mu_2(x)dx + (1-r)(1-p)\int_0^\infty P_{1,1}(x)\mu_1(x)dx$$

$$+(1-r)\int_0^\infty P_{2,1}(x)\mu_2(x)dx+(1-\theta)\int_0^\infty Q_{1,1}(x)\nu_1(x)dx+\int_0^\infty Q_{2,1}(x)\nu_2(x)dx \tag{13}$$

$$P_{1,n}(0)=r(1-p)\int_0^\infty P_{1,n}(x)\mu_1(x)dx+\int_0^\infty P_{2,n}(x)\mu_2(x)dx+(1-r)(1-p)\int_0^\infty P_{1,n+1}(x)\mu_1(x)dx$$

$$+(1-r)\int_0^\infty P_{2,n+1}(x)\mu_2(x)dx+(1-\theta)\int_0^\infty Q_{1,n+1}(x)\nu_1(x)dx+\int_0^\infty Q_{2,n+1}(x)\nu_2(x)dx,\quad n\ge 1 \tag{14}$$

$$P_{2,n}(0)=p\int_0^\infty P_{1,n}(x)\mu_1(x)dx,\qquad n\ge 0 \tag{15}$$

and the normalizing condition is

$$\sum_{n=1}^\infty\sum_{i=1}^2\int_0^\infty P_{i,n}(x)dx+\sum_{n=0}^\infty\sum_{i=1}^2\int_0^\infty Q_{i,n}(x)dx=1 \tag{16}$$

Now let us define the following PGF's

$$P_i(x,z)=\sum_{n=0}^\infty z^n P_{i,n}(x);\qquad x\ge 0,\qquad |z|\le 1,\qquad i\in\{1,2\} \tag{17}$$

$$P_i(0,z)=\sum_{n=0}^\infty z^n P_{i,n}(0);\qquad |z|\le 1,\qquad i\in\{1,2\} \tag{18}$$

$$Q_i(x,z)=\sum_{n=0}^\infty z^n Q_{i,n}(x);\qquad x\ge 0,\qquad |z|\le 1,\qquad i\in\{1,2\} \tag{19}$$

$$Q_i(0,z)=\sum_{n=0}^\infty z^n Q_{i,n}(0);\qquad |z|\le 1,\qquad i\in\{1,2\} \tag{20}$$

$$P_i(z)=\int_0^\infty P_i(x,z)dx,\qquad i\in\{1,2\} \tag{21}$$

$$Q_i(z)=\int_0^\infty Q_i(x,z)dx,\qquad i\in\{1,2\} \tag{22}$$

Multiplying (1) by z^n and summing over n=1 to ∞ and adding with (2), we get

$$\frac{d}{dx}P_1(x,z)+(\lambda-\lambda z+\mu_1(x))P_1(x,z)=0$$

$$P_1(x,z)=P_1(0,z)[1-B_1(x)]e^{-\lambda(1-z)x} \tag{23}$$

Multiplying (3) by z^n and summing over n=1 to ∞ and adding with (4), we get

$$\frac{d}{dx}P_2(x,z)+(\lambda-\lambda z+\mu_2(x))P_2(x,z)=0$$

$$P_2(x,z)=P_2(0,z)[1-B_2(x)]e^{-\lambda(1-z)x} \tag{24}$$

Multiplying (5) by z^n and summing over n=1 to ∞ and adding with (6), we get

$$\frac{d}{dx}Q_1(x,z)+(\lambda-\lambda z+\nu_1(x))Q_1(x,z)=0$$

$$Q_1(x,z)=Q_1(0,z)[1-V_1(x)]e^{-\lambda(1-z)x} \tag{25}$$

Multiplying (7) by z^n and summing over n=1 to ∞ and adding with (8), we get

$$\frac{d}{dx}Q_2(x,z)+(\lambda-\lambda z+\nu_2(x))Q_2(x,z)=0$$

$$Q_2(x,z)=Q_2(0,z)[1-V_2(x)]e^{-\lambda(1-z)x} \tag{26}$$

Multiplying (14) by z^{n+1} and summing over n=1 to ∞ and adding with z times (13), we get

$$zP_1(0,z)=(1-p)[1-r(1-z)]\int_0^\infty P_1(x,z)\mu_1(x)dx+[1-r(1-z)]\int_0^\infty P_2(x,z)\mu_2(x)dx$$
$$+(1-\theta)\int_0^\infty Q_1(x,z)\nu_1(x)dx+\int_0^\infty Q_2(x,z)\nu_2(x)dx-\lambda Q_{1,0} \tag{27}$$

