

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

Tapan Senapati¹, Manoranjan Bhowmik², Madhumangal Pal³

^{1,3}Department of Applied Mathematics with Oceanology and Computer Programming, Vidyasagar University, Midnapore -721 102, India.
E-mail: math.tapan@gmail.com, mmpalvu@gmail.com

² Department of Mathematics, V. T. T. College, Midnapore- 721 101, India.
E-mail: mbvttc@gmail.com

Received 4 February 2013; accepted 19 February 2013

Abstract. In this paper, we apply the concept of interval-valued fuzzy set to ideals and closed ideals in B-algebras. The notion of an interval-valued fuzzy closed ideal of a B-algebra is introduced, and some related properties are investigated. Also, the product of interval-valued fuzzy B-algebra is investigated.

Keywords: B-algebras, interval-valued fuzzy sets, interval-valued fuzzy ideals, interval-valued fuzzy closed ideals, homomorphism, equivalence relation, level cut.

AMS Mathematics Subject Classification (2010): 06F35, 03G25, 08A72

1. Introduction

Fuzzy set theory, which was introduced by Zadeh [28], is the oldest and most widely reported component of present day soft computing, allowing the design of more flexible information processing systems, with applications in different areas, such as artificial intelligence, multiagent systems, machine learning, knowledge discovery, information processing, statistics and data analysis, system modeling, control system, decision sciences, economics, medicine and engineering, as shown in the recent literature collected by Dubois et al [5,6].

Interval-valued fuzzy sets were introduced independently by Zadeh [29] and other authors (e.g., [8, 20, 30]) in the 70's, in order to treat intuitively vagueness jointly with uncertainty, which can be provided by combining fuzzy set theory with interval mathematics.

BCK-algebras and BCI-algebras are two important classes of logical algebras introduced by Imai and Iseki [9,10]. It is known that the class of BCK-algebra is a proper subclass of the class of BCI-algebras. Negger and Kim [14, 15] introduced a

new notion, called a B-algebra which is related to several classes of algebras of interest such as BCI/BCK-algebras. Cho and Kim [4] discussed further relations between B-algebras and other topics especially quasigroups. Park and Kim [16] obtained that every quadratic B-algebra on a field X with $|X| \geq 3$ is a BCI-algebra. Jun et al. [11] fuzzyfied (normal) B-algebras and gave a characterization of a fuzzy B-algebras. Saeid [18, 19] introduced fuzzy topological B-algebras and interval-valued fuzzy B-algebras. Kim and Kim [12] introduced the notion of BG-algebras. They have shown that if X is a B-algebra, then X is a BG-algebra but the converse is not true. Ahn and Lee [2] fuzzified BG-algebras. Senapati et al. [21, 22, 27] introduced fuzzy closed ideals of B-algebras, fuzzy B-subalgebras with respect to t -norm and fuzzy subalgebras of B-algebras with interval-valued membership function. They also [23-26] presented the concept and basic properties of intuitionistic fuzzy subalgebras, intuitionistic fuzzy ideals, interval-valued intuitionistic fuzzy subalgebras, interval-valued intuitionistic fuzzy closed ideals in BG-algebras. For the general development of the B-algebras, the ideal theory plays an important role.

In this paper, interval-valued fuzzy ideal (IVF-ideal) of B-subalgebras is defined. A lot of properties are investigated. We introduce the notion of equivalence relations on the family of all interval-valued IVF-ideals of a B-algebra and investigated some related properties. The product of interval-valued fuzzy (IVF) of B-subalgebra has been introduced and some important properties are of it are also studied.

The rest of this paper is organized as follows. The following section briefly reviews some background on B-algebra, B-subalgebra, refinement of unit interval, interval-valued fuzzy (IVF) B-subalgebras. In Section 3, we propose the concepts and operations of interval-valued fuzzy ideal (IVF-ideal) and interval-valued fuzzy closed ideal (IVFC-ideal) and discuss their properties in detail. In section 4, we investigate properties of IVF-ideals under homomorphisms. In section 5, we introduce equivalence relations on IVF-ideals. In section 6, product of IVF B-subalgebra and some of its properties are studied. Finally, in Section 7, the conclusion is made and presented some topics for future research.

2. Preliminaries

In this section, some elementary aspects that are necessary for this paper are included.

Definition 2.1. [14] (B-algebra) A B-algebra is a non-empty set X with a consonant 0 and a binary operation “ $*$ ” satisfying the following axioms:

- B1. $x * x = 0$,
- B2. $x * 0 = x$,
- B3. $(x * y) * z = x * (z * (0 * y))$,

for all $x, y, z \in X$.

Example 2.2. [14] Let X be the set of all real numbers except for a negative integer $-n$. Define a binary operation $*$ on X by $x * y = \frac{x(x-y)}{n+y}$. Then $(X, *, 0)$ is a B-algebra.

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

Lemma 2.3. [4,14] If X is a B-algebra, then $0 * (0 * x) = x$ for all $x \in X$.

Lemma 2.4. [14] If X is a B-algebra, then $(x * y) * (0 * y) = x$ for all $x, y \in X$.

Now, we introduce the concept of B-subalgebra, ideal over a crisp set X and the binary operation $*$ in the following. The definition of B-subalgebra and ideal are given below.

Definition 2.5. [1] (B-subalgebra) A non-empty subset S of a B-algebra X is called a B-subalgebra of X if $x * y \in S$ for any $x, y \in S$.

