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Weak Domination Alteration Sets in Fuzzy Graphs

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Abstract. In general γ_f(G) can be made to increase or decrease by the removal of nodes from fuzzy graph G. In this paper the effect of the removal of a node of a fuzzy graph on weak fuzzy dominating set is studied. Further the stability of weak fuzzy domination number of fuzzy graph is investigated.

Keywords: Fuzzy dominating set, weak fuzzy dominating set, weak fuzzy dominating critical node, weak fuzzy dominating stability, weak fuzzy domination number.

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1. Introduction

Harary et al [2] explained an interesting application in voting situations using the concept of domination. Hattingh and Renu [3] introduced on weak domination in graphs. Rosenfeld[7] introduced the notion of fuzzy graph and several fuzzy analogous of graph theoretic concepts such as paths, cycles, connectedness and etc. Somasundaram and Somasundaram [8] discussed domination in fuzzy graphs. Nagoor Gani and Basheer Ahamed [5] introduced strong and weak domination in fuzzy graphs. Ebadi and Ebrahimi [1] introduced weak domination critical and stability in graphs. In this paper we investigate the effect of removal of nodes from fuzzy graph G.

2. Preliminaries

A *fuzzy graph* G=(σ,μ) is a non-empty set V together with a pair of functions $\sigma: V \to [0,1]$ and μ:VxV→[0,1] such that μ(u,v)≤ $\sigma(u) \wedge \sigma(v)$ for all u,v $\in V$, where $\sigma(u) \wedge \sigma(v)$is the minimum of $\sigma(u)$ *and* $\sigma(v)$.The *underlying crisp graph* of the fuzzy graph G=(σ,μ) is denoted as $G^*=(\sigma^*, \mu^*)$ where σ^*={ u∈ V/ $\sigma(u)$ >0} and μ^*= {(u,v)∈VxV / μ(u,v)>0}. A fuzzy graph G=(σ,μ) is a complete fuzzy graph if μ(u,v) = $\sigma(u) \wedge \sigma(v)$for all u,v ∈ σ^*. Two nodes u and v are said to be neighbours if μ(u,v) > 0.The *strong neighbourhood* of u is $N_S(u)$ = { v ∈ V : (u,v) is a strong arc}. $N_S[u] = N_S(u) \cup \{u\}$ is the *closed strong neighbourhood* of u. A *path* ρ in a fuzzy graph is a sequence of distinct nodes $u_0, u_1, u_2, ... u_n$ such that $\mu(u_{i-1}, u_i) > 0$; $1 \leq i \leq n$ here $n \geq 0$ is called the *length* of the path ρ. The consecutive pairs (u_{i-1}, u_i) are called the *arcs* of the path. A path ρ is called a *cycle* if $u_o = u_n$ and $n \geq 3$. An arc (u,v) is said to be a *strong arc* if $\mu(u,v) \geq \mu^{\infty}(u,v)$ and the node v is said to be a *strong neighbour* of u. If $\mu(u,v) = 0$ for every v∈V then u is called *isolated node*. A fuzzy graph G is called *wheel fuzzy graph* ω_ρ if all the nodes of cycle strongly adjacent to a single node. A connected fuzzy graph G=(σ, μ) is a *fuzzy tree* if it has a fuzzy spanning subgraph F= (σ, ν) which is a tree where for all arcs (u,v) not in F, $\mu(u,v) < \nu\infty(u,v)$.

A fuzzy graph G=(σ, μ) is *fuzzy bipartite* if it has a spanning fuzzy subgraph H= (τ, π) which is bipartite where for all edges (u,v) not in H, weight of (u,v) in G is strictly less than the strength of pair (u,v) in H. i.e $\mu(u,v) < \pi^{\infty}(u,v)$. A fuzzy bipartite graph G with fuzzy bipartition (V_1, V_2) is said to be a complete fuzzy bipartite if for each node of V_1, every node of V_2 is a strong neighbor. Let G= (σ, μ) be a fuzzy graph and u be a node in G then there exist a node v such that (u,v) is a strong arc then we say that *u dominates v*. Let G=(σ, μ) be a fuzzy graph. A set D of V is said to be *fuzzy dominating set* of G if every v ∈ V-D there exist u ∈ D such that u dominates v. A fuzzy dominating set D of G is called a *minimal fuzzy dominating set* of G if no proper subset of D is a fuzzy dominating set. The *fuzzy domination number* γ_f(G) of the fuzzy graph G is the smallest number of nodes in any fuzzy dominating set of G. A fuzzy dominating set D of a fuzzy graph G such that $|D| = \gamma_f$(G) is called minimum fuzzy dominating set.

3. Critical node of fuzzy dominating set

Definition 3.1. Let G=(σ, μ) be a fuzzy graph. A node v of G is said to be fuzzy dominating critical node if its removal either increases (or) decreases the fuzzy domination number.