From equation (15), we get

$$P_2(0,z)=pP_1(0,z)B_1^*(\lambda-\lambda z) \tag{28}$$

From equation (23), we get

$$\int_0^\infty P_1(x,z)\mu_1(x)dx=P_1(0,z)B_1^*(\lambda-\lambda z) \tag{29}$$

From equation (24), we get

$$\int_0^\infty P_2(x,z)\mu_2(x)dx=pP_1(0,z)B_1^*(\lambda-\lambda z)B_2^*(\lambda-\lambda z) \tag{30}$$

From equation (25), we get

$$\int_0^\infty Q_1(x,z)\nu_1(x)dx=Q_1(0,z)V_1^*(\lambda-\lambda z) \tag{31}$$

From equation (26), we get

$$\int_0^\infty Q_2(x,z)\nu_2(x)dx=\theta Q_1(0,z)V_1^*(\lambda-\lambda z)V_2^*(\lambda-\lambda z) \tag{32}$$

From equation (11) and (12), we get

$$Q_1(0,z)=\lambda Q_{1,0} \tag{33}$$

$$Q_2(0,z)=\theta Q_1(0,z)V_1^*(\lambda-\lambda z) \tag{34}$$

Using (28) to (34) in (27), we get

$$P_1(0,z)=\left[\frac{\lambda\left\{[(1-\theta)+\theta V_2^*(\lambda-\lambda z)]V_1^*(\lambda-\lambda z)-1\right\}}{z-[(1-p)+pB_2^*(\lambda-\lambda z)]\,B_1^*(\lambda-\lambda z)\,[1-r(1-z)]}\right]Q_{1,0} \tag{35}$$

Using (35) in (28), we get

$$P_2(0,z)=\left[\frac{p\lambda\left\{[\,(1-\theta)+\theta V_2^*(\lambda-\lambda z)]\;V_1^*(\lambda-\lambda z)-1\,\right\}B_1^*(\lambda-\lambda z)}{z-[(1-p)+pB_2^*(\lambda-\lambda z)]\,B_1^*(\lambda-\lambda z)\,[1-r(1-z)]}\right]Q_{1,0} \tag{36}$$

Integrating (23) to (26) between 0 and ∞, we get

$$P_1(z)=\left[\frac{1-B_1^*(\lambda-\lambda z)}{(\lambda-\lambda z)}\right]P_1(0,z) \tag{37}$$

$$P_2(z)=p\left[\frac{1-B_2^*(\lambda-\lambda z)}{(\lambda-\lambda z)}\right]B_1^*(\lambda-\lambda z)P_1(0,z) \tag{38}$$

$$Q_1(z) = \left[\frac{1-V_1^*(\lambda-\lambda z)}{(\lambda-\lambda z)}\right]Q_1(0,z) \tag{39}$$

$$Q_2(z) = \left[\frac{\theta V_2^*(\lambda-\lambda z)[1-V_1^*(\lambda-\lambda z)]}{(\lambda-\lambda z)}\right]Q_1(0,z) \tag{40}$$

Using (35) in (37) and (38), we get

$$P_1(z) = \left[\frac{\{[\,(1-\theta)+\theta V_2^*(\lambda-\lambda z)\,]\,V_1^*(\lambda-\lambda z)-1\}(1-B_1^*(\lambda-\lambda z))}{(1-z)\{z-[(1-p)+pB_2^*(\lambda-\lambda z)]\,B_1^*(\lambda-\lambda z)\,[1-r(1-z)]\,\}}\right]Q_{1.0}$$