Definition 2.6. (Ideal) A non-empty subset I of a B-algebra X is called an ideal of X if for any $x, y \in X$ (i) $0 \in I$ (ii) $x * y \in I$ and $y \in I$ implies $x \in I$. An ideal I of a B-algebra X is called closed if $0 * x \in I$ for all $x \in I$.

A mapping $f: X \rightarrow Y$ of B-algebras is called a homomorphism ([15]) if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that if $f: X \rightarrow Y$ is a B-homomorphism, then $f(0) = 0$. A partial ordering " \leq " on X can be defined by $x \leq y$ if and only if $x * y = 0$.

Definition 2.7. [28] (Fuzzy set) Let X be the collection of objects denoted generally by x then a fuzzy set A in X is defined as $A = \{ \langle x, \alpha_A(x) \rangle : x \in X \}$ where $\alpha_A(x)$ is called the membership value of x in A and $0 \leq \alpha_A(x) \leq 1$.

Ahn et al. and Senapati et al. [1,21] extends the concepts of B-subalgebra, ideal from crisp set to fuzzy set respectively.

Definition 2.8. [1,11] (Fuzzy B-subalgebra) A fuzzy set A in X is called a fuzzy B-subalgebra if it satisfies the inequality $\alpha_A(x * y) \geq \min \{ \alpha_A(x), \alpha_A(y) \}$ for all $x, y \in X$.

Definition 2.9. [21] (Fuzzy ideal) Let A be a fuzzy set in a B-algebra X . Then A is called a fuzzy ideal of X if for all $x, y \in X$ it satisfies:

- (FI1) $\alpha_A(0) \geq \alpha_A(x)$
- (FI2) $\alpha_A(x) \geq \min \{ \alpha_A(x * y), \alpha_A(y) \}$.

The notion of interval-valued fuzzy sets (IVFS) was first introduced by Zadeh [28] in 1975 as a generalization of traditional fuzzy sets. This idea gives the simplest method to capture the imprecision of the membership grades for a fuzzy set. The membership value of an element of this is not a single number, it is an interval and this interval is a subinterval of the interval $[0, 1]$. Let $D[0, 1]$ be the set of all subintervals of the interval $[0, 1]$.

Definition 2.10. [29] (IVFS) An IVFS A over X is an object having the form $A = \{ \langle x, M_A(x) \rangle : x \in X \}$, where $M_A : X \rightarrow D[0, 1]$, where $D[0, 1]$ is the set of all subintervals of $[0, 1]$. The intervals $M_A(x)$ denote the intervals of the degree of membership of the element x to the set A , where $M_A(x) = [M_{AL}(x), M_{AU}(x)]$ for all $x \in X$.

The determination of maximum and minimum between two real numbers is very simple but it is not simple for two intervals. Biswas [2] described a method to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.11. [3] (Refinement of intervals) Consider two elements $D_1, D_2 \in D[0, 1]$. If $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$, then $\text{rmax}(D_1, D_2) = [\max(a_1, a_2), \max(b_1, b_2)]$ which is denoted by $D_1 \vee D_2$. Thus, if $D_i = [a_i, b_i] \in D[0, 1]$ for $i=1, 2, 3, 4, \dots$, then we define $\text{rsup}_i(D_i) = [\text{sup}_i(a_i), \text{sup}_i(b_i)]$, i.e. $\vee_i D_i = [\vee_i a_i, \vee_i b_i]$. Now we call $D_1 \geq D_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$. Similarly, the relation $D_1 = D_2$ are defined.

The intersection of two IVFS of X is defined by Zadeh [29] as follows

Definition 2.12. [29] Let A and B be two IVFSs on X , where $A = \{ \langle x, [M_{AL}(x), M_{AU}(x)] \rangle : x \in X \}$ and $B = \{ \langle x, [M_{BL}(x), M_{BU}(x)] \rangle : x \in X \}$. Then the intersection of A and B is denoted by $A \cap B$ and is given by $A \cap B = \{ \langle x, M_{A \cap B}(x) \rangle : x \in X \} = \{ \langle x, [\min(M_{AL}(x), M_{BL}(x)), \min(M_{AU}(x), M_{BU}(x))] \rangle : x \in X \}$.

Definition 2.13. [29] Let f be a mapping from the set X into the set Y . Let B be an IVFS in Y . Then the inverse image of B , is defined as $f^{-1}(B) = \{ \langle x, f^{-1}(M_B)(x) \rangle : x \in X \}$ with the interval-valued membership is given by $f^{-1}(M_B)(x) = M_B(f(x))$. It can be shown that $f^{-1}(B)$ is an interval-valued fuzzy set.

Combined the definition of B -subalgebra over crisp set and the idea of fuzzy set Senapati et al. [27] defined fuzzy B -subalgebra, which is defined below.

Definition 2.14. [27] Let A be an IVFS in X , where X is a B -algebra, then the set A is an IVF B -subalgebra over the binary operator $*$ if it satisfies

$$(F13) M_A(x * y) \geq \text{rmin}\{M_A(x), M_A(y)\} \text{ for all } x, y \in X.$$

Definition 2.15. [27] Let A is an IVF B -subalgebra of X . For $[s_1, s_2] \in [0, 1]$, the set $U(M_A : [s_1, s_2]) = \{x \in X : M_A(x) \geq [s_1, s_2]\}$ is called upper s -level of A .

