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Fuzzy Homomorphism and Ideal Flags on Incline

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Abstract. In this paper we prove that if a mapping μ from incline R into [0, 1] is a fuzzy homomorphism then the incline order relation is preserved and the crisp sets that is, complement of strong α-cuts are ideals of R. We provide sufficient conditions for μ to be a fuzzy homomorphism. Further we modify the concepts of chain, keychain, flag, pinned flag in terms of ideals of an incline R.

Keywords: incline, subincline, homomorphism, ideals, pinned ideal flags.

Mathematics Subject Classification (2010): 16D25, 16Y60

1. Introduction

Inclines are additively idempotent semirings in which products are less than (or) equal to either factor. The concept of incline was introduced by Cao and later it was developed by Cao et.al. in [2]. Recently a survey on incline was made by Kim and Roush in [4]. Incline algebra is a generalization of both Boolean and fuzzy algebras and it is a special type of semiring. It has both Semiring structure and Poset structure.

In [3], Jun, Ahn, and Kim have considered the fuzzification of subinclines (ideals) in incline and investigated some of the related properties. In [7], Murali and Makamba have used the keychains, index of keychains, flags and pinned flags to study the equivalence relation on fuzzy subsets of a finite set, by which the nature and number of fuzzy subsets are determined.

In this paper, we deal with fuzzy homomorphism and provide sufficient conditions for fuzzy subset of an incline R to be a fuzzy homomorphism. In section 2, we present the basic definitions, notations and required results of an incline R. In section 3, we prove that if a mapping μ from incline R into [0, 1] is a fuzzy homomorphism then the incline order relation is preserved and the crisp sets that is, complement of strong α-cuts are ideals of R. We provide sufficient conditions for fuzzy subset, μ to be a fuzzy homomorphism. Further we modify the concepts of chain, keychain, flag, pinned flag in terms of ideals of an incline R.

2. Preliminaries

In this section, we present some definitions, notations and required results on incline.

Definition 2.1. An *incline* is a non-empty set R with binary operations addition and multiplication denoted as +, $\cdot$ defined on R x R $\rightarrow$ R such that for all x, y, z $\in$ R,

$$x+y = y+x, \qquad x+(y+z) = (x+y)+z,$$
$$x(y+z) = xy+xz, \qquad (y+z)x = yx+zx,$$
$$x(yz) = (xy)z, \qquad x+xy = x,$$
$$x+x = x, \qquad y+xy = y.$$

An incline R is said to be *commutative* if xy = yx, for all x, y $\in$R.

Definition 2.2. An element 0_R in an incline R is the *zero element* of R if

$x + 0_R = 0_R + x = x$ and $\quad x \,.\, 0_R = 0_R \,.\, x = 0_R, \quad$ for all x $\in$ R.

Definition 2.3. (R, $\leq$) is an incline with order relation '$\leq$' defined as, x $\leq$ y if and only if x+y = y, for x, y $\in$ R. If x $\leq$ y then y is said to *dominate* x.

Property 2.4. For x, y in an incline R, x+y $\geq$ x and x+y $\geq$ y.

Property 2.5. For x, y in an incline R, xy $\leq$ x and xy $\leq$ y.

Definition 2.6. A *subincline* of an incline R is a non–empty subset I of R which is closed under the incline operations addition and multiplication.

Definition 2.7. A subincline I is said to be an *ideal* of an incline R if x $\in$ I and y $\leq$ x then y $\in$ I.

Definition 2.8. If R and R′ are inclines then an *incline homomorphism* is a mapping Φ : R $\rightarrow$ R′ such that

(i) $\Phi(x + y) = \Phi(x) + \Phi(y)$ for all x, y in R
(ii) $\Phi(xy) = \Phi(x)\,\Phi(y)$ for all x, y in R.

Definition 2.9. Let R be an incline. A mapping μ: R $\rightarrow$ [0, 1] is called a *fuzzy subset* of R.

