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Deletion of an Edge in Majority Independent Set of a Graph

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***Abstract*:** In this article we consider graphs which are critical with respect to their majority independence number. The effect of the removal of an edge on majority independence number of a graph is studied. There are several ways in which a graph could be critical or not. A graph may be critical in the sense that its majority independence number increases when any edge is deleted. Graphs for which the majority independence number remains unchanged when an edge is deleted have also been discussed.

Keywords: β_M -critical edge, β_M -redundant edge.

AMS Mathematics Subject Classification (2010): 05C69

1. Introduction

Claude Berge in 1980, introduced B graphs. These are graphs in which every vertex in the graph is contained in a maximum independent set of the graph. Fircke.et.al [1] in 2002 made a beginning of the study of graphs which are excellent with respect to various parameters. Also, Sridharan and Yamuna [7] made an extensive work in γ-excellent trees and total domination excellent trees. Swaminathan and Pushpalatha have defined β_o-excellent graphs and they have made a detailed study in this paper [6].

By a graph G , we mean a finite, simple graph which is undirected and nontrivial. Let $G=(V,E)$ be a graph of order p and size q. For every vertex $v\in V(G)$, the open neighbourhood of v is defined by $N(v)=\{u\in V(G)/uv\in E(G)\}$ and the closed neighbourhood of a vertex v is defined by $N[v]=N(v)\bigcup\{v\}$. Let S be a set of vertices, and let $u\in S$. The private neighbour set of u with respect to S is $pn[u,S]=\{v/N[v]\bigcap S=\{u\}\}$.

Definition 1.1. [2] A set of $S\subseteq V(G)$ of vertices of a graph G is said to be a Majority Independent set(or MI-set) if it induces a totally disconnected subgraph with

$|N[S]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,S]| > |N[S]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in S$. If any vertex set S' properly containing S is not majority independent, then S is called maximal majority independent set. The minimum cardinality of a maximal MI-set is called lower majority independence number of G and it is also called independent majority domination number of G. It is denoted by $i_M(G)$. The maximum cardinality of a maximal majority independent set of G is called Majority Independence number of G and it is denoted by $\beta_M(G)$. A β_M-set is a maximum cardinality of a maximal MI-set of G. This parameter has been studied by Joseline Manora and Swaminathan.

2. Definition and examples of edge removal.

Definition 2.1. Let G be a graph. Let $x \in E(G)$. Then $G-x$ is a spanning subgraph of G and hence $\beta_M(G-x) \geq \beta_M(G)$.

Definition 2.2. Let G be a graph. Let $x \in E(G)$. x is said to be a β_M-redundant edge of G if $\beta_M(G-x) = \beta_M(G)$ and a β_M-critical edge if $\beta_M(G-x) > \beta_M(G)$.

Example 2.3. In C_4, all edges are redundant edges. In C_6, all edges are critical edges.

Definition 2.4. For any graph G, the edge set can be partitioned with respect to β_M-set into two sets $E^0_{\beta M}(G)$ and $E^+_{\beta M}(G)$ is denoted by

$E^0_{\beta M}(G) = \{x \in E(G) : \beta_M(G-x) = \beta_M(G)\}$

$E^+_{\beta M}(G) = \{x \in E(G) : \beta_M(G-x) > \beta_M(G)\}$.

Results 2.5.

1. If $G = K_{1,p-1}, p \geq 3$. Then $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E(G)$.
2. Let $G = K_p, p \geq 3$. Then $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E(G)$.
3. Let $G = D_{r,s}, r < s$. Then $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E(G)$.
4. Let $G = mK_2$. Then if m is odd $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E(G)$ and m is even $\beta_M(G-x) > \beta_M(G)$ for $\forall x \in E(G)$.
5. If G is a caterpillar, then $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E(G)$.
6. If a β_M-set D of G contains pendant vertices then $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E_p(G)$, where $E_p(G)$ is the set of all pendant edges of G.

7. If G has atleast $\left\lceil \frac{p}{2} \right\rceil$ isolates then a β_M -set D of G contains only isolates and every edge x of G is $x \in E^0_{\beta M}(G)$.

Proposition 2.6. If G be a disconnected with isolates. Then

(i) $\beta_M(G-x) = \beta_M(G)$ for $\forall x \in E^0_{\beta M}(G)$ except pendant edges.

(ii) $\beta_M(G-x) > \beta_M(G)$ if x is a pendant edge.

Note 2.7.

1. $\mathrm{B}_M(G)$ is the set of all maximal MI-set of G .
2. $\beta_M(G) = \max_{D \in \mathrm{B}_M(G)} \{|D|\}$.

Proposition 2.8. Let G be a graph and H be a spanning subgraphof G . Then $\mathrm{B}_M(H) \subseteq \mathrm{B}_M(G)$.