Definition 2.16. [27] Let $A = \{ \langle x, M_A(x) \rangle : x \in X \}$ and $B = \{ \langle x, M_B(x) \rangle : x \in X \}$ be two IVFSs on X . The Cartesian product $A \times B = \{ \langle (x, y), M_A \times M_B(x, y) \rangle : x, y \in X \}$ is defined by $(M_A \times M_B)(x, y) = \text{rmin}\{M_A(x), M_B(y)\}$ where $M_A \times M_B : X \times X \rightarrow D[0, 1]$ for all $x, y \in X$.

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

In what follows, let X denote a B-algebra unless otherwise specified.

3. Interval-valued fuzzy closed ideals of B-algebras

In this section, IVF-ideals and IVFC-ideals of B-algebras are defined and some propositions and theorems are presented.

Definition 3.1. Let A be an IVFS in X . Then A is called an IVF-ideal of X if for all $x, y \in X$ it satisfies:

$$(FI4) \quad M_A(0) \geq M_A(x)$$

$$(FI5) \quad M_A(x) \geq \min \{M_A(x * y), M_A(y)\}.$$

Example 3.2. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table:

Table I:

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then $(X, *, 0)$ is a B-algebra. Define an IVFS A in X by $M_A(0) = M_A(2) = [1, 1]$ and $M_A(1) = M_A(3) = [m_1, m_2]$, where $[m_1, m_2] \in D[0, 1]$. Then A is an IVF-ideal of X .

Definition 3.3. An IVFS A in X is called an IVFC-ideal of X if it satisfies (FI4), (FI5) along with (FI6) $M_A(0 * x) \geq M_A(x)$ for all $x \in X$.

Example 3.4. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

Table II

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	1	2
4	4	5	3	2	0	1
5	5	3	4	1	2	0

Then $(X, *, 0)$ is a B-algebra (see [11], Example 3.5). We define A in X by $M_A(0) = [0.5, 0.7]$, $M_A(1) = M_A(2) = [0.4, 0.6]$ and $M_A(3) = M_A(4) = M_A(5) = [0.3, 0.4]$. By routine calculations, one can verify that A is IVFC-ideal of X .

Proposition 3.5. Every IVFC-ideal is an IVF-ideal.

The converse of above Proposition is not true in general as seen in the following example.

Example 3.6. Consider a B-algebra $X = \{0, 1, 2, 3\}$ with the table

Table III

*	0	1	2	3
0	0	2	1	3
1	1	0	1	1
2	2	2	0	2
3	3	3	3	0

Let us define an IVFS A in X by $M_A(0) = M_A(1) = [0.5, 0.7]$ and $M_A(2) = M_A(3) = M_A(5) = [0.3, 0.4]$. We know that A is an IVF-ideal of X . But it is not an IVFC-ideal of X since $M_A(0 * 1) < M_A(1)$ for some $x \in X$.

Corollary 3.7. Every IVF B-subalgebra satisfying (FI5) is an IVFC-ideal.

Theorem 3.8. Every IVFC-ideal of X is an IVF B-subalgebra of X .

Proof. If A is a IVFC-ideal of X , then for any $x \in X$ we have $M_A(0 * x) \geq M_A(x)$.

Now, $M_A(x * y) \geq \text{rmin}\{M_A((x * y) * (0 * y)), M_A(0 * y)\}$, by (FI5)
 $= \text{rmin}\{M_A(x), M_A(0 * y)\}$
 $\geq \text{rmin}\{M_A(x), M_A(y)\}$, by (FI6)

Hence the theorem.

Proposition 3.9. If an IVFS A in X is an IVFC-ideal, then $M_A(0) \geq M_A(x)$ for all $x \in X$.

Theorem 3.10. An IVFS $A = \{\langle x, M_A(x) \rangle : x \in X\} = \{\langle x, [M_{AL}, M_{AU}] \rangle : x \in X\}$ in X is an IVF-ideal of X if and only if M_{AL} and M_{AU} are fuzzy ideals of X .

Proof. Since $M_{AL}(0) \geq M_{AL}(x)$, $M_{AU}(0) \geq M_{AU}(x)$ therefore $M_A(0) \geq M_A(x)$. Let M_{AL} and M_{AU} are IVF-ideals of X . Let $x, y \in X$. Then $M_A(x) = [M_{AL}(x), M_{AU}(x)] \geq [\min\{M_{AL}(x * y), M_{AL}(y)\}, \min\{M_{AU}(x * y), M_{AU}(y)\}] = \text{rmin}\{[M_{AL}(x * y), M_{AU}(x * y)], [M_{AL}(y), M_{AU}(y)]\} = \text{rmin}\{M_A(x * y), M_A(y)\}$. Hence, $A = \{\langle x, M_A(x) \rangle : x \in X\}$ is an IVF-ideal of X .

Conversely, assume that, A is an IVIF-ideal of X . For any $x, y \in X$, we have

$$\begin{aligned} [M_{AL}(x), M_{AU}(x)] &= M_A(x) \\ &\geq \text{rmin}\{M_A(x * y), M_A(y)\} \\ &= \text{rmin}\{[M_{AL}(x * y), M_{AU}(x * y)], [M_{AL}(y), M_{AU}(y)]\} \\ &= [\min\{M_{AL}(x * y), M_{AL}(y)\}, \min\{M_{AU}(x * y), M_{AU}(y)\}]. \end{aligned}$$

Thus, $M_{AL}(x) \geq \min\{M_{AL}(x * y), M_{AL}(y)\}$ and $M_{AU}(x) \geq \min\{M_{AU}(x * y), M_{AU}(y)\}$. Hence, M_{AL} and M_{AU} are fuzzy ideals of X .

