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Inverse and Disjoint Restrained Domination in Graphs

V.R.Kulli

Department of Mathematics
Gulbarga University, Gulbarga 585106, India
e-mail: vrkulli@gmail.com

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Abstract. Let D be a minimum secure restrained dominating set of a graph $G = (V, E)$. If $V – D$ contains a restrained dominating set D' of G, then D' is called an inverse restrained dominating set with respect to D. The inverse restrained domination number $\gamma_r^{-1}(G)$ of G is the minimum cardinality of an inverse restrained dominating set of G. The disjoint restrained domination number $\gamma_r\gamma_r(G)$ of G is the minimum cardinality of the union of two disjoint restrained dominating sets in G. We also consider an invariant the minimum cardinality of the disjoint union of a dominating set and a restrained dominating set. In this paper, we initiate a study of these parameters and obtain some results on these new parameters.

Keywords: Inverse domination number, inverse restrained domination number, disjoint restrained domination number.

AMS Mathematics Subject Classification (2010): 05*C*69, 05*C*78

1. Introduction

By a graph, we mean a finite, undirected, without loops and multiple edges. Let G be a graph with $|V|=p$ vertices and $|E|=q$ edges. For all further notation and terminology we refer the reader to [1, 2].

A set $D\subseteq V$ is a dominating set if every vertex in $V – D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. Recently many new dominating parameters are given in the books by Kulli in [2,3,4].

Kulli and Sigarkanti [5] introduced the concept of the inverse domination as follows:

Let D be a minimum dominating set of G. If $V – D$ contains a dominating set D' of G, then D' is called an inverse dominating set of G with respect to D. The inverse domination number $\gamma^{-1}(G)$ of G is the minimum cardinality of an inverse dominating set of G.

Many other inverse domination parameters in domination theory were studied, for example, in [6, 7, 8, 9, 10, 11, 12, 13, 14, 15].

A dominating set D in G is a restrained dominating set if the induced subgraph $\langle V - D\rangle$ has no isolated vertices. Alternately, a set $D \subseteq V$ is a restrained dominating set if every vertex not in D is adjacent to a vertex in D and to a vertex in $V - D$. This concept was studied by Domke et al. in [16] and was also studied as cototal domination in graphs by Kulli et al. in [17].

The disjoint domination number $\gamma\gamma(G)$ of G is the minimum cardinality of the union of two disjoint dominating sets in G. This was introduced by Hedetniemi et al. in [18]. Many other disjoint domination parameters were studied, for example, in [19, 20, 21, 22, 23, 24, 25].

In this paper, we introduce inverse restrained domination number and the disjoint restrained domination number and study their some graph theoretical properties.

Let $\Delta(G)$ denote the maximum degree and $\lceil x \rceil$ $(\lfloor x \rfloor)$ the least (greatest) integer greater (less) than or equal to x. The complement of G is denoted by $\overline{G}$.

2. Inverse restrained domination

We introduce the concept of inverse restrained domination as follows:

Definition 1. Let D be a minimum restrained dominating set of a graph $G=(V, E)$. If $V - D$ contains a restrained dominating set D' of G, then D' is called an inverse restrained dominating set with respect to D. The inverse restrained domination number $\gamma_r^{-1}(G)$ of G is the minimum cardinality of an inverse restrained dominating set of G.

Definition 2. The upper inverse restrained domination number $\Gamma_r^{-1}(G)$ of G is the maximum cardinality of an inverse restrained dominating set of G.

A γ_r^{-1}-set is a minimum inverse restrained dominating set.

Example 3. Let K_3 be the complete graph. Then $\gamma_r(K_3) = 1$ and $\gamma_r^{-1}(K_3)=1$.

Remark 4. Not all graphs have an inverse restrained dominating set. For example, the cycle C_5 has a restrained dominating set, but no inverse restrained dominating set.

Proposition 5. For any path P_p, $\gamma_r^{-1}(P_p)$ does not exist.

Proof: Clearly if $p\leq 3$, then $\gamma_r(P_p) = p$. Thus $\gamma_r^{-1}(P_p)$ does not exist. Suppose $p\geq 4$ and D is a restrained dominating set of P_p. Let $V(P_p)=\{v_1, v_2, \ldots , v_p\}$. Clearly $v_1, v_p\in D$. Also any component $V - D$ is with exactly two vertices. We consider the following two cases.

Case 1. Let $v_1\in D$ and $v_2\notin D$. Let $D_1 \subseteq V - D$. If $v_2 \in D_1$, then D_1 is not a restrained dominating set of P_p.

Case 2. Let $v_1, v_2\in D$. Let $D_1 \subseteq V - D$. If $v_3 \in D_1$, then v_3 in not adjacent with v_1. Thus D_1 is not a dominating set of P_p. Hence D_1 is not a restrained dominating set of P_p.

