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Vertex Domination Critical in Circulant Graphs

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Abstract. A graph G is vertex domination critical if for any vertex v of G, the domination number of G – v is less than the domination number of G. We call these graphs γ-critical if domination number of G is γ. In this paper, we determine the domination and the total domination number of Cir(n,A) for two particular generating sets A of Zn, and then study γ-critical in these graphs.

Keywords: Domination, total domination, circulant graph.

AMS Mathematics Subject Classification (2010): 05C72

1. Introduction

A vertex in a graph G dominates itself and its neighbors. A set of vertices S in a graph G is a dominating set, if each vertex of G is dominated by some vertex of S. The domination number γ(G) of G is the minimum cardinality of a dominating set of G. A dominating set S is called a total dominating set if each vertex v of G is dominated by some vertex u ≠ v of S. The total domination number of G, denoted by γt(G), is the minimum cardinality of a total dominating set of G.

We denote the open neighborhood of a vertex v of G by N(v), or just N(v), and its closed neighborhood by N[v]. For a vertex set S ⊆ V(G), N(S) = ∪v∈S N(v) and N[S] = ∪v∈S N[v]. So, a set of vertices S in G is a dominating set, if N[S] = V(G). Also, S is a total dominating set, if N(S) = V(G). For notation and graph theory terminology in general we follow [3]. Rashmanlou and Pal et al. [5-17] studied different kinds of fuzzy graphs. We call a dominating set of cardinality γ(G), a γ(G) – set and a total dominating set of cardinality γt(G), a γt(G) – set. A graph G is called vertex domination critical if γ(G – v) < γ(G), for every vertex v in G. For references on the vertex domination critical graphs see [1,2,3].

Jafari Rad [4], determines the domination number and the total domination number of graph Cir(n, {1, 3}), for any integer n, and then study γ – criticality in Cir(n, {1, 3}).

Let n ≥ 7 be a positive integer. The circulant graph Cir(n, A) where A = {1, n – 1, 3, n – 3, 5, n – 5, …, 2k – 1, n – (2k – 1), 2k + 1, n – (2k + 1)} is the graph with vertex set {v0,
A.A.Talebi, M.Zameni and Hossein Rashmanlou

Let \( G = \{ v_0, v_1, \ldots, v_n \} \) and edge set \( \{ [v_i, v_j] : i \in \{ 0, 1, \ldots, n-1 \}, j \in \{ 1, n-1, 2, n-2, \ldots, 2k+1, n-(2k+2) \} \} \), \( k \) is an integer such that \( 0 \leq k < \left\lfloor \frac{n-3}{4} \right\rfloor \).

Let \( n \geq 9 \) be a positive integer. The circulant graph \( \text{Cir}(n,B) \) where \( B = \{ l, n-l, 2, n-2, 4, n-4, \ldots, 2k, n-2k, 2k+1, n-(2k+2) \} \) is the graph with vertex set \( \{ v_0, v_1, \ldots, v_n \} \), and edge set \( \{ [v_i, v_j] : i \in \{ 0, 1, \ldots, n-1 \}, j \in \{ 1, n-1, 2, n-2, \ldots, 2k+1, n-(2k+2) \} \} \), \( k \) is an integer such that \( 0 \leq k < \left\lfloor \frac{n-5}{4} \right\rfloor \).

All arithmetic on the indices is assumed to be modulo \( n \).

In this paper, we first determine the domination number and the total domination number in the circulant graphs \( \text{Cir}(n,A) \) and \( \text{Cir}(n,B) \) for any integer \( n \), and then study \( \gamma \) – criticality and \( \gamma_1(G) \) – criticality in these class of graphs.

For two vertices \( x, y \) in a graph \( G \) we denote the distance between \( x \) and \( y \) by \( d_G(x,y) \), or just \( d(x,y) \).

### 2. Domination and total domination

Let \( G \) be a circulant graph with \( n \) vertices. Let cycle \( C = C(G) \) be the subgraph of \( G \) with vertex set \( \{ v_0, v_1, \ldots, v_n \} \) and edge set \( \{ [v_i, v_{i+1}] : i \in \{ 0, 1, \ldots, n-1 \} \} \). For a subset \( S \subseteq V(G) \) with at least three vertices, we say that \( x, y \in S \) are consecutive if there is no vertex \( z \in S \) such that \( z \) lies between \( x \) and \( y \) in \( C \). For two consecutive vertices \( x, y \) in a subset of vertices \( S \), we define \( |x - y| = d_G(x,y) \). So, \( |x - y| \) equals to the number of edges in a shortest path between \( x \) and \( y \) in the cycle \( C \).