Proof: Let D be a maximal MI-setof H . ie., $D \in \mathrm{B}_M(H)$. Then $|N_H[D]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,D]| > |N_H[D]| - \left\lceil \frac{p}{2} \right\rceil$ for every $v \in D$ and $p = |V(H)|$. Since H is a spanning subgraph of G , $p = |V(H)| = |V(G)|$.$\therefore$ $|N_G[D]| \geq \left\lceil \frac{p}{2} \right\rceil$ and

$|pn[v,D]| > |N_G[D]| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall$ $v \in D \Rightarrow D$ is a maximal MI-set of G .

Hence $D \in \mathrm{B}_M(G)$. Then $\mathrm{B}_M(H) \subseteq \mathrm{B}_M(G)$.

Corollary 2.9. If H is a spanning subgraph of G then $\beta_M(H) \geq \beta_M(G)$.

Proof: Since $\mathrm{B}_M(H) \subseteq \mathrm{B}_M(G)$, $\beta_M(H) \geq \beta_M(G)$.

Corollary 2.10. Let G be a graph and x be any edge of G . Then $\beta_M(G-x) \geq \beta_M(G)$.

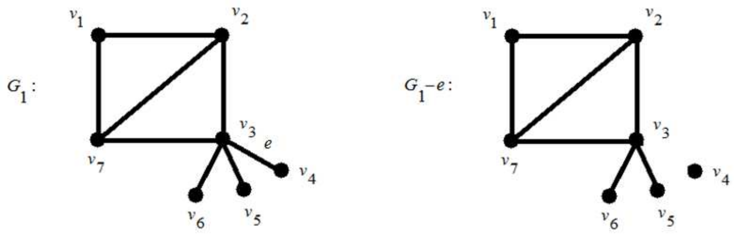
Proof: Since $(G-x) = H$ is a spanning subgraphof G , $\beta_M(G-x) \geq \beta_M(G)$.

Theorem 2.11. For any graph G , $\beta_M(G) \leq \beta_M(G-x) \leq \beta_M(G) + 1$.

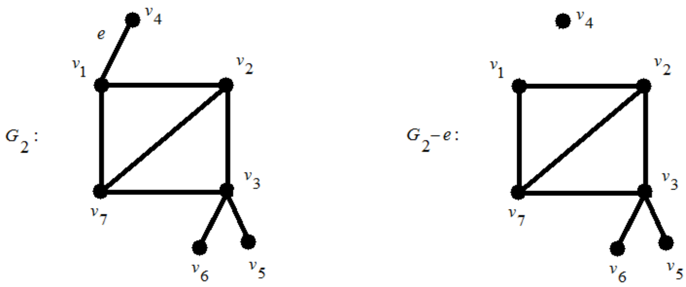
Proof: Suppose $\beta_M(G-x) \geq \beta_M(G) + 2$. Let $S = \{x_1, x_2, ..., x_t\}$ be a β_M -set of $G-x$, where $t \geq \beta_M(G) + 2$ and let $e = \{xy\}$. If x and y are not in S , then S is an independent set of G, a contradiction. (since $|S| > \beta_M(G)$). If $x \in S$ and $y \notin S$, then also S is an independent set of G, a contradiction. Suppose $x, y \in S$.

Then $|N[S]| > \left\lceil \frac{p}{2} \right\rceil$. So, $S - \{y\}$ is an independent set of G of cardinality $t - 1 = \beta_M(G) + 1$, a contradiction. Therefore $\beta_M(G - x) \leq \beta_M(G) + 1$.

Illustration 2.12.



$\beta_M(G_1) = 3$ $\beta_M(G_1 - e) = 3$



$\beta_M(G_2) = 2$ $\beta_M(G_2 - e) = 3$

Proposition 2.13. Let G be a graph and x be any edge of G. Then exactly one of the following is true

(i). $\beta_M(G - x) = \beta_M(G)$

(ii). $\beta_M(G - x) = \beta_M(G) + 1$.

Proof: Let $x = uv$ be any edge of G and $G - x$ is a spanning subgraph of G.

By corollary 2.10, $\beta_M(G - x) \geq \beta_M(G)$ (1)

ie., $\beta_M(G - x) = \beta_M(G) + k$, for some $k \geq 0$ (2)

Claim : $k \leq 1$. Let $H = (G - x)$ and D be a maximal MI-set of H.

Case (i): Let $x = uv$ such that u *and* $v \in D$.

Subcase (i): D contains no isolates and $\left|N_H[D]\right| = \left\lceil \frac{p}{2} \right\rceil$. Then $\left|N_G[D]\right| = \left\lceil \frac{p}{2} \right\rceil$ but u and v are adjacent in $G \Rightarrow D_1 = D - \{v\} \cup \{w\}$ such that $\left|N[D_1]\right| > \left\lceil \frac{p}{2} \right\rceil$ and $\left|pn[v,D_1]\right| > \left|N[D_1]\right| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\, v \in D_1$. $\therefore$ D_1 is a maximal MI-set of G.

Hence, $\beta_M(G) \geq \left|D_1\right| = \left|D\right| = \beta_M(H)$. Since $\beta_M(G) \leq \beta_M(H)$,
$\beta_M(G) = \beta_M(H) = \beta_M(G-x)$. $\therefore$ $\beta_M(G) = \beta_M(G) + k$,by (2) $\Rightarrow k = 0$.