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

The definition of intersections holds good for IVIF B-subalgebras.

Theorem 3.11. Let A_1 and A_2 be two IVF-ideals of X . Then $A_1 \cap A_2$ is also an IVF-ideal of X .

Proof. Let $x, y \in A_1 \cap A_2$. Then $x, y \in A_1$ and A_2 . Now, $M_{A_1 \cap A_2}(0) = M_{A_1 \cap A_2}(x * x) \geq \text{rmin}\{M_{A_1 \cap A_2}(x), M_{A_1 \cap A_2}(x)\} = M_{A_1 \cap A_2}(x)$. Also, $M_{A_1 \cap A_2}(x) = [M_{(A_1 \cap A_2)L}(x), M_{(A_1 \cap A_2)U}(x)] = [\min\{M_{A_1L}(x), M_{A_2L}(x)\}, \min\{M_{A_1U}(x), M_{A_2U}(x)\}] \geq [\min\{M_{(A_1 \cap A_2)L}(x * y), M_{(A_1 \cap A_2)L}(y)\}, \min\{M_{(A_1 \cap A_2)U}(x * y), M_{(A_1 \cap A_2)U}(y)\}] = \text{rmin}\{M_{A_1 \cap A_2}(x * y), M_{A_1 \cap A_2}(y)\}$. Hence, $A_1 \cap A_2$ is also an IVF-ideal of X .

The above theorem can be generalized as follows.

Theorem 3.12. Let $\{A_i \mid i = 1, 2, 3, 4 \dots\}$ be a family of IVF-ideals of X . Then $\cap A_i$ is also an IVF-ideal of X where, $\cap A_i = \{ \langle x, \text{rmin} M_{A_i}(x) \rangle : x \in X \}$.

Theorem 3.13. Let A be an IVF-ideals of X . If $x * y \leq z$ then $M_A(x) \geq \text{rmin}\{M_A(y), M_A(z)\}$.

Proof. Let $x, y, z \in X$ such that $x * y \leq z$. Then $(x * y) * z = 0$ and thus

$$\begin{aligned} M_A(x) &\geq \text{rmin}\{M_A(x * y), M_A(y)\} \\ &\geq \text{rmin}\{\text{rmin}\{M_A((x * y) * z), M_A(z)\}, M_A(y)\} \\ &= \text{rmin}\{\text{rmin}\{M_A(0), M_A(z)\}, M_A(y)\} \\ &= \text{rmin}\{M_A(y), M_A(z)\}. \end{aligned}$$

Theorem 3.14. Let A be an IVF-ideals of X . If $x \leq y$ then $M_A(x) \geq M_A(y)$ i.e, order reversing.

Proof. Let $x, y \in X$ such that $x \leq y$. Then $x * y = 0$ and thus $M_A(x) \geq \text{rmin}\{M_A(x * y), M_A(y)\} = \text{rmin}\{M_A(0), M_A(y)\} = M_A(y)$.

The above lemma can be generalized as

Lemma 3.15. Let A be an IVF-ideal of X , then $((\dots((x * a_1) * a_2) * \dots) * a_n) = 0$ for any $x, a_1, a_2, \dots, a_n \in X$, implies $M_A(x) \geq \text{rmin}\{M_A(a_1), M_A(a_2), \dots, M_A(a_n)\}$.

Proof. Using induction on n and by Theorem 3.14 and Theorem 3.15 we can easily prove the theorem.

Theorem 3.16. Let B be a crisp subset of X . Suppose that $A = \{ \langle x, M_A(x) \rangle : x \in X \}$ is an IVFS in X defined by $M_A(x) = [\lambda_1, \lambda_2]$ if $x \in B$ and $M_A(x) = [\tau_1, \tau_2]$ if $x \notin B$ for all $[\lambda_1, \lambda_2]$ and $[\tau_1, \tau_2] \in D[0, 1]$ with $\lambda_2 \geq \tau_2$. Then A is an IVFC-ideal of X if and only if B is an IVFC-ideal of X .

Proof. Assume that A is a IVFC-ideal of X . Let $x \in B$. Then, by (FI6), we have $M_A(0*x) \geq M_A(x) = [\lambda_1, \lambda_2]$ and so $M_A(0*x) = [\lambda_1, \lambda_2]$. It follows that $0*x \in B$. Let $x, y \in X$ be such that $x*y \in B$ and $y \in B$. Then $M_A(x*y) = [\lambda_1, \lambda_2] = M_A(y)$, and hence $M_A(x) \geq \text{rmin}\{M_A(x*y), M_A(y)\} = [\lambda_1, \lambda_2]$. Thus $M_A(x) = [\lambda_1, \lambda_2]$, that is, $x \in B$. Therefore B is an IVFC-ideal of X .

Conversely, suppose that B is a closed ideal of X . Let $x \in X$. If $x \in B$, then $0*x \in B$ and thus $M_A(0*x) = [\lambda_1, \lambda_2] = M_A(x)$. If $x \notin B$, then $M_A(x) = [\tau_1, \tau_2] \leq M_A(0*x)$. Let $x, y \in X$. If $x*y \in B$ and $y \in B$, then $x \in B$. Hence, $M_A(x) = [\lambda_1, \lambda_2] = \text{rmin}\{M_A(x*y), M_A(y)\}$. If $x*y \notin B$ and $y \notin B$, then clearly $M_A(x) \geq \text{rmin}\{M_A(x*y), M_A(y)\}$. If exactly one of $x*y$ and y belong to B , then exactly one of $M_A(x*y)$ and $M_A(y)$ is equal to $[\tau_1, \tau_2]$. Therefore, $M_A(x) \geq [\tau_1, \tau_2] = \text{rmin}\{M_A(x*y), M_A(y)\}$. Consequently, A is an IVFC-ideal of X .