(41)

$$P_2(z) = p\left[\frac{\{[(1-\theta)+\theta V_2^*(\lambda-\lambda z)]\,V_1^*(\lambda-\lambda z)-1\}(1-B_2^*(\lambda-\lambda z))B_1^*(\lambda-\lambda z)}{(1-z)\{z-[(1-p)+pB_2^*(\lambda-\lambda z)]B_1^*(\lambda-\lambda z)\,[1-r(1-z)]\}}\right]Q_{1.0} \tag{42}$$

Using (33) in (39) and (40), we get

$$Q_1(z) = \left[\frac{1-V_1^*(\lambda-\lambda z)}{(1-z)}\right]Q_{1,0} \tag{43}$$

$$Q_2(z) = \left[\frac{\theta V_1^*(\lambda-\lambda z)[1-V_2^*(\lambda-\lambda z)]}{(1-z)}\right]Q_{1,0} \tag{44}$$

Using the fact that $P_1(1) + P_2(1) + Q_1(1) + Q_2(1) = 1$, we arrived

$$Q_{1,0} = \frac{1-\rho}{\lambda(1-r)[E(V_1)+\theta E(V_2)]} \tag{45}$$

where

$$\rho = r+\lambda[E(B_1)+pE(B_2)] \tag{46}$$

and $B_1^{*'}(0) = -E(B_1)$, $B_2^{*'}(0) = -E(B_2)$, are the mean of service times of FES and SOS time respectively, $V_1^{*'}(0) = -E(V_1)$ and $V_2^{*'}(0) = -E(V_2)$ are the mean of vacation times of FRV and SOV respectively. Therefore

$$P(z) = \frac{N(Z)}{D(Z)} \tag{47}$$

where

$$N(z) = \{[(1-\theta)+\theta V_2^*(\lambda-\lambda z)]\,V_1^*(\lambda-\lambda z)-1\}\{1-[(1-p)+pB_2^*(\lambda-\lambda z)]B_1^*(\lambda-\lambda z)r\}Q_{1,0}$$

$$D(z) = \{z-[(1-p)+pB_2^*(\lambda-\lambda z)]B_1^*(\lambda-\lambda z)\}[1-r(1-z)]$$

4. Performance measures

Let L_q and L denote the steady state average queue size and system size respectively.

Then $L_q = \left[\frac{d}{dz}P(z)\right]_{z=1} = \left[\frac{d}{dz}\frac{N(z)}{D(z)}\right]_{z=1}$

Using the L'Hospital rule twice we obtain

$$L_q = \frac{D^{'}(1)N^{''}(1) - N^{'}(1)D^{''}(1)}{2(D^{'}(1))^2} \tag{48}$$

where

$$N^{'}(1) = \lambda(E(V_1) + \theta E(V_2))(1-r)Q_{1,0}$$

$$N^{''}(1) = -\lambda^2 \left\{ [E(V_1^2) + \theta E(V_2^2) + 2\theta\, E(V_1)E(V_2)](1-r) - 2r(E(V_1) + \theta E(V_2))(E(B_1) + pE(B_2)) \right\} Q_{1,0}$$

$$D^{'}(1) = 1 - r - \lambda\left(E(B_1) + pE(B_2)\right)$$

$$D^{''}(1) = -2\lambda\left\{E(B_1) + pE(B_2)\right\}r - \lambda^2\left\{E(B_1^2) + pE(B_2^2) + 2pE(B_1)E(B_2)\right\}$$

where $E(B_1^2)$, $E(B_2^2)$, $E(V_1^2)$, $E(V_2^2)$ are second moment of FES, SOS, FRV and SOV time respectively. Next, we can obtain $L = L_q + \rho$, where L_q and ρ have been found in (48) and (46) respectively. Then using Little's formulae, we obtain W_q, the average waiting time in the queue and W, the average waiting time in the system, as $W_q = \frac{L_q}{\lambda}$ and $W = \frac{L}{\lambda}$ respectively.