Subcase (ii): D contains isolatesand $\left|N_H[D]\right| = \left\lceil \frac{p}{2} \right\rceil$. Then $\left|N_G[D]\right| > \left\lceil \frac{p}{2} \right\rceil$ but u and v are adjacent in G. Choose $D_1 = D - \{v\} \cup \{w\}$ such that $\left|N[D_1]\right| > \left\lceil \frac{p}{2} \right\rceil$ and $\left|pn[v,D_1]\right| \leq \left|N[D_1]\right| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\, v \in D_1 \Rightarrow D_1$ is not a maximal MI-set of G and $D - \{v\}$ is a maximal MI-set of G. $\therefore$ $\beta_M(G) \geq \left|D\right| - 1 = \beta_M(H) - 1$
$\Rightarrow \beta_M(G) + 1 \geq \beta_M(G-x)$. Then by (2), $\beta_M(G) + 1 \geq \beta_M(G) + k \Rightarrow k \leq 1$.

Case (ii): D is a β_M-set of H and $\left|N_H[D]\right| > \left\lceil \frac{p}{2} \right\rceil$. Let $x = uv$ such that u *and* $v \in D$ .Then $\left|N_G[D]\right| > \left\lceil \frac{p}{2} \right\rceil$ but u and v are adjacent in G. $\Rightarrow D - \{v\} = D_1$ then $\left|N_G[D_1]\right| = \left\lceil \frac{p}{2} \right\rceil$ and $\left|pn[v,D_1]\right| > \left|N[D_1]\right| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\, v \in D_1$. $\Rightarrow D_1$ is a maximal MI-set of G. By the above argument, $k \leq 1$.

Case (iii): D is a β_M-set of H and $\left|N_H[D]\right| > \left\lceil \frac{p}{2} \right\rceil$. Let $x = uv$ such that $u \in D$, $v \in V - D \in N(D)$. Then $\left|N_G[D]\right| > \left\lceil \frac{p}{2} \right\rceil$ and $\left|pn[v,D]\right| \leq \left|N_G[D]\right| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\, v \in D \Rightarrow D - \{v\}$ is a maximal MI-set of G. $\therefore$ By the above argument, $k \leq 1$.

Case (iv): Let $x = uv$ such that $u \in D$, $v \in V - D$ and $\left|N_H[D]\right| = \left\lceil \frac{p}{2} \right\rceil$. Then $\left|N_G[D]\right| > \left\lceil \frac{p}{2} \right\rceil$ and $\left|pn[v,D]\right| > \left|N_G[D]\right| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\, v \in D \Rightarrow D$ is a maximal MI-setof G. $\therefore$ $k = 0$.

Case (v): Let $x = uv$ such that $u, v \in V - D$ and $|N_H[D]| > \left\lceil \frac{p}{2} \right\rceil$. Then $|N_G[D]| > \left\lceil \frac{p}{2} \right\rceil$ and $|pn\,[v,D]| > |N[D]| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\, v \in D \Rightarrow D$ is a maximal MI-setof G .∴ $\beta_M(G) \geq \beta_M(H)$ and by (2), $k = 0$.

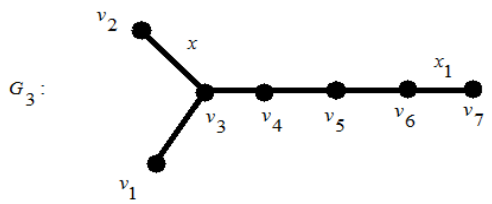
Case (vi): Let $x = uv$ such that $u, v \in V - D$ and $|N_H[D]| = \left\lceil \frac{p}{2} \right\rceil$. Then D contains pendants and isolates. So, $|N_G[D]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn\,[v,D]| \leq |N[D]| - \left\lceil \frac{p}{2} \right\rceil$, for $\forall\, v \in D$. ∴ $D - \{v\}$ is a maximal MI-set of G. By the above argument, $k \leq 1$.Hence, either $\beta_M(G - x) = \beta_M(G)$ or $\beta_M(G - x) = \beta_M(G) + 1$.

3. β_M - Critical and β_M -redundant edge

Definition 3.1. Let $G = (V, E)$ be any graph and x be any edge of G. An edge x is β_M -critical if $\beta_M(G - x) = \beta_M(G) + 1$.

Definition 3.2. An edge x is β_M -redundant if $\beta_M(G - x) = \beta_M(G)$.

Example 3.3.



$\beta_M(G) = 2$. $\beta_M(G - x) = 2$. Here x is a β_M -redundant edge of G.
$\beta_M(G - x_1) = 3$. Then x_1 is a β_M -critical edge of G.

Definition 3.4. A graph G may not have a β_M -critical edge at all. Such graphs are called β_M -durable graphs.

Example 3.5. (i) K_p, $p \geq 3$ (ii) $K_{1,p-1}$, $p \geq 3$ and (iii) $G = P_{11}$.

Remark 3.6. Let G be a β_M -critical graph. Then G is either connected or disconnected with isolates or disconnected without isolates.