Using the notion of level sets, we give a characterization of an IVFC-ideal.

Theorem 3.17. A fuzzy set A is an IVFC-ideal of X if and only if the set $U(M_A : [s_1, s_2])$ is IVFC-ideal of X for every $[s_1, s_2] \in D[0, 1]$.

Proof. Suppose that A is an IVFC-ideal of X . For $[s_1, s_2] \in [0, 1]$, obviously, $0 * x \in U(M_A : [s_1, s_2])$, where $x \in X$. Let $x, y \in X$ be such that $x * y \in U(M_A : [s_1, s_2])$ and $y \in U(M_A : [s_1, s_2])$. Then $M_A(x) \geq \text{rmin}\{M_A(x * y), M_A(y)\} \geq [s_1, s_2]$. Then $x \in U(M_A : [s_1, s_2])$. Hence, $U(M_A : [s_1, s_2])$ is closed ideal of X .

Conversely, assume that each non-empty level subset $U(M_A : [s_1, s_2])$ is closed ideals of X . For any $x \in X$, let $M_A(x) = [s_1, s_2]$. Then $x \in U(M_A : [s_1, s_2])$. Since $0 * x \in U(M_A : [s_1, s_2])$, it follows that $M_A(0*x) \geq [s_1, s_2] = M_A(x)$, for all $x \in X$.

If there exist $\lambda, \kappa \in X$ such that $M(\lambda) < \text{rmin}\{M_A(\lambda * \kappa), M_A(\kappa)\}$, then by taking $[s_1', s_2'] = \frac{1}{2}[M_A(\lambda * \kappa) + \text{rmin}\{M_A(\lambda), M_A(\kappa)\}]$, it follows that $\lambda * \kappa \in U(M_A : [s_1', s_2'])$ and $\kappa \in U(M_A : [s_1', s_2'])$, but $\lambda \notin U(M_A : [s_1', s_2'])$, which is a contradiction. Hence, $U(M_A : [s_1', s_2'])$ is not IVFC-ideal of X . Hence, A is an IVFC-ideal of X .

Theorem 3.18. Let A be an IVFC-ideal of X with the finite image. Then every descending chain of IVFC-ideals of X terminates at finite step.

Proof. Suppose that there exists a strictly descending chain $D_0 \supset D_1 \supset D_2 \supset \dots$ of IVFC-ideals of X which does not terminate at finite step. Define a fuzzy set A in X by $M_A(x) = [\frac{1}{n+1}, \frac{n}{n+1}]$ if $x \in D_n \setminus D_{n+1}$, $n = 0, 1, 2, \dots$ and $M_A(x) = [1, 1]$ if $x \in \bigcap_{n=0}^{\infty} D_n$, where $D_0 = X$. We prove that A is a IVFC-ideal of X . It is easy to show that A

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

satisfies (FI6). Let $x, y \in X$. Assume that $x * y \in D_n \setminus D_{n+1}$ and $y \in D_k \setminus D_{k+1}$ for $n = 0, 1, 2, \dots, k = 0, 1, 2, \dots$. Without loss of generality, we may assume that $n \leq k$. Then obviously $x * y$ and $y \in D_n$, so $x \in D_n$ because D_n is a IVFC-ideal of X . Hence

$$M_A(x) \geq [\frac{1}{n+1}, \frac{n}{n+1}] = \text{rmin}\{M_A(x * y), M_A(y)\}.$$

If $x * y, y \in \bigcap_{n=0}^{\infty} D_n$, then $x \in \bigcap_{n=0}^{\infty} D_n$. Thus

$$M_A(x) \geq [1, 1] = \text{rmin}\{M_A(x * y), M_A(y)\}.$$

If $x * y \notin \bigcap_{n=0}^{\infty} D_n$ and $y \in \bigcap_{n=0}^{\infty} D_n$, then there exists $k \in \mathbb{N}$ (the set of natural numbers) such that $x * y \in D_k \setminus D_{k+1}$. It follows that $x \in D_k$ so that

$$M_A(x) \geq [\frac{1}{k+1}, \frac{k}{k+1}] = \text{rmin}\{M_A(x * y), M_A(y)\}.$$

Finally suppose that $x * y \in \bigcap_{n=0}^{\infty} D_n$ and $y \notin \bigcap_{n=0}^{\infty} D_n$. Then $y \in D_r \setminus D_{r+1}$ for some $r \in \mathbb{N}$. Hence $x \in D_r$ and so

$$M_A(x) \geq [\frac{1}{r+1}, \frac{r}{r+1}] = \text{rmin}\{M_A(x * y), M_A(y)\}.$$

Consequently, we conclude that A is an IVFC-ideal of X and A has infinite number of different values, which is a contradiction. This completes the proof.

4. Investigation of interval-valued fuzzy ideals under homomorphisms

In this section, homomorphism of interval-valued fuzzy B-algebra is defined and some results are studied.