5. Particular cases

Case 1: Setting r =0 (no feedback) in (47) , we get

$$P(z) = \frac{\{[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]V_1^*(\lambda - \lambda z) - 1\}\,(1-\rho)}{\{z - [(1-p) + pB_2^*(\lambda - \lambda z)\,]B_1^*(\lambda - \lambda z)\}\,\lambda[E(V_1) + \theta E(V_2)]} Q_{1,0}$$

which coincides with the PGF of Manoharan [10] irrespective of the notations used .

Case 2: Setting p=0 (no SOS) and r=0 (no feedback) in (47) we get

$$P(z) = \frac{\{\,[(1-\theta) + \theta V_2^*(\lambda - \lambda z)]\;\; V_1^*(\lambda - \lambda z) - 1\}\,(1-\rho)}{\{\,z - B_1^*(\lambda - \lambda z)\}\,\lambda\,[\,E(V_1) + \theta\, E(V_2)\,]} Q_{1,0}$$

which coincides with the PGF of Choudhury [4] irrespective of the notations used.

6. Special cases

For validating our model it is important to analyze through specific distribution. By choosing some known distributions for service times and vacation times the validity of the system is examined in this section.

Model-1: Let the distribution of service and vacation times be assumed as Exponential. Then equations (45), (46), (47), (48) and (49) become

$$Q_{1,0} = \frac{(\mu_1\mu_2(1-r) - \lambda(\mu_2 + p\mu_1))\nu_1\nu_2}{(1-r)\mu_1\mu_2\lambda(\nu_2 + \theta\nu_1)} \qquad \rho = \frac{r\mu_1\mu_2 + \lambda(\mu_2 + p\mu_1)}{\mu_1\mu_2}$$

$$P(z)=\frac{\{(\lambda-\lambda z+\nu_2)[\nu_1-(\lambda-\lambda z+\nu_1)]-\theta(\lambda-\lambda z)\nu_1[(\lambda-\lambda z+\mu_2)][(\lambda-\lambda z+\mu_1)-\mu_1 r]]+p(\lambda-\lambda z)\mu_1 r\}}{\{z(\lambda-\lambda z+\mu_2)(\lambda-\lambda z+\mu_1)-[(1-p)(\lambda-\lambda z+\mu_2)+p\mu_2]\mu_1(1-r(1-z))\}(\lambda-\lambda z+\nu_1)(\lambda-\lambda z+\nu_2)}Q_{1,0}$$

$$L_q=\frac{\lambda^2\left\{\begin{array}{l}[\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1)]\{[\nu_2^2+\theta\nu_1^2+\theta\nu_1\nu_2]\mu_1\mu_2(1-r)\\ -r(\nu_2+\theta\nu_1)(\mu_2+p\mu_1)\nu_1\nu_2\}+(\nu_2+\theta\nu_1)(1-r)\nu_1\nu_2\{r\mu_1\mu_2(\mu_2+p\mu_1)\\ +\lambda(\mu_2^2+p\mu_1^2+p\mu_1\mu_2)\}\end{array}\right\}}{\nu_1^2\nu_2^2(\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1))^2}Q_{1,0}$$

$$W_q=\frac{\lambda\left\{\begin{array}{l}[\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1)]\{[\nu_2^2+\theta\nu_1^2+\theta\nu_1\nu_2]\mu_1\mu_2(1-r)\\ -r(\nu_2+\theta\nu_1)(\mu_2+p\mu_1)\nu_1\nu_2\}+(\nu_2+\theta\nu_1)(1-r)\nu_1\nu_2\{r\mu_1\mu_2(\mu_2+p\mu_1)\\ +\lambda(\mu_2^2+p\mu_1^2+p\mu_1\mu_2)\}\end{array}\right\}}{\nu_1^2\nu_2^2(\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1))^2}Q_{1,0}$$