Definition 2.10. For an incline R, a fuzzy subset μ: R $\rightarrow$ [0, 1] is called *fuzzy homomorphism* if it satisfies $\mu(x + y) = \mu(x) \vee \mu(y)$ and $\mu(xy) = \mu(x) \wedge \mu(y)$ for all x, y in R, where the operations '$\vee$' and '$\wedge$' are maximum and minimum in [0, 1].

Definition 2.11. A strong *α-cut* of a fuzzy subset μ is a crisp subset $\mu_\alpha = \{x \in R \,/\, \mu(x) > \alpha\}$ of incline R for some α in [0, 1].

3. Fuzzy Homomorphism and Flags on Incline

In this section, we prove that if a mapping $\mu : R \rightarrow [0, 1]$ is a fuzzy homomorphism then the incline order relation is preserved and the crisp sets that is, complement of strong α-cuts are ideals as a subincline of R. Further we provide sufficient conditions for μ to be a fuzzy homomorphism and we modify the concepts of chain, keychain, flag and pinned flag for ideals of an incline R.

Lemma 3.1. Let $\mu : R \to [0, 1]$ be a fuzzy homomorphism. Then for $x, y \in R$ if $x \leq y$ then $\mu(x) \leq \mu(y)$, $\mu(xy) = \mu(x)$ and $\mu(x + y) = \mu(y)$.

Proof: Let R be an incline and $\mu : R \to [0, 1]$ be a fuzzy homomorphism.
For $x, y \in R$, if $x \leq y$ then $x + y = y$

$$\Rightarrow \mu(x + y) = \mu(y)$$
$$\Rightarrow \mu(x) \vee \mu(y) = \mu(y)$$
$$\Rightarrow \mu(x) \leq \mu(y)$$

Therefore, the incline order relation is preserved.
Now, if $x \leq y$ then $\mu(x) \leq \mu(y)$

$$\Rightarrow \mu(x) \wedge \mu(y) = \mu(x)$$
$$\Rightarrow \mu(xy) = \mu(x), \text{ (by Definition 2.10).}$$

Definition 3.2. The *complement of a strong α-cut* of a fuzzy subset μ is a crisp subset $\mu_\alpha{}^c = \{x \in R \,/\, \mu(x) \leq \alpha\}$ of incline R for some α in [0, 1].

We remark that for $0 \leq \alpha \leq \beta \leq 1$, we have $\mu_\alpha{}^c \subseteq \mu_\beta{}^c$. Throughout this paper, by an α-cut we mean complement of a strong α-cut.

Theorem 3.3. If $\mu \in F(R)$ is a fuzzy homomorphism of an incline R, then the crisp subset $\mu_\alpha{}^c$ of μ is an ideal of R.

Proof: Let $\mu : R \to [0, 1]$ be a fuzzy homomorphism of R and $x, y \in \mu_\alpha{}^c$, for $\alpha \in [0, 1]$.
By the Definition of $\mu_\alpha{}^c$, we have $\mu(x) \leq \alpha$ and $\mu(y) \leq \alpha$

$$\Rightarrow \mu(x) \vee \mu(y) \leq \alpha$$
$$\Rightarrow \mu(x + y) \leq \alpha$$
$$\Rightarrow x + y \in \mu_\alpha{}^c$$

Also, $\mu(x) \wedge \mu(y) \leq \alpha$

$$\Rightarrow \mu(xy) \leq \alpha$$
$$\Rightarrow xy \in \mu_\alpha{}^c$$

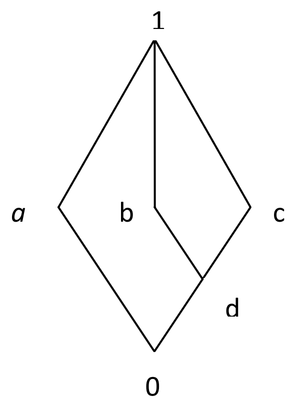
Thus $\mu_\alpha{}^c$ is a subincline of R.
Now, let $y \in \mu_\alpha{}^c$ and suppose $x \leq y$, for some $x \in R$, we have by Lemma 3.1,

$$\mu(x) \leq \mu(y) \leq \alpha$$
$$\Rightarrow \mu(x) \leq \alpha$$
$$\Rightarrow x \in \mu_\alpha{}^c$$

Thus $\mu_\alpha{}^c$ is an ideal of R.