We partition the nodes of G into three disjoint sets according to how their removal affects γ_f (G). Let $V = V_f^0 \cup V_f^+ \cup V_f^-$ for

$V_f^0 = \{ v \in V : \gamma_f(G-v) = \gamma_f(G) \}$

$V_f^+ = \{ v \in V : \gamma_f(G-v) > \gamma_f(G) \}$

$V_f^- = \{ v \in V : \gamma_f(G-v) < \gamma_f(G) \}$

Definition 3.2. Let G=(σ, μ) be a fuzzy graph. Two nodes u and v of G are *strong adjacent* if (u,v) is strong arc. Otherwise they are said to be weak.

The *strong degree* of a node v is the number of nodes that are strong adjacent to v. It is denoted by $d_S(v)$.

Let G=(σ, μ) be a fuzzy graph. For any u,v ∈ V, *u weakly dominates* v (i) If u is strong adjacent to v and (ii) $d_S(v) \geq d_S(u)$.

A set D⊆V is a *weak fuzzy dominating set* of G if every node in V-D is weakly dominated by at least one node in D. A weak fuzzy dominating set D of G is called a minimal weak fuzzy dominating set if no proper subset of D is a weak fuzzy dominating set of G.

The smallest number of nodes in any weak fuzzy dominating set is called the *weak fuzzy domination number* and is denoted by (γ_{wf} (G). A weak fuzzy dominating set D of G such that $|D|$ = (γ_{wf} (G) is called minimum weak fuzzy dominating set.

4. Critical nodes of weak fuzzy dominating set

Definition 4.1. Let G=(σ, μ) be a fuzzy graph. A node v of G is said to be weak fuzzy dominating critical node if its removal either increase (or) decrease the weak fuzzy domination number.

We partition the nodes of G into three disjoint sets according to how their removal affects γ_{wf} (G). Let V =$V_{wf}^0 \cup V_{wf}^+ \cup V_{wf}^-$

For V_{wf}^0 = { v∈V : γ_{wf} (G-v) = γ_{wf}(G)}

V_{wf}^+ = { v∈V : γ_{wf}(G-v) $>$ γ_{wf} (G)}

V_{wf}^-= { v∈V : γ_{wf} (G-v) $<$ γ_{wf} (G)} .

Definition 4.2. γ_{wf} - Stability of fuzzy graph G written γ_{wf} is the minimum number of nodes whose removal changes γ_{wf} (G).

γ_{wf}^+- Stability of a fuzzy graph G written $\gamma_{wf}{}^+$ is the minimum number of nodes whose removal increases γ_{wf} (G).

γ_{wf}^- - Stability of a fuzzy graph G written $\gamma_{wf}{}^-$ is the minimum number of nodes whose removal decreases γ_{wf} (G).

5. Critical node of weak fuzzy dominating set in fuzzy graphs

Results:

1. If I is the set of all isolated nodes of G then I ⊆ V_{wf}^-.
2. If D is a weak fuzzy dominating set , removing any node in V-D cannot increase the weak fuzzy domination number then $|V_{wf}^+| \leq \gamma_{wf}$ (G).

Theorem 5.1. If a node u of V(G) is in V_{wf}^+(G) then there is no weak fuzzy dominating set of G-u with γ_{wf} (G) nodes.

Proof: Let u ∈ V_{wf}^+(G). Then γ_{wf}(G-v) $>$ γ_{wf} (G). Suppose there exists a weak fuzzy dominating set of G-u with γ_{wf} (G) nodes. Then γ_{wf} (G-v) $\leq$ γ_{wf} (G), a contradiction.

Proposition 5.2. If a node u is in V_{wf}^{-}(G) , then N_S(u) ⊆ $\bigcup_{v\in D-\{u\}} N_S$[v], for some γ_{wf} - set D containing u.

Proof: Suppose there is w belong to N_S(u) such that w ∉ $\bigcup_{v\in D-\{u\}} N_s$[v]. So by removing u we see that v belong to weak fuzzy dominating set. Since there is no node in weak fuzzy dominating set such that dominate v, so γ_{wf} (G-u) = γ_{wf}(G), a contradiction.

Proposition 5.3. For any fuzzy graph G, if V_{wf}^{-} = {v} then A^*(v) = ∅. [Here A^*(v)= {u: u∉ D and N_s(u)∩D = {v}].

Proof: Suppose γ_{wf} (G-v) ≤ γ_{wf} (G) for some node v∈G, and let D be a minimum weak fuzzy dominating set for G-v. Clearly S = D∪ {v} is a minimum weak fuzzy dominating set for G with A^*(v) = ∅.