Theorem 4.1. Let $f: X \rightarrow Y$ be a homomorphism of B-algebras. If B is an IVF-ideal of Y , then the pre-image $f^{-1}(B)$ of B under f in X is an IVF-ideal of X .

Proof. For all $x \in X$, $f^{-1}(M_B)(x) = M_B(f(x)) \leq M_B(0) = M_B(f(0)) = f^{-1}(M_B)(0)$. Let $x, y \in X$. Then $f^{-1}(M_B)(x) = M_B(f(x)) \geq \text{rmin}\{M_B(f(x) * f(y)), M_B(f(y))\} \geq \text{rmin}\{M_B(f(x * y)), M_B(f(y))\} = \text{rmin}\{f^{-1}(M_B)(x * y), f^{-1}(M_B)(y)\}$. Hence, $f^{-1}(B) = \{x \in X, f^{-1}(M_B)(x) \geq f^{-1}(M_B)(0)\}$ is an IVF-ideal of X .

Theorem 4.2. Let $f: X \rightarrow Y$ be a homomorphism of B-algebras. Then B is an IVF-ideal of Y , if $f^{-1}(B)$ of B under f in X is an IVF-ideal of X .

Proof. For any $x \in Y$, there exist $a \in X$ such that $f(a) = x$. Then $M_B(x) = M_B(f(a)) = f^{-1}(M_B)(a) \leq f^{-1}(M_B)(0) = M_B(f(0)) = M_B(0)$. Let $x, y \in Y$. Then $f(a) = x$ and $f(b) = y$ for some $a, b \in X$. Thus $M_B(x) = M_B(f(a)) = f^{-1}(M_B)(a) \geq \text{rmin}\{f^{-1}(M_B)(a * b), f^{-1}(M_B)(b)\} = \text{rmin}\{M_B(f(a * b)), M_B(f(b))\} = \text{rmin}\{M_B(f(a) * f(b)), M_B(f(b))\} = \text{rmin}\{M_B(x * y), M_B(y)\}$. Then B is an IVF-ideal of Y .

5. Equivalence relations on interval-valued fuzzy ideals

Let $\text{IFI}(X)$ denote the family of all IVF-ideals of X and let $\rho = [\rho_1, \rho_2] \in D[0, 1]$. Define binary relation U^ρ on $\text{IFI}(X)$ as follows: $(A, B) \in U^\rho \Leftrightarrow U(M_A : \rho) = U(M_B : \rho)$ for A in $\text{IFI}(X)$. Then clearly U^ρ is equivalence relations on $\text{IFI}(X)$. For any $A \in \text{IFI}(X)$, let $[A]_{U^\rho}$ denote the equivalence class of A modulo U^ρ , and denote by $\text{IFI}(X)/U^\rho$, the collection of all equivalence classes modulo U^ρ , i.e., $\text{IFI}(X)/U^\rho := \{[A]_{U^\rho} \mid A \in \text{IFI}(X)\}$. These set is also called the quotient set.

Now let $T(X)$ denote the family of all ideals of X and let $\rho = [\rho_1, \rho_2] \in [0, 1]$. Define mappings f_ρ from $\text{IFI}(X)$ to $T(X) \cup \{\phi\}$ by $f_\rho(A) = U(M_A : \rho)$ for all $A \in \text{IFI}(X)$. Then f_ρ is clearly well-defined.

Theorem 5.1. For any $\rho = [\rho_1, \rho_2] \in D[0, 1]$, the map f_ρ is surjective from $\text{IFI}(X)$ to $T(X) \cup \{\phi\}$.

Proof. Let $\rho \in D[0, 1]$. Obviously $f_\rho(0) = U(0 : \rho) = U([0, 0] : [\rho_1, \rho_2]) = \phi$. Let $P (\neq \phi) \in \text{IFI}(X)$. For $P_- = \{< x, \chi_P(x) > : x \in X\} \in \text{IFI}(X)$, we have $f_\rho(P_-) = U(\chi_P : \rho) = P$. Hence f_ρ is surjective.

Theorem 5.2. The quotient set $\text{IFI}(X)/U^\rho$ is equipotent to $T(X) \cup \{\phi\}$ for every $\rho \in D[0, 1]$.

Proof. For $\rho \in D[0, 1]$, let f_ρ^* be a map from $\text{IFI}(X)/U^\rho$ to $T(X) \cup \{\phi\}$ defined by $f_\rho^*([A]_{U^\rho}) = f_\rho(A)$ for all $A \in \text{IFI}(X)$. If $U(M_A : \rho) = U(M_B : \rho)$ for A and B in $\text{IFI}(X)$, then $(A, B) \in U^\rho$; hence $[A]_{U^\rho} = [B]_{U^\rho}$. Therefore the maps f_ρ^* is injective. Now let $P (\neq \phi) \in \text{IFI}(X)$. For $P_- \in \text{IFI}(X)$, we have $f_\rho^*([P_-]_{U^\rho}) = f_\rho(P_-) = U(\chi_P : \rho) = P$. Finally, for $0 \in \text{IFI}(X)$ we get $f_\rho^*([0]_{U^\rho}) = f_\rho(0) = U(0 : \rho) = \phi$. This shows that f_ρ^* is surjective. This completes the proof.

6. Product of interval-valued fuzzy B-algebra

In this section, product of IVF-ideals in B-algebras are defined and some results are studied.

Theorem 6.1. Let A and B be IVF-ideals of X , then $A \times B$ is an IVF-ideal of $X \times X$.

