Another Theta Generalized Closed Sets in Fuzzy Topological Spaces

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Abstract. In this paper, we introduce a new class of fuzzy generalized closed sets called fuzzy \(\theta g^{\prime \prime}\)-closed and its properties are investigated. Further, new concept of fuzzy \(\theta g^s\)-closed, fuzzy \(g^{\prime \prime}\)-closed, fuzzy \(g^s\theta\)-closed, and fuzzy \(g^{\prime \prime}\alpha\theta\)-closed sets are introduced using it.

Keywords: fuzzy generalized closed sets, fuzzy \(\theta g\)-closed sets, fuzzy \(\theta gs\)-closed sets, fuzzy \(\theta g^{\prime \prime}\)-closed and fuzzy \(\theta\)-closed sets

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1. Introduction

In the present paper, we introduce fuzzy \(\theta g^{\prime \prime}\)-closed sets in fuzzy topological space and investigate certain basic properties of these fuzzy sets.

2. Basic concepts
Throughout the present paper, \((X, \tau)\) or simply \(X\) mean fuzzy topological space (abbreviated as fts) on which no separation axioms are assumed unless otherwise mentioned. We denote and define the closure and interior for a fuzzy set \(A\) by \(Cl(A) = \bigwedge\{\mu : \mu \geq A, 1 - \mu \in \tau\}\) and \(Int(A) = \bigvee\{\mu : \mu \leq A, \mu \in \tau\}\). fuzzy \(\theta\) closure of \(A\) [6] and fuzzy semi-\(\theta\) closure of \(A\) [12] are denoted by \(Cl_\theta(A) = \bigwedge\{\mu \in \tau\}\) and \(sCl_\theta(A) = \bigwedge\{\mu \in \tau\}\) respectively.
V. Chandrasekar and G. Anandajothi

**Definition 2.1.** A fuzzy set $A$ of $(X, \tau)$ is called

(i) fuzzy semi-open [1] if $A \leq Cl\left(\text{Int}\left(A\right)\right)$

(ii) fuzzy $\alpha$-open [3] if $A \leq \text{Int}\left(Cl\left(A\right)\right)$

(iii) fuzzy $\theta$-open if $A = \text{Int}_\theta(A)$ [6] and fuzzy $\theta$-closed [6] if $A = Cl_\theta(A)$

(iv) fuzzy semi-$\theta$-open [12] if $A = s\text{Int}_\theta(A)$ and fuzzy semi-$\theta$-closed [12] $A = sCl_\theta(A)$

The semi closure [11] (respectively $\alpha$-closure [10]) of a fuzzy set $A$ of $(X, \tau)$ is the intersection of all fs-Closed (respectively $f\alpha$-closed sets) that contain $A$ and is denoted by $sCl(A)$ (respectively $\alpha Cl(A)$).

**Definition 2.2.** A fuzzy set $A$ of $(X, \tau)$ is called

(i) fuzzy generalized closed (in short, fg-closed) [4] if $Cl A \leq H$, whenever $A \leq H$ and $H$ is fuzzy open set in $X$

(ii) fuzzy generalized semi closed (in short, fsg-closed) [2] if $sCl(A) \leq H$, whenever $A \leq H$ and $H$ is fs-open set in $X$.

(iii) fuzzy generalized closed (in short, fgs-closed) [9] if $sCl(A) \leq H$, whenever $A \leq H$ and $H$ is f-open set in $X$.

(iv) fuzzy $\theta$-generalized closed (in short, $f\theta g$-closed) [6] if $Cl_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is f-open set in $X$.

(v) fuzzy $\theta$-semi-generalized closed (in short, $f\theta sg$-closed) [12] if $sCl_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fs-open set in $X$.

(vi) fuzzy $\theta$-generalized semi closed (in short, $f\theta gs$-closed) [7] if $sCl_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is f-open set in $X$.

(vii) fuzzy $g^m$-closed [8] if $Cl A \leq H$, whenever $A \leq H$ and $H$ is fuzzy gs-open set in $X$

(viii) fuzzy $g^s$-closed [8] if $sCl A \leq H$, whenever $A \leq H$ and $H$ is fuzzy gs-open set in $X$

(ix) fuzzy $g^m_a$-closed [8] if $\alpha Cl A \leq H$, whenever $A \leq H$ and $H$ is fuzzy gs-open set in $X$

3. Fuzzy $\theta g^m$-closed sets in fuzzy topological spaces

**Definition 3.1.** Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $A$ of $(X, \tau)$ is called fuzzy $\theta g^m$-closed if $Cl_\theta(A) \leq H$ whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open in $X$.