From the above two cases, we conclude that $\gamma_r^{-1}(P_p)$ does not exist.

The following known results are used to prove our later results.

Proposition A[16]

1) $\gamma_r(K_p) = 1$ if $p \geq 3$ and $p = 1$.
2) $\gamma_r(K_{m,n}) = 2$, if $2 \leq m \leq n$.
3) $\gamma_r(C_p) = p - 2\left\lfloor \frac{p}{3} \right\rfloor$, if $p \geq 3$.

We obtain the exact values of $\gamma_r^{-1}(G)$ for some standard graphs.

Proposition 6. If K_p is a complete graph with $p \geq 3$ vertices, then

$$\gamma_r^{-1}(K_p) = 1.$$

Proof: Let D be a minimum restrained dominating set of K_p. By Proposition A(1), $|D|=1$. Let $D=\{u\}$. Then $D_1=\{x\}$ is a γ_r^{-1}-set of K_p for $x \in V(K_p) - \{u\}$. Thus $\gamma_r^{-1}(K_p) = 1$.

Proposition 7. If $K_{m,n}$, is a complete bipartite graph with $2 \leq m \leq n$, then

$$\gamma_r^{-1}(K_{m,n}) = 2.$$

Proof: Let $V(K_{m,n}) = V_1 \cup V_2$ where $V_1 = \{u_1, u_2, ..., u_m\}$ and $V_2 = \{v_1, v_2, ..., v_n\}$. By Proposition A(2), $D=\{u_1, v_1\}$ is a minimum restrained dominating set of $K_{m,n}$. Then $S = \{u_2, v_2\}$ is a γ_r^{-1}-set of $K_{m,n}$ for $u_2, v_2 \in V(K_{m,n}) - \{u_1, v_1\}$. Thus $\gamma_r^{-1}(K_{m,n}) = 2$.

Proposition 8. If C_p is a cycle with $p \geq 3$ vertices, then

$$\gamma_r^{-1}(C_p) = p - 2\left\lfloor \frac{p}{3} \right\rfloor.$$

Theorem 9. If a γ_r^{-1}-set exists in a graph G, then

$$\gamma_r(G) \leq \gamma_r^{-1}(G) \qquad (1)$$

and this bound is sharp.

Proof: Clearly every inverse restrained dominating set is a restrained dominating set. Thus (1) holds.

The complete graphs K_p, $p \geq 3$, achieve this bound.

Theorem 10. If a γ_s^{-1}-set exists in a graph G with p vertices, then

$$\gamma_r(G) + \gamma_r^{-1}(G) \leq p$$

and this bound is sharp.

Proof: This follows from the definition of $\gamma_r^{-1}(G)$.

The graph $K_{2,2}$ achieves this bound.

Theorem 11. For any graph G with a γ_r^{-1}-set and with p vertices,

$$\gamma(G) + \gamma_r^{-1}(G) \leq p \qquad (2)$$

and this bound is sharp.

Proof: By definition, $\gamma(G) \leq \gamma_r(G)$. By Theorem 10, $\gamma_r(G) + \gamma_r^{-1}(G) \leq p$. Thus (2) holds.

The graph $K_{2,2}$ achieves this bound.

We establish lower and upper bounds on $\gamma_r^{-1}(G)$.

Theorem 12. For any graph G with p vertices and with a γ_s^{-1}-set,

$$\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma_r^{-1}(G) \leq \left\lceil \frac{p\Delta(G)}{\Delta(G)+1} \right\rceil. \tag{3}$$

Proof: It is known that $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma(G)$ and since $\gamma(G) \leq \gamma_r^{-1}(G)$, we see that the lower bound in (3) holds.

By Theorem 11, we have

$$\gamma_r^{-1}(G) \leq p - \gamma(G).$$

Since $\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq \gamma(G)$ and the above inequality,

$$\gamma_r^{-1}(G) \leq p - \left\lceil \frac{p}{\Delta(G)+1} \right\rceil.$$

Thus the upper bound in (3) holds.

The complete graph K_3 achieves the lower bound.

3. Disjoint restrained domination

The inverse restrained domination number inspires us to introduce the concept of disjoint restrained domination number.

Definition 13. The disjoint restrained domination number $\gamma_r\gamma_r(G)$ of a graph G is defined as follows: $\gamma_r\gamma_r(G) = \min\{|D_1|+|D_2|: D_1, D_2$ are disjoint restrained dominating sets of $G\}$. We say that two disjoint restrained dominating sets, whose union has cardinality $\gamma_r\gamma_r(G)$, is a $\gamma_r\gamma_r$-pair of G.