Model-II: Let the service and vacation times be taken as Erlang distributions. Then equations (45), (46), (47), (48) and (49) become

$$Q_{1,0}=\frac{(\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1))\ \nu_1\nu_2}{(1-r)\mu_1\mu_2\lambda(\nu_2+\theta\nu_1)}\qquad \rho=\frac{r\ \mu_1\mu_2+\lambda(\mu_2+p\mu_1)}{\mu_1\mu_2}$$

$$P(z)=\frac{\left[\begin{array}{l}\{(\lambda-\lambda z+\nu_2 k)^k[(1-\theta)(\nu_1 k)^k-(\lambda-\lambda z+\nu_1 k)^k]+\theta(\nu_2 k)^k(\nu_1 k)^k\}\\ \times\{(\lambda-\lambda z+\mu_2 k)^k[(\lambda-\lambda z+\mu_1 k)^k-(1-p)(\mu_1 k)^k r]-p(\mu_1 k)^k(\mu_2 k)^k r\}\end{array}\right]}{\left[\begin{array}{l}\{(\lambda-\lambda z+\mu_2 k)^k[z(\lambda-\lambda z+\mu_1 k)^k-(1-p)(\mu_1 k)^k(1-r(1-z))]\\ -p(\mu_1 k)^k(\mu_2 k)^k(1-r(1-z))\}(\lambda-\lambda z+\nu_1 k)^k(\lambda-\lambda z+\nu_2 k)^k\end{array}\right]}Q_{1,0}$$

$$L_q=\frac{\left[\lambda^2\left\{\begin{array}{l}[(1-r)\mu_1\mu_2-\lambda(\mu_2+p\mu_1)]\{\mu_1\mu_2(1-r)[(k+1)\nu_2^2+\theta(k+1)\nu_1^2+2\theta\nu_1\nu_2 k]\\ -2rk\nu_1\nu_2(\nu_2+\theta\nu_1)(\mu_2+p\mu_1)\}\mu_1\mu_2+(\nu_2+\theta\nu_1)(1-r)\{2k\mu_1\mu_2(\mu_2+p\mu_1)r\\ -\lambda[(k+1)\mu_2^2+p(k+1)\mu_1^2+p(k+1)\mu_1^2+2p\mu_1\mu_2]\}\nu_1\nu_2\end{array}\right\}\right]}{2k\nu_1^2\nu_2^2(\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1))^2}Q_{1,0}$$

$$W_q=\frac{\left[\lambda\left\{\begin{array}{l}[(1-r)\mu_1\mu_2-\lambda(\mu_2+p\mu_1)]\{\mu_1\mu_2(1-r)[(k+1)\nu_2^2+\theta(k+1)\nu_1^2+2\theta\nu_1\nu_2 k]\\ -2rk\nu_1\nu_2(\nu_2+\theta\nu_1)(\mu_2+p\mu_1)\}\mu_1\mu_2+(\nu_2+\theta\nu_1)(1-r)\{2k\mu_1\mu_2(\mu_2+p\mu_1)r\\ -\lambda[(k+1)\mu_2^2+p(k+1)\mu_1^2+p(k+1)\mu_1^2+2p\mu_1\mu_2]\}\nu_1\nu_2\end{array}\right\}\right]}{2k\nu_1^2\nu_2^2(\mu_1\mu_2(1-r)-\lambda(\mu_2+p\mu_1))^2}Q_{1,0}$$

7. Numerical results

Assigning particular values to the parameters of the system as $p=0.02$, $\mu_1=2.5$, $\mu_2=1.5$, $\nu_1=2,\nu_2=1,\theta=0.3$, $r=0.05$ and varying the value of λ from 0.1 to 1 in steps of 0.1 we calculated the values of Lq and Wq which are tabulated in Table-1 and the corresponding graphs are drawn for Model-I and Model-II in Figure 1 and Figure 2 respectively. We observe that when λ increases, there is a steady increase in Lq as well as in Wq for both Model-I and Model-II as can be expected.