**Definition 3.2.** Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $A$ of $(X, \tau)$ is called
fuzzy $\theta g^s$- closed if $s\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open set in $X$.

**Definition 3.3.** Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $A$ of $(X, \tau)$ is called fuzzy $g'''\theta$- closed if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open set in $X$.

**Definition 3.4.** Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $A$ of $(X, \tau)$ is called fuzzy $g^s\theta$- closed if $s\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open set in $X$.

**Definition 3.5.** Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $A$ of $(X, \tau)$ is called fuzzy $g''\alpha\theta$- closed if $\alpha \text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open set in $X$.

**Theorem 3.1.** Every fuzzy $\theta$-closed set is fuzzy $\theta g'''$- closed set in a fuzzy topological space $(X, \tau)$.

**Proof:** Let $A$ be fuzzy $\theta$ closed set in $X$. Let $H$ be a $f\theta gs$- open set in $X$ such that $A \leq H$. Since $A$ is fuzzy $\theta$- closed, $\text{Cl}_\theta(A) = A$. Therefore $\text{Cl}_\theta(A) \leq H$ whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open set in $X$. Hence $A$ is fuzzy $\theta g'''$- closed set in $X$.

**Remark 3.1.** The converse of the above theorem need not be true as shown in the following example.

**Example 3.1.** Let $X = \{a\}$. Fuzzy sets $A$ and $B$ are defined by $A(a) = 0.6$; $B(a) = 0.5$. Let $\tau = \{0, A, 1\}$. Then $B$ is a fuzzy $\theta g'''$- closed set but it is not a fuzzy $\theta$- closed set in $(X, \tau)$.

**Theorem 3.2.** Every fuzzy semi-$\theta$-closed set is fuzzy $\theta g^s$- closed set in a fuzzy topological space $(X, \tau)$.

**Proof:** Let $A$ be fuzzy semi-$\theta$ closed set in $X$. Let $H$ be a $f\theta gs$- open set in $X$ such that $A \leq H$. Since $A$ is fuzzy semi-$\theta$- closed, $s\text{Cl}_\theta(A) = A$. Therefore $s\text{Cl}_\theta(A) \leq H$ whenever $A \leq H$ and $H$ is fuzzy $\theta gs$-open set in $X$. Hence $A$ is fuzzy $\theta g^s$- closed set in $X$.

**Theorem 3.3.** Every fuzzy $\alpha$-closed set is fuzzy $g'''\alpha\theta$- closed set in a fuzzy topological space $(X, \tau)$.

**Proof:** Obvious from Definition 3.5 and Definition 3.1 (ii)

**Theorem 3.4.** Every fuzzy closed set is fuzzy $g'''\theta$- closed set in a fuzzy topological space $(X, \tau)$.
V.Chandrasekar and G.Anandajothi

**Proof:** Obvious from Definition 3.3.

**Theorem 3.5.** Every fuzzy semi-closed set is fuzzy $g^{s}\theta$- closed set in a fuzzy topological space $(X, \tau)$.

**Proof:** Obvious from Definition 3.4

**Remark 3.2.** The converse of the theorems 3.2, 3.3, 3.4 and 3.5 need not be true as shown in the following example.

**Example 3.2.** Let $X = \{a\}$. Fuzzy sets $A$, $B$ and $D$ are defined by

$A(a) = 0.5$ ; $B(a) = 0.4$ ; $D(a) = 0.7$.

Let $\tau = \{0, A, B, 1\}$. Then $D$ is

(i) fuzzy $\theta g^{s}$- closed set but it is not a fuzzy semi-$\theta$-closed set in $(X, \tau)$.

(ii) fuzzy $g^{m}\alpha$- closed set but it is not a fuzzy $\alpha$-closed set in $(X, \tau)$.

(iii) fuzzy $g^{m}\theta$- closed set but it is not a fuzzy closed set in $(X, \tau)$.

(iv) fuzzy $g^{s}\theta$- closed set but it is not a fuzzy semi-closed set in $(X, \tau)$.

**Theorem 3.6.** Every fuzzy $\theta g^{m}$-closed set is fuzzy $\theta g^{s}$- closed set in a fuzzy topological space $(X, \tau)$.