Remark 14. Not all graphs have a disjoint restrained domination number. For example, the cycle C_5 does not have two disjoint restrained dominating sets.

Theorem 15. If a γ_r^{-1}-set exists in a graph G with p vertices, then

$$2\gamma_r(G) \leq \gamma_r\gamma_r(G) \leq \gamma_r(G) + \gamma_r^{-1}(G) \leq p.$$

We also consider an invariant the minimum cardinality of a disjoint union of a dominating set D and a restrained dominating set D' and it is denoted by $\gamma\gamma_r(G)$. We call such a pair of dominating sets (D, D'), a $\gamma\gamma_r$–pair. A $\gamma\gamma_r$-pair can be found by letting D' be any restrained dominating set, and then noting that the complement $V - D'$ is a dominating set. Thus $V - D'$ contains a minimal dominating set D.

Remark 16. Not all graphs have a $\gamma\gamma_r$-pair. For example, the cycle C_5 does not have a $\gamma\gamma_r$-pair.

Proposition 17. If both $\gamma\gamma_r$-pair and $\gamma_r\gamma_r$-pair exist, then

$$\gamma\gamma(G) \le \gamma\gamma_r G) \le \gamma_r\gamma_r(G).$$

Proposition 18. If K_p is a complete graph with $p \ge 3$ vertices, then

$$2\gamma_r(K_p) = \gamma_r\gamma_r(K_p) = \gamma\gamma(K_p) = \gamma\gamma_r(K_p) = 2.$$

Proposition 19. For the complete bipartite graph $K_{m,n}$, $2 \le m \le n$,

$$2\gamma_r(K_{m,n}) = \gamma_r\gamma_r(K_{m,n}) = \gamma\gamma(K_{m,n}) = \gamma\gamma_r(K_{m,n}) = 4.$$

Definition 20. A graph G is called $\gamma_r\gamma_r$-minimum if $\gamma_r\gamma_r(G) = 2\gamma_r(G)$.

Definition 21. A graph G is called $\gamma_r\gamma_r$-maximum if $\gamma_r\gamma_r(G) = p$.

The following classes of graphs are $\gamma_r\gamma_r$-minimum.

i) The complete graphs K_p, $p\ge3$, are $\gamma_r\gamma_r$-minimum.

ii) The complete bipartite graphs $K_{m,n}$, $2 \le m \le n$, are $\gamma_r\gamma_r$-minimum.

One can see that the graph $K_{2,2}$ is $\gamma_r\gamma_r$-maximum.

Theorem 22. For each integer $n \ge 1$, there exists a connected graph G such that $\gamma_r^{-1}(G) - \gamma_r(G) = 2n$ and $|V(G)| = \gamma_r(G) + \gamma_r^{-1}(G)$.

Proof: Let $n \ge 1$. Consider the graph G with $2n$+4 vertices as in Figure 1.

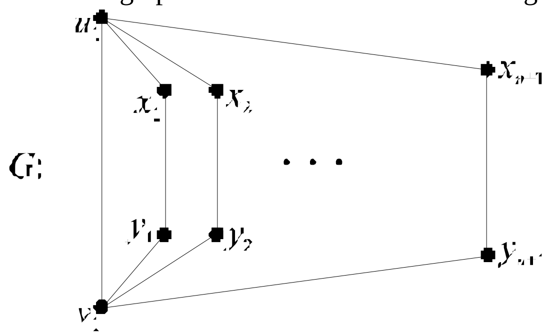

Figure 1:

Then $D = \{u_1, v_1\}$ is a restrained dominating set in G, which is minimum. Thus $\gamma_r(G) = 2$. Since x_i, y_i are adjacent in G for $i = 1, 2, \ldots, n+1$, it implies that $D_1 = V(G) - \{u_1, v_1\}$ is a restrained dominating set in $V - D$. Therefore $\gamma_r^{-1}(G) \le |D_1| = 2n+2$. Since $N_G[S] \ne V(G)$ for all proper subsets of S of D_1, it implies that $\gamma_r^{-1}(G) = |D_1| = 2n+2$. Hence $\gamma_r^{-1}(G) - \gamma_r(G) = 2n$ and also $|V(G)| = \gamma_r(G) + \gamma_r^{-1}(G)$.

Theorem 23. For each integer $n \ge 1$, there exists a connected graph G such that $\gamma_r(G) + \gamma_r^{-1}(G) - \gamma_r\gamma_r(G) = 2n$.

Proof: Consider the graph G as in Figure 2 obtained by adding to the corona C_4oC_4 $2n$ vertices $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$ and the edges x_iu_j, y_iu_j, x_iy_i, $i = 1,2, \ldots, n$, $j = 1, 2, 3, 4$.