Proposition 5.4. If the removal of a node u from G increases γ_{wf} (G), then (i) u is not isolated node and end node and (ii) there is no weak fuzzy dominating set for G-N_s[u] having γ_{wf} (G) nodes which also dominates N_s[u] for some γ_{wf} - set D containing u.

Proof:

(i) Suppose γ_{wf}(G-v) > γ_{wf} (G) and u∈D. Then clearly u is not an isolated, and also u is not end –node, sine for any fuzzy graph G if u is end-node then γ_{wf} (G-v) ≤ γ_{wf} (G), a contradiction.

(ii) Suppose there exists a weak fuzzy dominating set of G-N_s[u] with γ_{wf} (G) nodes. Then γ_{wf} (G-v) ≤ γ_{wf} (G), a contradiction.

Proposition 5.5. If there exists at least u_1,u_2 ∈ A^*(v), then γ_{wf}(G-v) > γ_{wf} (G), for some γ_{wf} - set D containing v.

Proof: Suppose there exists at least u_1,u_2 ∈ A^*(v) such that u_1 and u_2 are not strong adjacent so by removing v, there are no nodes in weak fuzzy dominating set such that at least dominates u_1 and u_2 (and maybe more). Hence γ_{wf}(G-v) > γ_{wf} (G).

Proposition 5.6. Let G=(σ,μ) be a fuzzy graph where G^* = (V,E) is a wheel graph of p≥ 5 vertices then $\gamma_{sf}{}^{+}$(ω_ρ) = 1.

Proof: Let G be a fuzzy graph whose underlying crisp graph is a wheel graph of n≥ 5 vertices, then ω_ρ has a vertex v of strong degree p-1 and hence γ_{sf}(ω_ρ) = 1. Since ω_ρ -1 is a cycle of length p-1. Hence γ_{sf}(ω_ρ-v) = γ_{sf}(C_p-1) > 1= γ_{sf}(ω_ρ). Thus $\gamma_{sf}{}^{+}$ (ω_ρ) = 1.

Proposition 5.7. For any fuzzy graph G (G≠$K_{m,n}$) then $|V_{wf}^{0}|$ ≥2$|V_{wf}^{+}|$.

Proof: For each v∈ V_{wf}^{+}(G), we consider the following two cases.

Case (i) If v is in minimum weak fuzzy dominating set, then $N_s(v)$ contains at least two non strong adjacent nodes $u_1$and u_2, since $u_1,u_2 \notin D$, if γ_{wf} (G- u_1,u_2) = γ_{wf}(G) we have done. If not, again u_1 or u_2 (and may be both) is strong adjacent with another node such that, that node is not in minimum weak fuzzy dominating set, so by removing that node we see that γ_{wf}(G) is not change , otherwise we continue this process, then we see that $|V_{wf}^0| \geq 2|V_{wf}^+|$.

Case (ii) Suppose v is not in minimum weak fuzzy dominating set , then $N_s[v]$ contains at least one (and maybe more) node, such that that is not in minimum weak fuzzy dominating set that is and v. If not, we use case (i).Hence $|V_{wf}^0| \geq 2|V_{wf}^+|$.

Theorem 5.8. For any fuzzy graph G ($G \neq K_{m,n}$), γ_{wf} (G-v) $\neq$ γ_{wf}(G) for all v∈V if and only if $V=V_{wf}^-$.

Proof: Obviously, if $V=V_{wf}^-$ then γ_{wf} (G-v) $\neq$ γ_{wf}(G) for all v∈V.Assume that γ_{wf} (G-v) $\neq$ γ_{wf}(G) for all v∈V. Then V_{wf}^+ and V_{wf}^- partition V. But if v∈ V_{wf}^+, then proposition 8.5, V_{wf}^0 is not empty, a contradiction. Hence $V=V_{wf}^-$.

6. Stability of weak fuzzy dominating set in fuzzy paths

Theorem6.1. For any path ρ_n with n≥9, $\gamma_{wf}^+(\rho_n)+ \gamma_{wf}^-(\rho_n) = \begin{cases} 3 \ if \ n \equiv 0,2(mod 3) \\ 4 \ if \ n \equiv 1 \ (mod 3) \end{cases}$

Proof: We consider the following three cases.