**Proof:** Let $A$ be a fuzzy $\theta g^{m}$-closed set in $X$. Let $H$ be a $f \theta$ gs- open set in $X$ such that $A \leq H$. Since, $s Cl_{\theta}(A) \leq Cl_{\theta}(A) \leq H$. Therefore $s Cl_{\theta}(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta$ gs-open set in $X$. Hence $A$ is fuzzy $\theta g^{s}$-closed set in $X$.

**Remark 3.3.** The converse of the above theorem need not be true as shown in the following example.

**Example 3.3.** Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows

$A(a) = 0.1$ , $A(b) = 0.2$ ; $B(a) = 0.5$ , $B(b) = 0.4$. Let $\tau = \{0, B, 1\}$. Then $A$ is fuzzy $\theta g^{s}$- closed but it is not $f \theta g^{m}$-closed.

**Theorem 3.7.** Every fuzzy $\theta g^{m}$-closed set is fuzzy $g^{m}\theta$- closed set in a fuzzy topological space $(X, \tau)$.

**Proof:** Let $A$ be a fuzzy $\theta g^{m}$-closed set in $X$. Let $H$ be a $f \theta$ gs- open set in $X$ such that $A \leq H$. Since, $Cl(A) \leq Cl_{\theta}(A) \leq H$. Therefore $Cl A \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta$ gs-open set in $X$. Hence $A$ is fuzzy $g^{m}\theta$-closed set in $X$.

**Remark 3.4.** The converse of the above theorem need not be true as shown in the following example.

**Example 3.4.** Let $X = \{a, b\}$ and the fuzzy sets $A$, $B$ and $D$ be defined as follows $A(a) = 0.4$ , $A(b) = 0.4$ ; $B(a) = 0.5$ , $B(b) = 0.4$ ; $D(a) = 0.5$ , $D(b) = 0.6$. 

42
Another Theta Generalized Closed Sets in Fuzzy Topological Spaces

Let \( \tau = \{0, A, B, 1\} \). Then A is fuzzy \( g'' \theta \)-closed but it is not \( f \theta g''' \)-closed.

**Theorem 3.8.** Every fuzzy \( \theta g''' \)-closed set is fuzzy \( g'' \alpha \theta \)-closed set in a fuzzy topological space \((X, \tau)\).

**Proof:** Let \( A \) be a fuzzy \( \theta g''' \)-closed set in \( X \). Let \( H \) be a \( f \theta g \)-open set in \( X \) such that \( A \leq H \). Since, \( \alpha Cl(A) \leq Cl_\theta(A) \leq H \). Therefore \( \alpha Cl A \leq H \), whenever \( A \leq H \) and \( H \) is fuzzy \( \theta g \)-open set in \( X \). Hence \( A \) is fuzzy \( g'' \alpha \theta \)-closed set in \( X \).

**Remark 3.5.** The converse of the above theorem need not be true as shown in the following example.

**Example 3.5.** Let \( X = \{a, b\} \) and the fuzzy sets \( A \) and \( B \) be defined as follows
\[
A(a) = 0.3, \quad A(b) = 0.4; \quad B(a) = 0.7, \quad B(b) = 0.6
\]
Let \( \tau = \{0, A, 1\} \). Then \( B \) is fuzzy \( g'' \alpha \theta \)-closed but it is not \( f \theta g''' \)-closed.

**Theorem 3.9.** Every fuzzy \( \theta g \)-\( g \)-closed set is fuzzy \( g \)-\( g \)-closed set in a fuzzy topological space \((X, \tau)\).

**Proof:** Let \( A \) be a fuzzy \( \theta g \)-\( g \)-closed set in \( X \). Let \( H \) be a \( f \theta g \)-open set in \( X \) such that \( A \leq H \). Since, \( sCl(A) \leq sCl_\theta(A) \leq H \). Therefore \( sCl A \leq H \), whenever \( A \leq H \) and \( H \) is fuzzy \( \theta g \)-open set in \( X \). Hence \( A \) is fuzzy \( g \)-\( g \)-closed set in \( X \).

**Remark 3.6.** The converse of the above theorem need not be true as shown in the following example.

**Example 3.6.** Let \( X = \{a\} \) and the fuzzy sets \( A \), \( B \) and \( D \) be defined as follows
\[
A(a) = 0.6; \quad B(a) = 0.5; \quad D(a) = 0.1
\]
Let \( \tau = \{0, A, B, 1\} \). Then \( D \) is fuzzy \( g \)-\( g \)-\( g \)-closed but it is not \( f \theta g \)-\( g \)-closed.

**Theorem 3.11.** Every fuzzy \( \theta g''' \)-closed set is fuzzy \( g \)-\( g \)-closed set in a fuzzy topological space \((X, \tau)\).