Proof. For any $(x, y) \in X \times X$, we have $(M_A \times M_B)(0, 0) = \text{rmin} \{M_A(0), M_B(0)\} \geq \text{rmin} \{M_A(x), M_B(y)\} = (M_A \times M_B)(x, y)$. Let (x_1, y_1) and $(x_2, y_2) \in X \times X$. Then $(M_A \times M_B)(x_1, y_2) = \text{rmin} \{M_A(x_1), M_B(y_2)\} \geq \text{rmin} \{\text{rmin} \{M_A(x_1 * x_2), M_A(x_2)\}, \text{rmin} \{M_B(y_1 * y_2), M_B(y_2)\}\} = \text{rmin} \{\text{rmin} \{M_A(x_1 * x_2), M_B(y_1 * y_2)\}, \text{rmin} \{M_A(x_2), M_B(y_2)\}\} = \text{rmin} \{(M_A \times M_B)(x_1 * x_2, y_1 * y_2), (M_A \times M_B)(x_2, y_2)\} = \text{rmin} \{(M_A \times M_B)((x_1, y_1) * (x_2, y_2)), (M_A \times M_B)(x_2, y_2)\}.$

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

Hence, $A \times B$ is an IVF-ideal of $X \times X$.

The converse of Theorem 6.1 may not be true as seen in the following example.

Example 6.2. Let $[s_1, s_2], [t_1, t_2] \in D[0, 1]$ such that $[s_1, s_2] \leq [t_1, t_2]$. Define IVFS A and B in X by $M_A(x) = [s_1, s_2]$, $M_B(x) = [t_1, t_2]$ if $x = 0$ and $M_B(x) = [1, 1]$ otherwise, for all $x \in X$, respectively.

If $x \neq 0$, then $M_B(x) = [1, 1]$, and thus

$$(M_A \times M_B)(x, x) = \text{rmin} \{M_A(x), M_B(x)\} = \text{rmin} \{[s_1, s_2], [1, 1]\} = [s_1, s_2].$$

If $x = 0$, then $M_B(x) = [t_1, t_2] < [1, 1]$, and thus

$$(M_A \times M_B)(x, x) = \text{rmin} \{M_A(x), M_B(x)\} = \text{rmin} \{[s_1, s_2], [t_1, t_2]\} = [s_1, s_2].$$

That is, $A \times B$ is a constant function and so $A \times B$ is an IVF-ideal of $X \times X$. Now A is a IVF-ideal of X , but B is not an IVF-ideal of X since for $x \neq 0$, we have $M_B(0) = [t_1, t_2] < [1, 1] = M_B(x)$.

Proposition 6.3. Let A and B are IVFC-ideals of X , then $A \times B$ is an IVFC-ideal of $X \times X$.

Proof. Now, $(M_A \times M_B)((0, 0) * (x, y)) = (M_A \times M_B)(0 * x, 0 * y) = \text{rmin} \{M_A(0 * x), M_B(0 * y)\} \geq \text{rmin} \{M_A(x), M_B(y)\} = (M_A \times M_B)(x, y)$. Hence, $A \times B$ is an IVFC-ideal of $X \times X$.

Definition 6.4. Let A and B is IVF-ideals of X . For $[s_1, s_2] \in D[0, 1]$, the set $U(M_A \times M_B : [s_1, s_2]) = \{(x, y) \in X \times X \mid (M_A \times M_B)(x, y) \geq [s_1, s_2]\}$ is called upper $[s_1, s_2]$ -level of $A \times B$.

Lemma 6.5. Let A and B be fuzzy sets in X such that $A \times B$ is an IVF-ideal of $X \times X$, then

(i) Either $M_A(0) \geq M_A(x)$ or $M_B(0) \geq M_B(x)$ for all $x \in X$.

(ii) If $M_A(0) \geq M_A(x)$ for all $x \in X$, then either $M_B(0) \geq M_A(x)$ or $M_B(0) \geq M_B(x)$.

(iii)) If $M_B(0) \geq M_B(x)$ for all $x \in X$, then either $M_A(0) \geq M_A(x)$ or $M_A(0) \geq M_B(x)$.

Proof. (i) Assume that $M_A(x) > M_A(0)$ and $M_B(y) > M_B(0)$ for some $x, y \in X$. Then $(M_A \times M_B)(x, y) = \text{rmin} \{M_A(x), M_B(y)\} > \text{rmin} \{M_A(0), M_B(0)\} = (M_A \times M_B)(0, 0)$ which implies $(M_A \times M_B)(x, y) > (M_A \times M_B)(0, 0)$ for all $x, y \in X$, which is a contradiction. Hence (i) is proved.

(ii) Again assume that $M_B(0) < M_A(x)$ and $M_B(0) < M_B(y)$ for all $x, y \in X$. Then $(M_A \times M_B)(0, 0) = \text{rmin} \{M_A(0), M_B(0)\} = M_B(0)$. Now, $(M_A \times M_B)(x, y) = \text{rmin} \{M_A(x), M_B(y)\} > M_B(0) = (M_A \times M_B)(0, 0)$, which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

Theorem 6.6. For any fuzzy set A and B , $A \times B$ is a IVF-ideal of $X \times X$ if and only if the non-empty upper $[s_1, s_2]$ -level cut $U(M_A \times M_B : [s_1, s_2])$ is IVFC-ideal of $X \times X$ for any $[s_1, s_2] \in D[0, 1]$.