Theorem 3.7. An edge $x = uv$ of a graph G is β_M -critical edge if and only if for every maximum MI-set D of G, the following four conditions hold:

i) u and v do not belong to D.

ii) If u and v both belong to $V-D$ then D is not β_M -set of $(G-x)$.

iii) If $v \in V-D$ then $N(v) \cap D = \{u\}$. A similar result holds if $u \in V-D$.

iv) Suppose $u \in D$ and $v \in V-D$. Then $|N[D]-\{v\}| < \left\lceil \frac{p}{2} \right\rceil$ and

$$|pn[v, D_1]| > |N[D_1]| - \left\lceil \frac{p}{2} \right\rceil \text{for } \forall\ v \in D_1 \text{ if } D_1 = D \cup \{w\},\ w \in V-D.$$

Proof: Let $\mathrm{B}_M(G)$ be the set of all maximal MI-setof G. Let $x = uv$ be a β_M -critical edge. Let D be a β_M - set of G.

Suppose the condition (i) is not true then u and v both belong to D. $\Rightarrow$ D is not an independent set, which is a contradiction to D is a MI-set of G.

Suppose the condition(ii) is not true. If u and v both belong to $V-D$ then D is again a β_M -set of $(G-x) \Rightarrow x$ is β_M -redundant edge, which is a $\Rightarrow\Leftarrow$.

Suppose condition (iii) is not true in the case of $u \in D$ and $v \in V-D$, for $v \in V-D$, $N(v) \cap D \neq \{u\}$ then D is a β_M -set of $(G-x) \Rightarrow x$ is β_M -redundant edge, which is a $\Rightarrow\Leftarrow$.

Suppose (iv) is not true. When $u \in D$ and $v \in V-D$, $|N[D]-\{v\}| \geq \left\lceil \frac{p}{2} \right\rceil$ and also, for $\forall$ $v \in D$, $|pn[v,D_1]| > |N[D_1]| - \left\lceil \frac{p}{2} \right\rceil$ if $D_1 = ([D]-\{v\}) \Rightarrow D$ is a β_M -set of $(G-x) \Rightarrow x$ is β_M -redundant edge, $\Rightarrow\Leftarrow$. Hence, an edge $x = uv$ satisfies all four conditions.

Conversely, let the condition (i) to (iv) be true for $\forall$ $D \in \mathrm{B}_M(G)$ such that x is β_M -critical edge. Let $x = uv$, by (i) either u and v both belong to D and $V-D$ or $u \in D$ and $v \in V-D$ and vice versa.

Case (i): By(ii), if u and v both belong to $V-D$ then D is not a β_M -set of $(G-x)$. Choose a MI-set $D_1 = D \cup \{w\}$ which contains isolates and pendants such that $|N[D_1]| \geq \left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,D_1]| > |N[D_1]| - \left\lceil \frac{p}{2} \right\rceil$, for $\forall$ $v \in D_1 \Rightarrow D_1$ is a β_M -set $(G-x)$. $\therefore$ $\beta_M(G-x) = |D| = \beta_M(G)+1 \Rightarrow x$ is β_M -critical edge.

Case (ii): Either $u \in D$ and $v \in V-D$ or $u \in V-D$ and $v \in D$. Let $u \in D$ and $v \in V-D$. Then $N(v) \cap D = \{u\}$ and D is nota maximum MI-setof $(G-x)$. Then $D \cup \{v\}$ is a maximal MI-setof $(G-x)$. $\therefore$ $\beta_M(G-x) \geq |D|+1 = \beta_M(G)+1$.

Claim : $\beta_M(G-x)=\beta_M(G)+1$. Suppose this is not true, then by proposition 2.13, $\beta_M(G-x)=\beta_M(G)$. Hence $|N[v]\cap D|\geq 2$ and $|N[D]-\{v\}|\geq\left\lceil\frac{p}{2}\right\rceil$. Also, $|pn[v,D]|\leq|N[D]|-\left\lceil\frac{p}{2}\right\rceil$, for any $v\in D$. Which is a contradictionto (iii) and (iv).

Hence $\beta_M(G-x)=\beta_M(G)+1\Rightarrow x$ is β_M-critical edge of G.

Definition 3.8. A graph G is called β_M-critical graph if $\beta_M(G-x)=\beta_M(G)+1$,for every edge x of G.

Corollary 3.9. A graph G is β_M-critical if and only if for every edge $x=uv$ in G and for every maximum MI-set D of G, the following four conditions hold:

i) u and v do not belong to D.

ii) If u and v both belong to $V-D$ then D is not β_M-set of $(G-x)$.

iii) If $v\in V-D$ then $N(v)\cap D=\{u\}$. A similar result holds if $u\in V-D$.

iv) Suppose $u\in D$ and $v\in V-D$. Then $|N[D]-\{v\}|<\left\lceil\frac{p}{2}\right\rceil$ and $|pn[v,D_1]|>|N[D_1]|-\left\lceil\frac{p}{2}\right\rceil$, for $v\in D_1$ if $D_1=D\cup\{w\}$.

Result 3.10.