**Proof:** It is clear from Theorem 3.6 and Theorem 3.10

**Remark 3.7.** The converse of the above theorem need not be true as shown in the following example.

**Example 3.7.** \( X = \{a, b\} \). Consider the fuzzy topology \( \tau \) as in Example 3.3 where \( A \) is defined by \( A(a) = 0.1, \quad A(b) = 0.2 \). Clearly \( A \) is fuzzy \( g \)-\( g \)-\( g \)-closed set but not fuzzy \( \theta g''' \)-closed set.

**Theorem 3.8.** Every fuzzy \( g'' \theta \)-closed set is fuzzy \( g'' \alpha \theta \)-closed set in a fuzzy topological space \((X, \tau)\).
Proof: Let $A$ be a fuzzy $g^\alpha$-closed set in $X$. Let $H$ be a $f\theta$-gs-open set in $X$ such that $A \subseteq H$. Since, $\alpha Cl(A) \leq Cl(A) \leq H$. Therefore $\alpha Cl A \leq H$, whenever $A \subseteq H$ and $H$ is fuzzy $\theta$gs-open set in $X$. Hence $A$ is fuzzy $g^\alpha$-closed set in $X$.

Remark 3.8. The converse of the above theorem need not be true as shown in the following example.

Example 3.8. Let $X = \{a\}$ and the fuzzy sets $A$ and $D$ be defined as follows

$A(a) = 0.5$; $B(a) = 0.3$; $D(a) = 0.6$. Let $\tau = \{0, A, B, 1\}$. Then $D$ is fuzzy $g^\alpha$-closed but it is not $f g^\alpha$-

Remark 3.9. The following examples shows that the fuzzy $\theta g^\alpha$-closed set is independent of fuzzy $g^\alpha$-closed, fuzzy $g^\alpha$-s-closed, fuzzy $g^\alpha$-closed, fuzzy semi-$\theta$-closed, fuzzy $\theta$-g-closed, fuzzy $\theta$-gs closed, fuzzy $\theta$g$s$-closed.

Example 3.9. Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows

$A(a) = 0.6$; $A(b) = 0.6$; $B(a) = 0.5$; $B(b) = 0.6$.

Let $\tau = \{0, A, B, 1\}$. Then $B$ is fuzzy $\theta g^\alpha$-closed but it is not fuzzy $g^\alpha$-closed.

Example 3.10. Let $X = \{a, b\}$ Consider the Fuzzy topology $\tau$ as in Example 3.4 where $D$ is defined by $D(a) = 0.5$, $D(b) = 0.6$. Clearly $D$ is both fuzzy closed and fuzzy $g^\alpha$-closed but it is not fuzzy $\theta g^\alpha$-closed.

Example 3.11. Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows

$A(a) = 0.6$, $A(b) = 0.6$; $B(a) = 0.7$, $B(b) = 0.8$.

Let $\tau = \{0, A, B, 1\}$. Then $B$ is fuzzy $\theta g^\alpha$-closed but it is neither $g^\alpha$-closed nor fuzzy $g^\alpha$s-closed.

Example 3.12. Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows

$A(a) = 0.6$, $A(b) = 0.6$; $B(a) = 0.3$, $B(b) = 0.4$.

Let $\tau = \{0, A, B, 1\}$. Then $B$ is both fuzzy $g^\alpha$-closed and fuzzy $g^\alpha$s-closed but it is not fuzzy $g^\alpha$-closed.

Example 3.13. Let $X = \{a, b\}$ Consider the Fuzzy topology $\tau$ as in Example 3.4 where $E$ is defined by $E(a) = 0.6$, $E(b) = 0.6$. Clearly $E$ is fuzzy semi-$\theta$-closed but it is not fuzzy $\theta g^\alpha$-closed and $D$ is defined by $D(a) = 0.5$, $D(b) = 0.6$ is fuzzy closed but it is not fuzzy $\theta g^\alpha$-closed.

Example 3.14. Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows

$A(a) = 1$, $A(b) = 0.2$; $B(a) = 0.5$, $B(b) = 0.9$. 

44
Another Theta Generalized Closed Sets in Fuzzy Topological Spaces

Let \( \tau = \{0, A, 1\} \). Then B is fuzzy \( \theta g''' \)-closed but it is neither fuzzy closed nor fuzzy semi-\( \theta \)-closed.