Case (i) Let n≡0 (mod3). Let Let $v_1,v_2,...v_n$ be the nodes of ρ_n, then ρ_n-{v_4} consists of ρ_3 and ρ_{n-4}. Thus γ_{wf} (ρ_n-{v_4}) = γ_{wf} (ρ_3)+ γ_{wf} (ρ_{n-4}) = 2+ $\left\lceil\frac{n-4}{3}\right\rceil$ + 1 = $\left\lceil\frac{n-1}{3}\right\rceil$ + 2 = $\left\lceil\frac{n}{3}\right\rceil$ + 2 > $\left\lceil\frac{n}{3}\right\rceil$ +1 = γ_{wf} (ρ_n). Hence $\gamma_{wf}^+(\rho_n)$ = 1(if n≡0 (mod3) . To see that $\gamma_{wf}^-(\rho_n)$ = 2 first note that γ_{wf} (ρ_{n-2}) = γ_{wf} (ρ_n) –1. Hence $\gamma_{wf}^-(\rho_n) \leq 2$. Since γ_{wf} (ρ_{n-1}) = γ_{wf} (ρ_n) the only way to lower the weak fuzzy domination number of ρ_n by removing either one or two nodes is to disconnect ρ_n.

Case (ii) Let n≡2 (mod3). Now γ_{wf} (ρ_{n-1}) < γ_{wf} (ρ_n) and hence $\gamma_{wf}^-(\rho_n)$ = 1. If we remove v_4 from ρ_n we obtain ρ_3 and ρ_{n-4}. Thus γ_{wf} (ρ_n-{v_4}) = γ_{wf} (ρ_3)+ γ_{wf} (ρ_{n-4}) = 2+ $\left\lceil\frac{n-4}{3}\right\rceil$ = $\left\lceil\frac{n-1}{3}\right\rceil$ + 1 = $\left\lceil\frac{n}{3}\right\rceil$ + 1= γ_{wf} (ρ_n).Since n-4≡1 (mod3) by case (i) $\gamma_{wf}^+(\rho_{n-4})$ = 1. Hence $\gamma_{wf}^+(\rho_n)$ = 2.

Case (iii) Let n≡1 (mod3). If we remove v_4 from ρ_n we obtain ρ_3 and ρ_{n-4}. Thus γ_{wf} (ρ_n-{v_4}) = γ_{wf} (ρ_3)+ γ_{wf} (ρ_{n-4}) = 2+ $\left\lceil\frac{n-4}{3}\right\rceil$ + 1 = $\left\lceil\frac{n+2}{3}\right\rceil$ + 1 = $\left\lceil\frac{n}{3}\right\rceil$ + 1> $\left\lceil\frac{n}{3}\right\rceil$ = γ_{wf} (ρ_n). Hence $\gamma_{wf}^+(\rho_n)$ = 1 if n≡1 (mod 3). To see that $\gamma_{wf}^-(\rho_n)$ = 3 first note that γ_{wf} (ρ_{n-3}) = γ_{wf} (ρ_n) – 1. Hence $\gamma_{wf}^-(\rho_n) \leq 3$. Since γ_{wf} (ρ_{n-2}) = γ_{wf} (ρ_{n-1}) = γ_{wf} (ρ_n)

the only way to lower the weak fuzzy domination number of ρ_n by removing either one or two nodes is to disconnect ρ_n.

7. Stability of weak fuzzy dominating set in fuzzy cycles

Theorem 7.1. For n≥9,

$$\gamma^{+}_{wf}(C_n)+\quad \gamma^{-}_{wf}(C_n) = \begin{cases} 6 \; if \; n \equiv 0,2(mod3) \\ 5 \; if \; n \equiv 1 \; (mod3) \end{cases}$$

Proof: It suffices to show that for n≡0,1 and 2(mod3), we have respectively $\gamma^{+}_{wf}(C_n)$ =1, $\gamma^{-}_{wf}(C_n)$ = 5 and $\gamma^{+}_{wf}(C_n)$ = 2 , $\gamma^{-}_{wf}(C_n)$ = 3 and $\gamma^{+}_{wf}(C_n)$=2, $\gamma^{-}_{wf}(C_n)$ = 4. We indicate how to prove that $\gamma^{+}_{wf}(C_n)$ = 2 when n≡1 (mod3). The remaining cases follow easily from the proof of proposition 7.2. Let n≡1 (mod3) and we denote C_n by $v_0,v_1,...,v_n$, then removal of the set of nodes $\{v_0, v_4\}$ leaves ρ_3 and ρ_{n-5}. Hence $\gamma_{wf}(C_n-\{v_0,v_4\})$ = $\gamma_{wf}(\rho_3) + \gamma_{wf}(\rho_{n-5}) = 2 + \left\lceil \frac{n-5}{3} \right\rceil +1= \left\lceil \frac{n+1}{3} \right\rceil + 1 = \left\lceil \frac{n}{3} \right\rceil +1> \left\lceil \frac{n}{3} \right\rceil = \gamma_{wf}(C_n)$. Thus $\gamma^{+}_{wf}(C_n)$ = 2.

8. Conclusion

In this paper, we apply critical concept to weak fuzzy dominating set. Also we introduced fuzzy wheel graph and apply this concept to fuzzy wheel graph and obtained some results.

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