1. If the graph G is β_M-critical then either G is connected or disconnected.
2. Any edge of a cycle C_6 is β_M-critical but this is not so in the case of independent set. ie., Every edge of C_6 is β_0-redundant.

Examples 3.11.

1. Let $G=P_9$. For $\forall\ x\in E(G)$, $\beta_M(G-x)=\beta_M(G)+1$ $\therefore$ $G=P_9$ is a β_M-critical and a connected graph.
2. Let $G=K_{1,5}\cup 5K_1$ be a disconnected with isolates, for $\forall\ x\in E(G)$, $\beta_M(G-x)=\beta_M(G)+1$.$\therefore$ This G is β_M-critical.
3. Let $G=mK_2$, $m=even$ be a disconnected graph without isolates. Then for $\forall\ x\in E(G)$, $\beta_M(G-x)=\beta_M(G)+1$.$\therefore$ This graph G is β_M-critical.

Theorem 3.12. A disconnected graph G with isolates is β_M-critical graph if and only if $G=\left(K_{1,r_1}\cup K_{1,r_2}\cup\ldots\cup K_{1,r_s}\right)\cup\left[p-(r_1+r_2+\ldots+r_s+s)\right]K_1$, where $\left\lceil\frac{p}{2}\right\rceil\leq\left(r_1+r_2+\ldots+r_s+s\right)\leq\binom{p}{2}+1$.

Proof: Let G be a disconnected graph with isolates which is a β_M -critical graph. Let D be a β_M -set of G. Then for every edge x of G, $\beta_M(G-x)>\beta_M(G)$. Since D is a MI-set of G, D is independent set of G. Suppose $V-D$ is independent then there is an edge $x=uv$ such that $u, v\in V-D$. Then, $\beta_M(G-x)=|D|=\beta_M(G)$, which is a contradiction. $\therefore$ $V-D$ is not independent. Thus, D is independent and $(V-D)$ is not independent.

Subcase (i): $d(u)<1$ for all $u\in D$. Then D contains only isolates. Then $\beta_M(G-x)=|D|=\beta_M(G)$ which contradicts the condition (1).

Subcase (ii): $d(u)=1$ for all $u\in D$. Then D contains only pendant vertices. If $|N[D]|>\left\lceil \frac{p}{2}\right\rceil$ and $\left\lceil \frac{p}{2}\right\rceil$ is odd, then $\beta_M(G-x)=|D|=\beta_M(G)$, which is a contradiction. If $|N[D]|=\left\lceil \frac{p}{2}\right\rceil$ and $\left\lceil \frac{p}{2}\right\rceil$ is even then $\beta_M(G-x)>|D|=\beta_M(G)$, which contradict to u with isolates.

Subcase (iii): $d(u)\leq 1$, for some $u\in D$. Then D contains pendants and isolates. Let $V-D$ may also contains isolates and $d(v)\geq 1$, for some $v\in V-D$.

Therefore $G=\left(K_{1,r_1}\cup K_{1,r_2}\cup\ldots\cup K_{1,r_s}\right)\cup\left[p-(r_1+r_2+\ldots+r_s+s)\right]K_1$

Claim: $\left\lceil \frac{p}{2}\right\rceil\leq\left(r_1+r_2+...+r_s+s\right)\leq\left(\frac{p}{2}\right)+1$. When p is even.

Suppose $(r_1+r_2+...+r_s+s)>\left(\frac{p}{2}\right)+1$, then $(r_1+r_2+...+r_s+s)\geq\left(\frac{p}{2}\right)+2$.

$p-(r_1+r_2+...+r_s+s)\leq\left(\frac{p}{2}\right)-2$. ie., G contains atmost $\left\lceil \frac{p}{2}\right\rceil-2$ isolates. Let D be a β_M -set of G. $D=\left\{i_1, i_2, \ldots, i_{\left\lceil \frac{p}{2}\right\rceil-2}, u\right\}$. Let $x=uv$ such that $u\in D$, $v\in V-D$.

Then $G-x=H$. $|N_H[D]|<\left\lceil \frac{p}{2}\right\rceil$. So, $D_1=D\cup\{w\}$. Then $|N[D_1]|>\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D_1]|=|N[D_1]|-\left\lceil \frac{p}{2}\right\rceil$ for $\forall\, v\in D_1\Rightarrow D_1$ is not a MI-setof H.

$D_1=D-(u)\cup\{w\}\Rightarrow\beta_M(G-x)=|D|=\beta_M(G)$, which is a contradiction.

$\therefore$ $\left(r_1+r_2+...+r_s+s\right)\leq\left(\frac{p}{2}\right)+1$, if p is even.

When p is odd. Suppose $(r_1+r_2+...+r_s+s)<\left\lceil \frac{p}{2} \right\rceil$. Then

$[p-(r_1+r_2+...+r_s+s)]\geq\left\lceil \frac{p}{2} \right\rceil$. Therefore G contains atleast $\left\lceil \frac{p}{2} \right\rceil$ isolates.

$\Rightarrow \beta_M$-set D of G contains only $\left\lceil \frac{p}{2} \right\rceil$ isolates. Let $x=uv$ such that $u,v\in V-D$.