**Theorem 2.1.** For any integer \( n \geq 7 \),

\[
\gamma(\text{Cir}(n,A)) = \begin{cases} 
\left\lfloor \frac{n}{2k+3} \right\rfloor + 1 & n \equiv 4,6,8, \ldots, 2k+2 \pmod{2k+3} \\
\left\lfloor \frac{n}{2k+3} \right\rfloor & \text{otherwise}
\end{cases}
\]

**Proof:** Let \( S \) be a \( \gamma(G) \)-set of \( G = \text{Cir}(n,A) \). Any vertex of \( G \) dominates \( 2k+3 \) vertices of \( G \) including itself, so \( |S| \geq \left\lfloor \frac{n}{2k+3} \right\rfloor + 1 \).

We claim that if \( n = 2t \pmod{2k+3} \), for an integer \( t \) such that \( 2 \leq t \leq k+1 \), then \( |S| \geq \left\lfloor \frac{n}{2k+3} \right\rfloor + 1 \).

To see this, assume to the contrary that \( n \equiv 2t \pmod{2k+3} \), and \( |S| = \left\lfloor \frac{n}{2k+3} \right\rfloor \). There are two consecutive vertices \( v, v' \in S \) such that \( |v-v'| < 2k+3 \). Let \( v'' \neq v \) be a consecutive vertex of \( v' \). Without loss of generality assume that \( |v'' - v| = 2k+3+2t \). Then there are \( 2k+2+2t \) possibilities for \( v'' \) to lie between \( v \) and \( v' \). In each possibly there exists a vertex between \( v \) and \( v'' \) which is not dominated by \( \{ v, v', v'' \} \), a contradiction. Hence, for \( n \equiv 2t \pmod{2k+3} \), \( |S| \geq \left\lfloor \frac{n}{2k+3} \right\rfloor + 1 \).

Now it is sufficient to get a dominating set \( S \) of required cardinality. We consider the following cases:

1. For \( n \equiv 4 \pmod{2k+3} \), \( S = \{ v_{2k+3i} : 0 \leq i < \left\lfloor \frac{n}{2k+3} \right\rfloor \} \cup \{ v_{n-2} \} \).
Vertex Domination Critical in Circulant Graphs

2. For \( n \equiv 6, 8, 10, 12, 14, \ldots, 2k+2 \pmod{2k+3} \), \( S = \{ v_{(2k+3)i}; 0 \leq i < \left\lfloor \frac{n}{2k+3} \right\rfloor \} \cup \{ v_{n,1} \} \).

3. For \( n \equiv 4, 6, 8, 10, \ldots, 2k+2 \pmod{2k+3} \), \( S = \{ v_{(2k+3)i}; 0 \leq i < \left\lfloor \frac{n}{2k+3} \right\rfloor \} \).

In each of the above cases, \( S \) is a dominating set for \( Cir(n, A) \) of cardinality \( \left\lceil \frac{n}{2k+3} \right\rceil + 1 \) when \( n \equiv 4, 6, 8, \ldots, 2k+2 \pmod{2k+3} \), and of cardinality \( \left\lfloor \frac{n}{2k+3} \right\rfloor \) when \( n \equiv 6, 8, 10, \ldots, 2k+2 \pmod{2k+3} \). Hence, the result follows.

**Theorem 2.2.** For any integer \( n \geq 9 \),

\[
\gamma(Cir(n, B)) = \begin{cases} 
\left\lceil \frac{n}{2k+5} \right\rceil + 1, & n \equiv 6, 8, 10, \ldots, 2k+4 \pmod{2k+5} \\
\left\lfloor \frac{n}{2k+5} \right\rfloor, & \text{otherwise}
\end{cases}
\]

**Proof:** Let \( S \) be a \( \gamma(G) \)-set of \( G = Cir(n, B) \). Any vertex of \( G \) dominates \( 2k+5 \) vertices of \( G \) including itself, so \( |S| \geq \left\lceil \frac{n}{2k+5} \right\rceil \).