**Example 3.15.** Let \( X = \{a\} \) and the fuzzy sets \( A, B \) and D be defined as follows
\[
A(a) = 0.7; \quad B(a) = 0.8; \quad D(a) = 0.4.
\]
Let \( \tau = \{0, A, B, 1\} \). Then D is fuzzy \( \theta g''' \)-closed but it is not fuzzy \( \theta g \)-closed.

**Example 3.16.** Let \( X = \{a, b\} \) and the fuzzy sets \( A \) and \( B \) be defined as follows
\[
A(a) = 0.5, \quad A(b) = 0.5; \quad B(a) = 0.2, \quad B(b) = 0.3.
\]
Let
\[
\tau = \{0, A, 1\}.
\]
Then B is fuzzy \( \theta g \)-closed but it is not fuzzy \( \theta g''' \)-closed.

**Example 3.17.** Let \( X = \{a\} \) and the fuzzy sets \( A, B \) be defined as follows
\[
A(a) = 0.4; \quad B(a) = 0.3.
\]
Let \( \tau = \{0, A, 1\} \). Then B is both fuzzy \( \theta sg \)-closed and fuzzy \( \theta gs \)-closed, but it is not fuzzy \( \theta g''' \)-closed.

**Example 3.18.** Let \( X = \{a\} \) and the fuzzy sets \( A, B \) be defined as follows
\[
A(a) = 0.7; \quad B(a) = 0.9.
\]
Let \( \tau = \{0, A, 1\} \). Then B is fuzzy \( \theta g''' \)-closed but it is not fuzzy \( \theta sg \)-closed.

**Example 3.19.** Let \( X = \{a\} \) and the fuzzy sets \( A, B \) be defined as follows
\[
A(a) = 0.6; \quad B(a) = 0.5.
\]
Let \( \tau = \{0, A, 1\} \). Then B is fuzzy \( \theta g''' \)-closed but it is not fuzzy \( \theta gs \)-closed.

**Remark 3.10.** From the above results and examples we have the following diagram of implications.
4. Properties of fuzzy $\theta g''$-closed sets

**Theorem 4.1** Let $X$ be a fts, then the union of two fuzzy $\theta g''$-closed sets is fuzzy $\theta g''$-closed set.

**Proof:** Suppose that $A$ and $B$ are fuzzy $\theta g''$-closed sets in $X$ and let $H$ be a fuzzy $\theta gs$ open set in $X$ such that $A \lor B \leq H$. Since $A$ and $B$ are fuzzy $\theta g''$-closed sets, we have $Cl_{\theta}(A) \lor Cl_{\theta}(B) \leq H$.

Since $Cl_{\theta}(A \lor B) \leq Cl_{\theta}(A) \lor Cl_{\theta}(B)$, therefore $Cl_{\theta}(A \lor B) \leq H$, whenever $A \lor B \leq H$ and $H$ is fuzzy $\theta gs$ open set in $X$.

**Theorem 4.2.** If $A$ is a fuzzy $\theta g''$-closed set in $(X, \tau)$ and $A \leq B \leq Cl_{\theta}(A)$, then $B$ is fuzzy $\theta g''$-closed set in $(X, \tau)$.

**Proof:** Let $A$ be a fuzzy $\theta g''$-closed set in $(X, \tau)$. Let $B \leq H$ where $H$ is a fuzzy $\theta gs$-open set in $X$. Then $A \leq H$. Since $A$ is fuzzy $\theta g''$-closed set in, it follows that $Cl_{\theta}(A) \leq H$.

Now $B \leq Cl_{\theta}(A)$ implies $Cl_{\theta}(B) \leq Cl_{\theta}(Cl_{\theta}(A)) = Cl_{\theta}(A)$. We get, $Cl_{\theta}(B) \leq H$.

Hence, $B$ is fuzzy $\theta g''$-closed set in $(X, \tau)$.

**Theorem 4.3.** If a fuzzy set $A$ of a fuzzy topological space $X$ is both fuzzy $\theta gs$-open and fuzzy $\theta g''$-closed then it is fuzzy $\theta$-closed.
Another Theta Generalized Closed Sets in Fuzzy Topological Spaces

**Proof:** Suppose that a fuzzy set $A$ of $X$ is both fuzzy $\theta$-gs-open and fuzzy $\theta$-gs-closed. Now $A \supseteq Cl_\theta(A)$ whenever $A \supseteq A$ and $A$ is fuzzy $\theta$-gs-open. Since $A \subseteq Cl_\theta(A)$. We get $A = Cl_\theta(A)$. Hence $A$ is fuzzy $\theta$-closed in $X$.

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