Then D is the MI-setof $G-x \Rightarrow \beta_M(G-x)=|D|=\beta_M(G)$, whichcontradicts (1).

Hence $(r_1+r_2+...+r_s+s)\geq\left\lceil \frac{p}{2} \right\rceil$. Thus, $\left\lceil \frac{p}{2} \right\rceil \leq (r_1+r_2+...+r_s+s) \leq \left(\frac{p}{2}\right)+1$.

Conversely, Let $G=\left(K_{1,r_1}\cup K_{1,r_2}\cup\ldots\cup K_{1,r_s}\right)\cup\left[p-(r_1+r_2+\ldots+r_s+s)\right]K_1$,

where $\left\lceil \frac{p}{2} \right\rceil \leq (r_1+r_2+...+r_s+s) \leq \left(\frac{p}{2}\right)+1$.

Case (i): When p is even. $(r_1+r_2+...+r_s+s)=\left(\frac{p}{2}\right)+1$.

$[p-(r_1+r_2+...+r_s+s)]=p-\left(\frac{p}{2}+1\right)=\frac{p}{2}-1$. $\therefore$ G contains only $\left(\frac{p}{2}-1\right)$ isolates.

Let $D=\left\{i_1,i_2,\ldots,i_{\left\lceil \frac{p}{2}\right\rceil-2},u\right\}$ be a β_M-set of G. Then $|N[D]|=\left\lceil \frac{p}{2} \right\rceil$ and $v\in D$,

$|pn[v,D]|>1$.

Subcase (i): Let $x=uv$ such that $u\in D$, $v\in V-D$. Then u is an isolate in $(G-x)$ and $|N[D]|<\left\lceil \frac{p}{2} \right\rceil$. Let $D_1=\left\{i_1,i_2,\ldots,i_{\left\lceil \frac{p}{2}\right\rceil-1},u\right\}$. $|N[D_1]|=\frac{p}{2}$ and

$|pn[v,D_1]|>|N[D_1]|-\left(\frac{p}{2}\right)$ for $\forall\, v\in D_1 \Rightarrow D_1$ is a MI-set of $(G-x)$.

$\beta_M(G-x)=|D_1|=\beta_M(G)+1$ for $\forall\, x$.

Subcase (ii): Let $x=vw$ such that $v,w\in V-D$. In $(G-x)$, w is an isolate.

Choose $D_1=D-\{u\}\cup\left\{w,i_{\left(\frac{p}{2}\right)-2}\right\}$. ie., $D_1=\left\{i_1,i_2,\ldots,i_{\left(\frac{p}{2}\right)-2},w\right\}$.

$\therefore |D_1|=|D|-1+2\Rightarrow |D|+1$, by the above argument, $\beta_M(G-x)=\beta_M(G)+1$ for $\forall\, x\in E(G)$.

Case (ii): p is odd. When $(r_1+r_2+...+r_s+s)=\left\lceil \frac{p}{2} \right\rceil$.

Then $\left[p-(r_1+r_2+...+r_s+s)\right]=\left\lfloor \frac{p}{2} \right\rfloor=\left\lceil \frac{p}{2} \right\rceil-1$ and G contains $\left\lceil \frac{p}{2} \right\rceil-1$ isolates.

By the similar argument, $\beta_M(G-x)=\beta_M(G)+1$.

Subcase (i): Let $x=uv$ such that $u\in D$, $v\in V-D$. Then $\beta_M(G-x)=\beta_M(G)+1$.

Subcase (ii): Let $x=uv$ such that $u,v\in V-D$. Then $\beta_M(G-x)=\beta_M(G)+1$ for $\forall\, x\in E(G)$. Hence for every $\forall\, x\in E(G)$, $\beta_M(G-x)=\beta_M(G)+1$.

$\therefore G$ is β_M -critical.

Case (iii): Let $x=uv$ such that both $u,v\in D$. Since D is a MI-set. This case does not exist.

Theorem 3.13. For any disconnected graph $G(p,q)$ with isolates, the following statements are equivalent.

a. G is β_M -critical.

b. $G=\left(K_{1,r_1}\cup K_{1,r_2}\cup\ldots\cup K_{1,r_s}\right)\cup\left[p-(r_1+r_2+\ldots+r_s+s)\right]K_1$ where $\left\lceil \frac{p}{2} \right\rceil\le(r_1+r_2+...+r_s+s)\le\left(\frac{p}{2}\right)+1$.

c. $\beta_M(G)=\left\lceil \frac{p}{2} \right\rceil-1$.

Proof : By theorem 3.12, (a) $\Rightarrow$ (b).Next to prove (b) $\Rightarrow$(c).