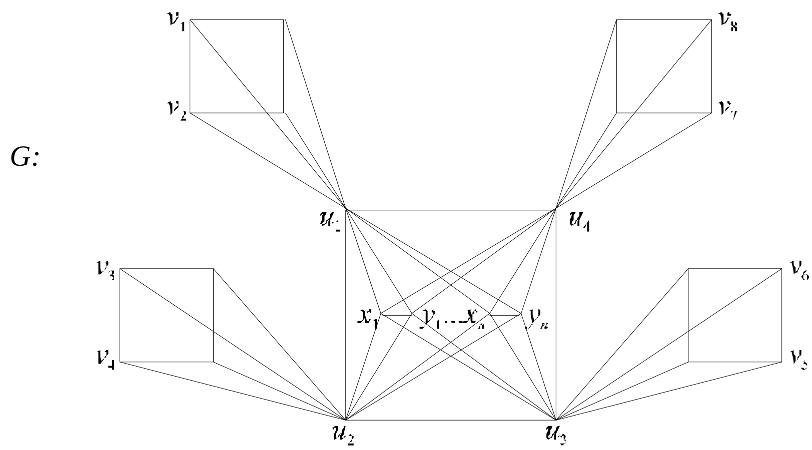

Figure 2:

Then $\{u_1, u_2, u_3, u_4\}$ is the unique minimum restrained dominating set in G and $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\} \cup \{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ is a γ_r^{-1}-set in G. Thus $\gamma_r(G) = 4$ and $\gamma_r^{-1}(G) = 8 + 2n$. Also the sets $D_1=\{u_1, u_2, v_5, v_6, v_7, v_8\}$ and $D_2=\{u_3, u_4, v_1, v_2, v_3, v_4\}$ constitute a $\gamma_r\gamma_r$-pair in G. Hence $\gamma_r\gamma_r(G) = |D_1|+|D_2| = 12$. Thus $\gamma_r(G) + \gamma_r^{-1}(G) - \gamma_r\gamma_r(G) = 2n$.

Corollary 24. The difference $\gamma_r(G) + \gamma_r^{-1}(G) - \gamma_r\gamma_r(G)$ can be made arbitrarily large.

Theorem 25. Let G and H be complete graphs such that $V(G+H) \geq 3$. Then $\gamma_r\gamma_r(G+H) = 2$.

Proof: Let G and H be complete graphs such that $V(G+H) \geq 3$. In $G + H$, each vertex of G is adjacent to every vertex of H and vice versa. Thus $G+H$ is a complete graph with at least 3 vertices and hence $\gamma_r(G+H)=1$ and $\gamma_r^{-1}(G+H)=1$. Thus

$$2\gamma_r(G+H) = \gamma_r\gamma_r(G+H) = \gamma_r(G + H) + \gamma_r^{-1}(G+H) = 2.$$

Theorem 26. Let G and H be connected non-complete graphs. Then $\gamma_r\gamma_r(G+H) = 4$.

Proof: Let G and H be connected non-complete graphs. In $G + H$, each vertex of G is adjacent to every vertex of H and vice versa. Thus pick $u \in G$, $v \in H$ and choose $x \in V(G) - \{u\}$ and $y \in V(H) - \{v\}$. Then $D = \{u, v\}$ and $D_1 = \{x, y\}$ are disjoint γ_r-sets in $G + H$. Thus $\gamma_r(G+H)=2$ and $\gamma_r^{-1}(G+H)=2$. Thus

$$2\gamma_r(G+H) = \gamma_r\gamma_r(G+H) = \gamma_r(G + H) + \gamma_r^{-1}(G+H) = 4.$$

4. Open problems

Many questions are suggested by this research, among them are the following:

Problem 1. Characterize graphs G for which $\gamma_r(G) = \gamma_r^{-1}(G)$.

Problem 2. Characterize graphs G for which $\gamma_r(G) + \gamma_r^{-1}(G) = p$.

Problem 3. Characterize graphs G for which $\gamma(G) + \gamma_r^{-1}(G) = p$.

Problem 4. Characterize graphs G for which $\gamma\gamma_r(G) = \gamma_r\gamma_r(G)$.

Problem 5. When is $\gamma\gamma(G) = \gamma\gamma_r(G)$?

Problem 6. Characterize the class of $\gamma_r\gamma_r$ -minimum graphs.

Problem 7. Characterize the class of $\gamma_r\gamma_r$ -maximum graphs.

Problem 8. Obtain bounds for $\gamma_r\gamma_r(G) + \gamma_r\gamma_r(\overline{G})$.

Problem 9. What is the complexity of the decision problem corresponding to $\gamma_r\gamma_r(G)$?

Problem 10. Is DISJOINT RESTRAINED DOMINATION NP-complete for a class of graphs?

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