Table 1:

λ	Model-I (L_q)	Model-II (L_q)	Model-I (W_q)	Model-II (W_q)
0.1	0.089516	0.072952	0.895163	0.729519
0.2	0.282463	0.229563	0.941543	0.765210
0.3	0.499119	0.404420	0.998238	0.808840
0.4	0.748385	0.604372	1.069122	0.863388
0.5	1.044253	0.840186	1.160281	0.933540
0.6	1.410055	1.129817	1.281868	1.027107
0.7	1.887819	1.505609	1.452169	1.158161
0.8	2.561687	2.032311	1.707791	1.354874
0.9	3.628291	2.861243	2.134289	1.683084
1.0	5.679119	4.447581	2.989010	2.340832

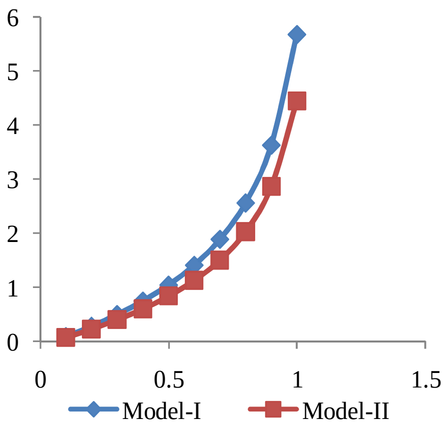

Figure 1: Arrival rate versus L_q

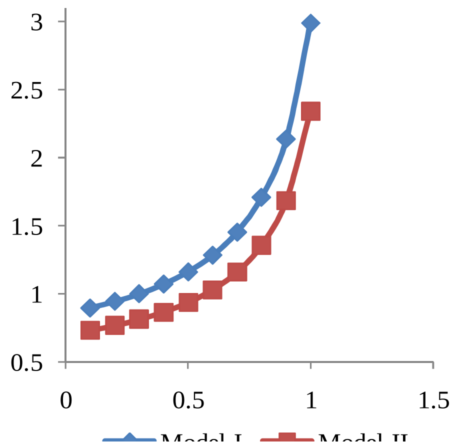

Figure 2: Arrival rate versus W_q

Again assigning particular values as $\lambda = 2,\ \mu_1 = 8,\ \mu_2 = 5,\ \nu_1 = 5,\ \nu_2 = 7,\ \theta = 0.4,\ r = 0.2$ and varying the value of p from 0.1 to 1 in steps of 0.1 we calculated the values of Lq and Wq which are tabulated in Table-2 and the corresponding graphs are drawn for Model-I and Model-II in Figure 3 and Figure 4 respectively. We observe that when p increases, there is a steady increase in Lq as well as in Wq for both Model-I and Model-II as expected.

Table 2:

P	Model-I (L_q)	Model-II (L_q)	Model-I (W_q)	Model-II (W_q)
0.1	0.678247	0.582366	0.339123	0.291183
0.2	0.765035	0.662368	0.382517	0.331184
0.3	0.869829	0.758184	0.434915	0.379092
0.4	0.998172	0.874681	0.499086	0.437341
0.5	1.158135	1.018948	0.579068	0.509474
0.6	1.369160	1.201736	0.680980	0.600868
0.7	1.629140	1.440166	0.814570	0.720083
0.8	1.992731	1.763265	0.996366	0.881633
0.9	2.513624	2.224513	1.256812	1.112256
1.0	3.317661	2.934426	1.658830	1.467213