Let $G=\left(K_{1,r_1}\cup K_{1,r_2}\cup\ldots\cup K_{1,r_s}\right)\cup\lambda K_1$, where $\lambda=\left[p-(r_1+r_2+\ldots+r_s+s)\right]$ where $\left\lceil \frac{p}{2} \right\rceil\le p-\lambda\le\left(\frac{p}{2}\right)+1$.Let $V(K_{1,ri})=\{v_i,v_{i1},v_{i2},...,v_{i\,ri}\}$, $1\le i\le s$ and $V(\lambda K_1)=\{u_1,u_2,...,u_\lambda\}$ and $\lambda=\left\lceil \frac{p}{2} \right\rceil-1$. Take $(r_1+r_2+...+r_s)=q$. Since $\left\lceil \frac{p}{2} \right\rceil\le(r_1+r_2+...+r_s)\le\left(\frac{p}{2}\right)+1$, $\left\lceil \frac{p}{2} \right\rceil\le q+s\le\left(\frac{p}{2}\right)+1\Rightarrow\left\lceil \frac{p}{2} \right\rceil-s\le q\le\left(\frac{p}{2}\right)+1-s$.

When p is odd, $q\ge\left\lceil \frac{p}{2} \right\rceil-s.\Rightarrow q+s\ge\left\lceil \frac{p}{2} \right\rceil$ and when p is even, $q\le\left(\frac{p}{2}\right)+1-s$ $\Rightarrow q+s\le\left(\frac{p}{2}\right)+1$.

Case (i): Suppose $q+s=\left\lceil \frac{p}{2} \right\rceil$, then $s=\left\lceil \frac{p}{2} \right\rceil-q$, p is odd.

$q+s=\frac{p}{2}+1$, then $s=\left(\frac{p}{2}\right)+1-q$, p is even.

Let $D=\{u_1,u_2,...,u_{\lambda-1},v_{11}\}$, $v_{11}\in V_{i r_i}$, $i=1$ and $|D|=\lambda$. Then $|N[D]|=|\lambda-1|+N[v_{11}]=\left\lceil \frac{p}{2}\right\rceil-1-1+2$ if p is odd.

Sub case (i): When p is odd, $|N[D]|=\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D]|>|N[D]|-\left\lceil \frac{p}{2}\right\rceil$, for $\forall\ v\in D \Rightarrow D$ is a maximal MI-set of G and $\beta_M(G)\geq|D|=|\lambda|=\left\lceil \frac{p}{2}\right\rceil-1$. Since $|N[D]|=\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D]|>0$ for $\forall\ v\in D$, $\beta_M(G)\leq|D|=\left\lceil \frac{p}{2}\right\rceil-1$. Hence $\beta_M(G)=\left\lceil \frac{p}{2}\right\rceil-1$ if p is odd.

Sub case (ii): When p is even, $|N[D]|=\lambda-1+N[v_{11}]=\left\lceil \frac{p}{2}\right\rceil-1-1+2=\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D]|>|N[D]|-\left\lceil \frac{p}{2}\right\rceil$ for $\forall\ v\in D \Rightarrow D$ is a maximal MI-set of G and $\beta_M(G)\geq|D|=\left\lceil \frac{p}{2}\right\rceil-1$. Since $|N[D]|=\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D]|>0$ for $\forall\ v\in D$, $\beta_M(G)\leq \frac{p}{2}-1$. Hence $\beta_M(G)=\frac{p}{2}-1$ if p is even. $\therefore\ \beta_M(G)=\left\lceil \frac{p}{2}\right\rceil-1$.

Case (ii): Suppose $q+s>\left\lceil \frac{p}{2}\right\rceil$ if p isodd and $q+s<\left(\frac{p}{2}\right)+1$ if p is even.

Subcase (i): If $q+s>\left\lceil \frac{p}{2}\right\rceil$ then $|\lambda|<\left\lceil \frac{p}{2}\right\rceil-1$. i.e., $|\lambda|=\left\lceil \frac{p}{2}\right\rceil-2$.

Let $D=\{u_1,u_2,\ldots,u_\lambda,v_{11}\}$. Then $|N[D]|=\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D]|>0$, for $\forall\ v\in D$ $\Rightarrow D$ is a MI-set of G. Let $x=uv$ such that $u\in D$ and $v\in V-D$.
Then $|N[D]|<\left\lceil \frac{p}{2}\right\rceil$. So, choose $D_1=D\bigcup\{v_{12}\}$. Then $|N[D_1]|>\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D_1]|=|N[D_1]|-\left\lceil \frac{p}{2}\right\rceil$ for any $v\in D_1$. Now, choose $D_1=D-\{v_{11}\}\bigcup\{v_{12}\}$ such that $|N[D_1]|=\left\lceil \frac{p}{2}\right\rceil$ and $|pn[v,D_1]|>0$ for $\forall\ v\in D_1 \Rightarrow |D_1|=|D|$ is a maximal MI-

setof $(G-x)$. $\Rightarrow x$ is not a β_M -critical edge of G, which contradicts the condition (i).

Thus $q+s=\left\lceil \frac{p}{2} \right\rceil$ if p is odd.