Remark 3.4. We observe that in Theorem (3.3) the condition on μ to be a fuzzy homomorphism cannot be relaxed. This is illustrated in the following example.

Example 3.5. Let $R = \{0, a, b, c, d, 1\}$ be a commutative incline. Define $\bullet: R\bullet R \to R$ by $x\bullet y = d$ for all $x, y \in \{b, c, d, 1\}$ and 0 otherwise. Let $\mu: R \to [0, 1]$ be a fuzzy subset of R, defined as $\mu(0) = 0$, $\mu(a) = 0.2$, $\mu(b) = 0.4$, $\mu(c) = 0.5$, $\mu(d) = 0.7$, $\mu(1) = 1$.



For $b, c \in R$, $\mu(b + c) = \mu(1) = 1$

$\Rightarrow \mu(b) \vee \mu(c) = 0.4 \vee 0.5 = 0.5$

$\Rightarrow \mu(b + c) \neq \mu(b) \vee \mu(c)$

Also, $\mu(bc) = \mu(d) = 0.7$

$\Rightarrow \mu(b) \wedge \mu(c) = 0.4 \wedge 0.5 = 0.4$

$\Rightarrow \mu(bc) \neq \mu(b) \wedge \mu(c)$.

Therefore, μ is not a fuzzy homomorphism.

The crisp set $\mu_{0.5}^{c} = \{x \in R \,/\, \mu(x) \leq 0.5\}$, $0.5 \in [0, 1]$.

$\Rightarrow \mu_{0.5}^{c} = \{0, a, b, c\}$

Here $a + b = 1 \notin \mu_{0.5}^{c}$

Therefore, $\mu_{0.5}^{c}$ is not a subincline.

Thus, by Definition (2.7), $\mu_{0.5}^{c}$ is not an ideal of R.

Proposition 3.6. Let I be an ideal of an incline R and $\mu: R \to [0, 1]$ be a fuzzy subset defined by,

$$\mu(x) = \begin{cases} \alpha_0, & \text{if } x \in I \\ \alpha_1, & \text{if } x \notin I \end{cases}, \text{ for all if } x \in R. \text{ where } \alpha_0, \alpha_1 \in [0, 1],\ \alpha_0 < \alpha_1.$$

Then μ is a fuzzy homomorphism of R and $\mu_{\alpha 0}^{c} = I$.

Proof: Let I be an ideal of an incline R and $\mu: R \to [0, 1]$ be a fuzzy subset of R. For $x, y \in R$, if any one of x and y does not belong to I that is, if $x \in I$ and $y \notin I$, then

$\mu(x) \vee \mu(y) = \alpha_0 \vee \alpha_1 = \alpha_1$

Since $x \in I$, $x \leq y \Rightarrow x + y = y$

$\Rightarrow \mu(x + y) = \mu(y) = \alpha_1$.

Therefore, $\mu(x + y) = \mu(x) \vee \mu(y)$.
Now, $\mu(x) \wedge \mu(y) = \alpha_0 \wedge \alpha_1 = \alpha_0$
Since, $xy \leq x \in I$, by Definition (2.7) $xy \in I$, which implies $\mu(xy)= \alpha_0$.
Therefore, $\mu(xy) = \mu(x) \wedge \mu(y)$.
If $x, y \in I$, then $x + y \in I$ and $xy \in I$ which implies $\mu(x) = \mu(y) = \mu(x + y) = \mu(xy) = \alpha_0$.

Thus, μ is a fuzzy homomorphism and it is clear that $\mu_{\alpha 0}^c = I$.