We claim that if \( n \equiv 2t \pmod{2k+5} \), \( t \) is an integer such that \( 3 \leq t \leq k+1 \), then \( |S| \geq \left\lceil \frac{n}{2k+5} \right\rceil + 1 \). To see this, assume to the contrary that \( n \equiv 2t \pmod{2k+5} \), and \( |S| = \left\lfloor \frac{n}{2k+5} \right\rfloor \). There are two consecutive vertices \( v, v' \in S \) such that \( |l - l'| < 2k+5 \). Let \( v'' = v' \neq v \) is a consecutive vertex of \( v' \). Without loss of generality we assume that \( |v'' - v| = 2k+5+2t \). Then there are \( 2k+4+2t \) possibilities for \( v'' \) to lies between \( v \) and \( v'' \). In each possibly there exists a vertex between \( v \) and \( v'' \) which is not dominated by \( \{v, v', v''\} \), a contradiction. Hence, for \( n \equiv 2t \pmod{2k+5} \), \( |S| \geq \left\lceil \frac{n}{2k+5} \right\rceil + 1 \).

Now it is sufficient to get a dominating set \( S \) of required cardinality. We consider the following cases:

1. For \( n \equiv 6, 8, 10, 12, 14, \ldots, 2k+4 \pmod{2k+5} \), \( S = \{ v_{(2k+5)i}; 0 \leq i < \left\lfloor \frac{n}{2k+5} \right\rfloor \} \cup \{ v_{n,1} \} \).

2. For \( n \equiv 6, 8, 10, \ldots, 2k+4 \pmod{2k+5} \), \( S = \{ v_{(2k+5)i}; 0 \leq i < \left\lfloor \frac{n}{2k+5} \right\rfloor \} \).

In each of the above cases \( S \) is a dominating set for \( Cir(n, B) \) of cardinality \( \left\lceil \frac{n}{2k+5} \right\rceil + 1 \) when \( n \equiv 6, 8, \ldots, 2k+4 \pmod{2k+5} \), and of cardinality \( \left\lfloor \frac{n}{2k+5} \right\rfloor \) when \( n \equiv 6, 8, 10, \ldots, 2k+4 \pmod{2k+5} \). Hence, the result follows.

**Theorem 2.3.** For any integer \( n \geq 7 \),

\[
\gamma_t(Cir(n, A)) = \begin{cases} 
\left\lceil \frac{2n}{4k+4} \right\rceil + 1, & n \equiv 2, 4, 6, \ldots, 2k+2 \pmod{4k+4} \\
\left\lfloor \frac{2n}{4k+4} \right\rfloor, & \text{otherwise}
\end{cases}
\]

**Proof.** Let \( S \) be a \( \gamma_t \)-set of \( G = Cir(n, A) \). Note that \( |A| = 2k+2 \) and \( G \) is \( 2k+2 \)-regular.

From the definition of the total domination number, it follows that \( \left\lfloor \frac{n}{2k+2} \right\rfloor \leq \gamma_t(G) \), \( \gamma_t(G) = |S| \).

For \( n \equiv 2j \pmod{4k+4} \), \( j \) is an integer such that \( 0 \leq j < 2k+2 \), we have
A.A. Talebi, M. Zameni and Hossein Rashmanlou

\[ \frac{n}{2k+2} = \frac{2n}{4k+4}, \text{ so } \frac{2n}{4k+4} \leq \gamma(G). \]

For \( n \equiv 2j \pmod{4k+4}, j \) is an integer such that \( 0 \leq j < 2k+1 \), \( n \) can be written as \( n = (4k+4)l + j = 2((2k+2)l + j) \), which \( l \) is integer and \( n \) is an even number. We partition \( V(G) \) into two disjoint sets \( I_1 = \{v_j, v_{j+1}, v_{j+2}, \ldots, v_{j+(n-3)}\} \) and \( I_2 = \{v_{n-1}, v_{n-2}, \ldots, v_{n-d}\} \). Note that \( |I_1| = |I_2| = (2k+2)l + j \). For any \( x \in I_1, N(x) \subseteq I_2, \) for any \( y \in I_2, N(y) \subseteq I_1 \). It follows that \( G \) is a balanced bipartite graph with bipartition sets \( I_1 \) and \( I_2 \). We can write \( S = S_1 \cup S_2 \), such that \( S_1 \subseteq I_2, \ S_2 \subseteq I_1, \) is dominated by \( S_1 \), \( l \leq i \leq 2 \) and \( |S_1| = |S_2| \).