Subcase (ii): When p is even. Suppose $(q+s)<\left(\frac{p}{2}\right)+1$. Then $p-(q+s)=\lambda$ and $|\lambda|\geq\frac{p}{2}$. ie., $q+s\leq\frac{p}{2}$. $\therefore$ G contains λK_1 isolates $\Rightarrow D=\left\{u_1,u_2,\ldots,u_{\frac{p}{2}}\right\}$ is a MI-set of G and $(G-x) \Rightarrow x$ is not a β_M -critical edge, $\Rightarrow\Leftarrow$ to condition (i).

$\therefore (q+s)=\frac{p}{2}+1$ if p is even.

$$\text{Thus}\quad \left.\begin{aligned} q+s&=\left\lceil \frac{p}{2} \right\rceil \quad \textit{if}\ \ p\ \ \textit{is}\ \ \textit{odd} \\ \textit{and}\ \ q+s&=\left\lceil \frac{p}{2} \right\rceil+1 \quad \textit{if}\ \ p\ \ \textit{is}\ \ \textit{even} \end{aligned}\right\} \qquad \text{(a)}$$

If the result (a) is true then by case (i) $\beta_M(G)=\left\lceil \frac{p}{2} \right\rceil-1$.

(c) $\Rightarrow$(a) Assume that $\beta_M(G)=\left\lceil \frac{p}{2} \right\rceil-1$. To prove : A disconnected graph G with isolates is β_M -critical. By assumption, $D=\{u_1,u_2,\ldots,u_{\lambda-1},v_{11}\}$ be a β_M -set of G and $|D|=\lambda=\left\lceil \frac{p}{2} \right\rceil-1$.

Case (i): Let $x\in E(G)$. ie., $x_1=(v_1v_{11})$ such that $v_1\in V-D$ and $v_{11}\in D$.

In $G-x$, $|N[D]|<\left\lceil \frac{p}{2} \right\rceil$. Choose $D_1=D\bigcup\{u_\lambda\}$. Then $|N[D_1]|=\left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,D_1]|=1>0$ for $\forall\, v\in D_1 \Rightarrow D_1$ is a maximal MI-set of $G-x$.

$\therefore \beta_M(G-x)\geq|D_1|=|D|+1=\beta_M(G)+1$. Since $\beta_M(G)\leq\beta_M(G-x)\leq\beta_M(G)+1$, $\beta_M(G-x)=\beta_M(G)+1 \Rightarrow x$ is a β_M -critical edge.

Case (ii): Let $x_2=(v_1v_{1j})$, $j=2,\ldots,r_1$ such that $v_1\in V-D$, $v_{1j}\in V-D$. Then there exists $(\lambda+1)=\left\lceil \frac{p}{2} \right\rceil$ isolates in $(G-x_2)$. Let $D_1=D-\{v_{11}\}\bigcup\{u_\lambda,v_{1j}\}$

ie., $D_1=\{u_1,u_2,\ldots,u_\lambda,v_{1j}\}$ such that $|N[D_1]|=\left\lceil \frac{p}{2} \right\rceil$ and $|pn[v,D_1]|=1>0$, for $\forall v \in D_1 \Rightarrow D_1$ is a maximal MI-set of $(G - x_2)$. $\Rightarrow \beta_M(G - x_2) \geq |D_1|$ $= |D| - 1 + 2 = |D| + 1 = \beta_M(G) + 1$. By the above argument, $\beta_M(G - x_2)$ $= \beta_M(G) + 1 \Rightarrow x_2$ is a β_M -critical edge of G. Hence, every edge $x = uv$ lies either both u and $v \in V - D$ or $u \in D$ and $v \in V - D$ but u and $v \notin D$. Then x is a β_M -critical edge, for $\forall x \in E(G)$. $\Rightarrow G$ is β_M -critical graph.

Theorem 3.14. Let G be a disconnected without isolates. Then the graph G is β_M -critical if and only if the β_M -set D of G satisfies the following condition hold.

i) $|N_G[D]| = \left\lceil \frac{p}{2} \right\rceil$, p is even and $|N_H[D]| < \left\lceil \frac{p}{2} \right\rceil$, $|N_G[D] - \{v\}| < \left\lceil \frac{p}{2} \right\rceil$ for $\forall\ v \in V - D$.

ii) $|N_H[D_1]| = \left\lceil \frac{p}{2} \right\rceil$, where $D_1 = D \cup \{u\}$ and $|pn[v, D_1]| > |N_H[D_1]| - \left\lceil \frac{p}{2} \right\rceil$ for $\forall\ v \in V - D$.

iii) $|N(v_1) \cap N(v_2)| = \phi$, for $v_1, v_2 \in D$.

iv) Let $e = uv$ such that either u and $v \in V - D$ or $u \in D$ and $v \in V - D$.

Proposition 3.15. Every tree T need not to have a β_M -redundant edge.

Example 3.16. Let $G = P_9$. This graph has no β_M -redundant edge but every edge of G is β_M -critical edge.

Proposition 3.17. For any graph G, there may or may not have a β_M -critical edge.

Example 3.18.

1. Let $G = K_{1,p-1}$, $p \geq 3$. Every edge of G is a β_M -redundant edge.
2. Let $G = C_6 o K_1$. Every pendant of G is β_M -critical edge and every edge $x \in E(C_6)$ is β_M -redundant.

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