Proposition 3.7. Let μ: $R \rightarrow [0, 1]$ be a fuzzy homomorphism and $M = \{\mu_\alpha^c / \alpha \in [0,1]\}$ be a family of ideals of R. Then $\cup_{\alpha\in[0, 1]} \mu_\alpha^c$ and $\cap_{\alpha\in[0, 1]} \mu_\alpha^c$, $\alpha \in [0,1]$ are ideals of R.

Proof: Let μ be a fuzzy homomorphism and $M = \{\mu_\alpha^c / \alpha \in [0,1]\}$ be a family of ideals as a subincline of R. Now, let $\alpha_0, \alpha_1, \ldots\ldots\ldots, \alpha_n \in [0, 1]$ such that $\alpha_0 < \alpha_1 < \ldots\ldots\ldots < \alpha_n$.
Then $\mu_{\alpha 0}^c \subseteq \mu_{\alpha 1}^c \subseteq \ldots\ldots\ldots \subseteq \mu_{\alpha n}^c$.
Therefore, $\cup_{\alpha\in[0, 1]} \mu_\alpha^c = \mu_{\alpha n}^c$.
Since $\mu_{\alpha n}^c$ is an ideal of R, we have $\cup_{\alpha\in[0, 1]} \mu_\alpha^c$ is an ideal of R.
Also, $\cap_{\alpha\in[0, 1]} \mu_\alpha^c = \mu_{\alpha 0}^c$.
Since $\mu_{\alpha 0}^c$ is an ideal of R, we have $\cap_{\alpha\in[0, 1]} \mu_\alpha^c$ is an ideal of R.

Proposition 3.8. If μ is a fuzzy homomorphism of an incline R and δ is a homomorphism of R into itself, then $\mu^\delta : R \rightarrow [0, 1]$ defined as $\mu^\delta(x) = \mu(\delta(x))$, for all $x \in R$ is a fuzzy homomorphism of R.

Proof: Let μ be a fuzzy homomorphism of an incline R, δ be a homomorphism of R into itself and $x, y \in R$.

We have, $\mu^\delta(x + y) = \mu(\delta(x + y)) = \mu(\delta(x) + \delta(y)) = \mu(\delta(x)) \vee \mu(\delta(y)) = \mu^\delta(x) \vee \mu^\delta(y)$.
Also, $\mu^\delta(xy) = \mu(\delta(xy)) = \mu(\delta(x)\delta(y)) = \mu(\delta(x)) \wedge \mu(\delta(y)) = \mu^\delta(x) \wedge \mu^\delta(y)$.
Thus, μ^δ is a fuzzy homomorphism.

Here we modify the definitions and notations of chain, keychain, flag, pinned flag studied in [7], for a set, in terms of ideals for an incline.

Definition 3.9. An n-chain is an (n + 1) tuple of distinct real numbers $\lambda_i \in [0, 1]$ always including 0 and not necessarily including 1 of the form, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots\ldots < \lambda_n$, written in ascending order of magnitude. The length of n-chain is (n + 1).

Definition 3.10. By a keychain k of an n-chain we mean a set of real numbers $\lambda_i \in [0, 1]$ of the form $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots\ldots\ldots \leq \lambda_n \leq 1$. The λ_i's are called pins and we write a keychain k as $k = 0\lambda_1\lambda_2\ldots\ldots\lambda_n$.

Definition 3.11. By an ideal flag C on an incline R, we mean a maximal chain C of ideals of R of the form $I_0 \subset I_1 \subset I_2 \subset \ldots\ldots \subset I_n = R$.

Definition 3.12. By a pinned ideal flag on an incline R, we mean a pair (C, k) of an ideal flag C on R and a keychain k from [0, 1] written as $I_0^{\lambda 0} \subset I_1^{\lambda 1} \subset I_2^{\lambda 2} \subset \ldots\ldots\ldots \subset I_n^{\lambda n} = R$.