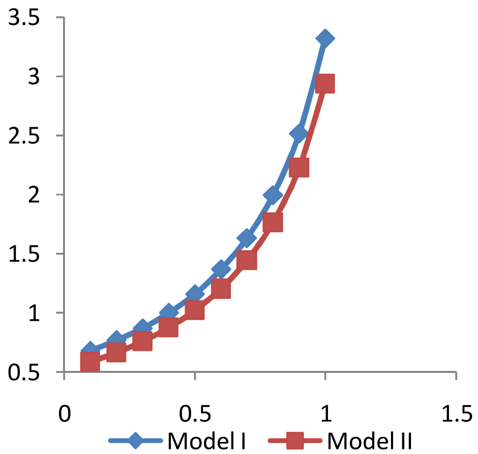

Figure 3: p (probability of SOS) versus L_q

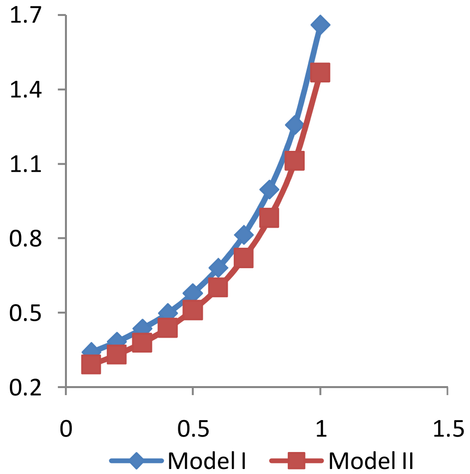

Figure 4: p (probability of SOS) versus W_q

Taking the values of the parameters of as λ=2, p = 0.2, μ_1 =9, μ_2 =8, ν_1 =4, ν_2 =2, r =0.5 and varying the value of θ from 0.1 to 1 in steps of 0.1 we calculated the values of Lq and Wq which are tabulated in Table-3 and the corresponding graphs are drawn for Model-I and Model-II in Figure 5 and Figure 6 respectively. We observe that when θ increases, there is a steady increase in Lq and Wq for both Model-I and Model-II as can be expected.

Table 3:

θ	Model-I (L_q)	Model-II (L_q)	Model-I (W_q)	Model-II (w_q)
0.1	1.312466	1.234824	0.656233	0.617412
0.2	1.431514	1.338990	0.715757	0.669495
0.3	1.520799	1.417115	0.760400	0.708558
0.4	1.590799	1.477879	0.795122	0.738940
0.5	1.645800	1.526490	0.822900	0.763245
0.6	1.691254	1.566263	0.845627	0.783132
0.7	1.729133	1.599407	0.864566	0.799704
0.8	1.761184	1.627452	0.880592	0.813726
0.9	1.788657	1.651491	0.894328	0.825745
1.0	1.812466	1.672324	0.906233	0.836162

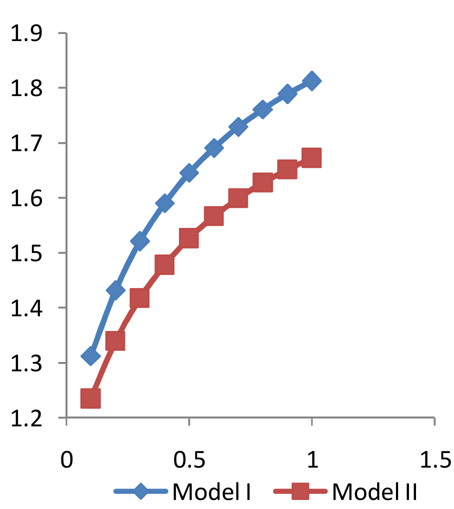

Figure 5: θ (probability of SOV) versus L_q

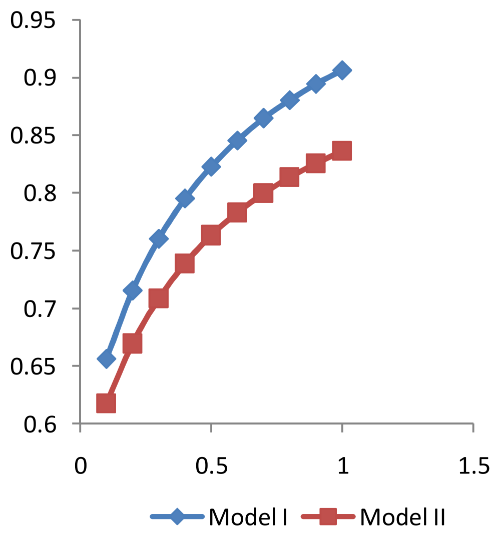

Figure 6: θ (probability of SOV) versus W_q

8. Conclusion

The analysis carried out in "An M/G/1 feedback Queueing system with second optional service and with second optional vacation" is to obtain the probability generating function for the number of customers in the system and also to obtain waiting time of a customer in the system. Numerical work is carried out to study the effect of some parameters on the operating characteristics of the system.

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