7. Conclusions

In the present paper, we have presented some extended results of IVF-ideal called IVFC-ideals of B-algebras and investigated some of their useful properties. The product of B-subalgebra has been introduced and some important properties are of it are also studied. In our opinion, these definitions and main results can be similarly extended to some other algebraic systems such as BF-algebras, lattices and Lie algebras. It is our hope that this work would other foundations for further study of the theory of B-algebras. In our future study of fuzzy structure of B-algebra, may be the following topics should be considered:

- To find T-IVFC-ideals of B-algebras, where T are given imaginable triangular norm,
- To get more results in IVFC-ideals of B-algebras and application,
- To find $(\square, \square \vee q)$ -IVF-ideals of B-algebras.

Acknowledgements. The authors would like to express sincere appreciation to the referees for their valuable suggestions and comments helpful in improving this paper.

REFERENCES

1. S.S. Ahn and K. Bang, On fuzzy subalgebras in B-algebras, *Commun. Korean Math. Soc.*, 10 (3) (2003), 429-437.
2. S.S. Ahn and H.D. Lee, Fuzzy subalgebras of BG-algebras, *Commun. Korean Math. Soc.*, 19(2) (2004), 243-251.
3. R. Biswas, Rosenfeld's fuzzy subgroups with interval valued membership function, *Fuzzy Sets and Systems*, 63 (1994), 87-90.
4. J.R. Cho and H.S. Kim, On B-algebras and quasigroups, *Quasigroups and Related Systems*, 7 (2001), 1-6.
5. D. Dubois, H. Prade and S. Sessa, Recent literature, *Fuzzy Sets and Systems*, 160(20) (2009), 3017-3026.
6. D. Dubois, H. Prade and S. Sessa, Recent literature, *Fuzzy Sets and Systems*, 161(7) (2010), 1039-1046.
7. D.H. Foster, Fuzzy topological groups, *J. Math. Anal. Appl.*, 67(2) (1979), 549-564.
8. I. Grattan-Guinness, Fuzzy membership mapped onto interval and manyvalued quantities, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 22(1) (1976), 149-160.
9. Y. Imai and K. Iseki, On axiom system of propositional calculi, *Proc. Japan Academy*, 42 (1966), 19-22.
10. K. Iseki, An algebra related with a propositional calculus, *Proc. Japan Academy*, 42 (1966), 26-29.

Fuzzy Closed Ideals of B-algebras with Interval-Valued Membership Function

11. Y.B. Jun, E.H. Roh and H.S. Kim, On fuzzy B-algebras, *Czech. Math. J.*, 52(2002), 375-384.
12. C.B. Kim and H.S. Kim, On BG-algebras, *Demonstratio Mathematica*, 41 (2008), 497-505.
13. W.J. Liu, Fuzzy invariant subgroups and IVF-ideals, *Fuzzy Sets and Systems*, 8 (1982), 132-139.
14. J. Neggers and H.S. Kim, On B-algebras, *Math. Vensik*, 54 (2002), 21-29.
15. J. Neggers and H.S. Kim, A fundamental theorem of B-homomorphism for B-algebras, *Int. Math. J.*, 2 (2002), 215-219.
16. H.K. Park and H.S. Kim, On quadratic B-algebras, *Quasigroups and Related Systems*, 7 (2002), 67-72.
17. A. Rosenfeld, Fuzzy Groups, *J. Math. Anal. Appl.*, 35 (1971), 512-517.
18. A.B. Saeid, Fuzzy topological B-algebras, *International Journal of Fuzzy Systems*, 8(3) (2006), 160-164.
19. A.B. Saeid, Interval-valued fuzzy B-algebras, *Iranian Journal of Fuzzy Systems*, 3(2)(2006), 63-73.
20. R. Sambuc, Fonctions \square -floues. Application l'aide au diagnostic en pathologiethyroïdienne, Ph.D. Thesis, Université Marseille, Marseille, 1975.
21. T. Senapati, M. Bhowmik and M. Pal, Fuzzy closed ideals of B-algebras, *International Journal of Computer Science, Engineering and Technology*, 1(10) (2011), 669-673.
22. T. Senapati, M. Bhowmik and M. Pal, Fuzzy B-subalgebras of B-algebra with respect to t-norm, *Journal of Fuzzy Set Valued Analysis*, 2012 (2012) 11 Pages.
23. T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy BG-subalgebras, *The Journal of Fuzzy Mathematics*, 20(3) (2012), 707-720.
24. T. Senapati, M. Bhowmik and M. Pal, Interval-valued intuitionistic fuzzy closed ideals of BG-algebra and their products, *International Journal of Fuzzy Logic Systems*, 2(2) (2012), 27-44.
25. T. Senapati, M. Bhowmik and M. Pal, Intuitionistic fuzzifications of ideals in BG-algebras, *Mathematica Aeterna*, 2(9) (2012), 761-778.
26. T. Senapati, M. Bhowmik and M. Pal, Intuitionistic L-fuzzy ideals in BG-algebras, *Afrika Matematika*, (Accepted).
27. T. Senapati, M. Bhowmik and M. Pal, Fuzzy subalgebras of B-algebras with interval-valued membership function, Communicated.
28. L.A. Zadeh, Fuzzy sets, *Inform. and Control*, 8 (1965), 338-353.
29. L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning. I, *Information Sciences*, 8 (1975), 199-249.
30. K.U. Jahn, Intervall-wertigemengen, *Mathematische Nachrichten*, 68(1) (1975), 115-132.