Theorem 3.13. For an incline R, μ is a fuzzy subset of R if and only if μ can be decomposed into a pinned ideal flag.

Proof: Let R be an incline and μ be a fuzzy subset of R.
Now a μ can be associated with such a pinned ideal flag (C, k) as follows:

$$\mu(x) = \begin{cases} 0 \ , x \in I_0 \\ \lambda_1 , x \in I_1 \setminus I_0 \\ \lambda_2 , x \in I_2 \setminus I_1 \\ \vdots \\ \vdots \\ \lambda_n , x \in I_n \setminus I_{n-1} \end{cases}$$

where the ideal I_n is the full incline R. We denote this simply by $I_n^{\lambda n} = I^{\lambda n}$. Therefore, the α-cuts of μ corresponding to $\lambda_{i-1} \le \alpha \le \lambda_i$ are $I_{i-1,}$ for i = 1, 2, ….. , n. Thus μ as defined above is a fuzzy subset of R.

Conversely, let μ be a fuzzy subset of R and $\mu(R) = \{\lambda_0, \lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n\}$ where the sequence is increasing (strictly). Let $I_i = \mu_{\lambda i}{}^c$ be the α-cut corresponding to $\alpha = \lambda_i$, i = 0, 1, 2, …. ,n.
Then three facts are well known:

i) Every I_i is an ideal of R.
ii) $\lambda_i < \lambda_j$ implies that $I_i \subset I_j$, for $0 \le i, j \le n$.
iii) The chain C: $I_0 \subset I_1 \subset I_2 \subset \ldots\ldots \subset I_n$ is an ideal flag.

Suppose we refine the chain C to yield a flag, then we may have to repeat some of the pins λ_i's correspondingly. So that the components of the chain will not be an ideal of R. Hence C is an ideal flag and we arrive a pinned flag $I_0^{\lambda 0} \subset I_1^{\lambda 1} \subset I_2^{\lambda 2} \subset \ldots\ldots\ldots \subset I_n^{\lambda n} = R$, where all the λ_i's are distinct.

Remark 3.14. Theorem (3.13) is illustrated by an example.
Let us consider the incline R in Example 3.3 and a fuzzy subset μ of R is defined as

$$\mu(R) = \begin{cases} 0 & \text{for } x = 0 \\ 0.3 & \text{for } x = d \\ 0.5 & \text{for } x = b \\ 0.7 & \text{for } x = a, c, 1. \end{cases}$$

The α-cuts for λ = 0, 0.3, 0.5 and 0.7 are

$\mu_0{}^c = \{0\} = I_0$
$\mu_{0.3}{}^c = \{0, d\} = I_1$
$\mu_{0.5}{}^c = \{0, b, d\} = I_2$

$\mu_{0.7}{}^{c} = R = I_3$

Therefore, $C = I_0 \subset I_1 \subset I_2 \subset I_3 = R$ is sequence of ideals of R and it is an ideal flag. Suppose we refine it, we have

$X_0 = I_0 = \{0\}, \lambda = 0$

$X_1 = I_1 = \{0, d\}, \lambda = 0.3$

$X_2 = I_2 = \{0, b, d\}, \lambda = 0.5$

$X_3' = I_3' = \{0, b, d, 1\}, \lambda' = 0.7$

$X_3 = I_3 = R, \lambda = 0.7$

Here $X_3' = I_3'$ is not an ideal of R. Therefore, we cannot refine the ideal flags in an incline R.

Thus $C = I_0 \subset I_1 \subset I_2 \subset I_3 = R$ is an ideal flag and $I_0{}^{0} \subset I_1{}^{0.3} \subset I_2{}^{0.5} \subset I_3{}^{0.7}$ is a pinned ideal flag of R.

Here the $\lambda = 0, 0.3, 0.5$ and 0.7 are all distinct.

Remark 3.15. If μ is a fuzzy subset of R and the subsets of an incline R are subinclines, then the chain C whose components are subinclines of R can be refined to yield a subincline flag further a pinned subincline flag is arrived.

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