If \( 0 \leq j \leq k+1 \), then \( |S_1| = |S_2| \leq \frac{(2k+2)l+j}{2k+2} = l + 1 \) and \( \gamma(G) = |S| = |S_1| + |S_2| \geq 2\left(\frac{(2k+2)l+j}{2k+2}\right) = 2l + 2 \). On the other hand \( 2l + 2 = \frac{(4k+4)(2l+j)}{4k+4} + 1 \), and so

\[ \gamma(G) \geq \frac{(4k+4)(2l+j)}{4k+4} + 1. \]

If \( k+2 \leq j \leq 2k+1 \), then \( |S_1| = |S_2| \geq \frac{(2k+2)l+j}{2k+2} = l + 1 \) and \( \gamma(G) = |S| = |S_1| + |S_2| \geq 2\left(\frac{(2k+2)l+j}{2k+2}\right) = 2l + 2 \). On the other hand \( 2l + 2 = \frac{(4k+4)(2l+j)}{4k+4} \), and so

\[ \gamma(G) \geq \frac{(4k+4)(2l+j)}{4k+4}. \]

If \( j = 0 \), then \( |S_1| = |S_2| \geq \frac{(2k+2)l+j}{2k+2} = l \) and \( \gamma(G) = |S| = |S_1| + |S_2| \geq 2\left(\frac{(2k+2)l+j}{2k+2}\right) = 2l \)

On the other hand \( 2l = \frac{(4k+4)(2l+j)}{4k+4} \), and so \( \gamma(G) \geq \frac{(4k+4)(2l+j)}{4k+4} \).

Now it is sufficient to define a total dominating set \( S \) of required cardinality. We consider the following cases:

1. For \( n \equiv 0 \pmod{4k+4}, \) \( S = \{v_{i+2k+2j}, \ldots, v_{i+4k+4j+2k+2}\} \), \( 0 \leq i < \frac{n}{4k+4} \).
2. For \( n \equiv 1, 3, 5, 7, \ldots, 2k+1 \pmod{4k+4}, \) \( S = \{v_{i+2k+2j+2k+1}, \ldots, v_{i+4k+4j+2k+2}\} \), \( 0 \leq i < \frac{n}{4k+4} \). \( \cup \{v_0\} \).
3. For the cases \( n \equiv 2, 4, 6, \ldots, 2k+2 \pmod{4k+4} \) and \( n \equiv 2k+3, 2k+4, 2k+5, 2k+6, \ldots, 4k+2, 4k+3 \pmod{4k+4} \), \( S = \{v_{i+2k+2j+2k+1}, \ldots, v_{i+4k+4j+2k+2}\} \), \( 0 \leq i < \frac{n}{4k+4} \). \( \cup \{v_{n-2k-1}, v_{n-2k-2}\} \).

In each of the above cases, \( S \) is a total dominating set of \( Cir(n, A) \), cardinality of \( S \) is \( \frac{n}{2k+2} + 1 \) when \( n = 2, 4, \ldots, 2k+2 \pmod{4k+4} \), and cardinality of \( S \) is \( \frac{n}{2k+2} \) when \( n = 2, 4, 6, \ldots, 2k+2 \pmod{4k+4} \). Hence, the result follows.

**Lemma 2.1.** Let \( S \) be a subset of vertices of \( G = Cir(n, B) \) with \( k \geq 3 \) and \( G[S] \) has no isolated vertices. If \( |S| \) is even, then \( S \) dominates at most \((2k+3)|S|\) vertices of \( G \).

**Proof:** Let \( S \) be a subset of vertices of \( G \) with \(|S| = t\), where \( t \) is even. Any two adjacent vertices of \( S \) dominate \( 4k+6 \) vertices of \( G \) including themselves. \( S \) dominates at most \((4k+6)|S| - 2 \cdot (2k+3)|S| = 2k+6 \) vertices of \( G \).

**Lemma 2.2.** Let \( S \) be a subset of vertices of \( G = Cir(n, B) \) with \( k \geq 3 \) and \( G[S] \) has no isolated vertices. If \( |S| \) is odd, then \( S \) dominates at most \((2k+3)|S| - (k+1)\) vertices of \( G \).

**Proof:** Let \( S \) be a subset of vertices of \( G \) with \(|S| = t\), where \( t \) is odd. Without loss of generality we may assume that \( G[S] \) has \( d = \left|\frac{|S|-3}{2}\right| + 1 \) components \( G_1, G_2, \ldots, G_d \) where \( |V(G_i)| = 3 \) and \( |V(G_j)| = 2 \) for \( i = 2, 3, 4, \ldots, d \). Let \( V(G_i) = \{x, y, z\} \), then \( \{x, y, z\} \)
Vertex Domination Critical in Circulant Graphs
dominishes at most 5k+8 vertices of G. S dominates at most (4k+6) \left(\frac{|S|-3}{2}\right) + 5k + 8 = (2k+3)t-(k+1) vertices of G.\blacksquare

**Theorem 2.4.** For any integer \( n \geq 21 \) and \( k \geq 3 \),

\[ \gamma(Cir(n, B)) = \begin{cases} \frac{n}{\gcd(3,4,5)} & n \equiv 3, 4, 5, \ldots, 2k+3 \pmod{4k+6} \\ \frac{n}{\gcd(3,4,5)} & \text{otherwise} \end{cases} \]

**Proof:** Let \( S \) be a \( \gamma_t \)-set of \( G = Cir(n, B) \). It follows from Lemma 2.1 and Lemma 2.2 that \( |S| \geq \frac{n}{\gcd(3,4,5)} \).

We claim that if \( n \equiv 3, 4, 5, \ldots, 2k+3 \pmod{4k+6} \) and \( S \) is a total dominating for \( G \), then \( |S| \geq \frac{n}{\gcd(3,4,5)} + 1 \).

To see this, assume to the contrary that \( |S| = \frac{n}{\gcd(3,4,5)} \). We have \( n = (4k+6)l+j \), where \( l \) is a positive integer, \( j \in \{3, 4, 5, \ldots, 2k+3\} \) then \( |S| = \frac{(4k+6)(l+j)}{\gcd(3,4,5)} = 2l + j \) is an odd number. So, the induced subgraph \( G[S] \) has an odd component \( H \) with at least three vertices. We proceed to prove the following facts.

(i) Any component of \( G[S] \) has at most three vertices.
Assume to the contrary that \( G_1 \) is a component of \( G[S] \) and \( G_1 \) has at least 4 vertices. Without loss of generality assume that \( G_1 \) has 4 vertices. Then \( S \) dominates at most \( 6k+10+(4k+6)\left(\frac{|S|-3}{2}\right) + 5k + 8 = (4k+6)l-k \) vertices of \( G \), a contradiction.

(ii) \( H \) is the only odd component of \( G[S] \).
Assume to the contrary that \( H' \neq H \) is a component of \( G[S] \) with \( |V(H')| \) odd. It follows from fact (i) that \( |V(H')| = 3 \). Since \( |S| \) is odd, there is another component \( H' \) with three vertices. Now \( S \) dominates at most \( (4k+6)\left(\frac{|S|-3}{2}\right) + 3(5k+8) = (4k+6)l-k \) vertices of \( G \), a contradiction.

For \( n \equiv k+3+k+4+k+5, \ldots, 2k+3 \pmod{4k+6} \) and we have \( V(H) = \{v_{ij} \} \) and \( S \) dominates \( (4k+6)\left(\frac{|S|-3}{2}\right) + 5k + 8 = (4k+6)(l-k) \) vertices of \( G \), a contradiction.

For \( n \equiv 3, 4, 5, \ldots, k+2 \pmod{4k+6} \). We have \( n = (4k+6)l+j \), where \( l \) is a positive integer, \( j \in \{3, 4, 5, \ldots, k+2\} \). It follows from facts that \( G[S] \) has \( l = \left(\frac{|S|-3}{2}\right) + 1 \) components \( G_1, G_2, \ldots, G_l \) where \( |V(G_i)| = 2 \) for \( i = 2, 3, 4, \ldots, l \) and \( |V(G_l)| = 3 \). Any two adjacent vertices of \( G \) dominates at most \( 4k+6 \) consecutive vertices of \( V(G) \).

\( V(G) \) can be partitioned into \( l \) subsets \( I_1 = \{v_{0l}, v_{1l}, v_{2l}, \ldots, v_{dl}\} \), \( I_2 = \{v_{dl+1}, v_{dl+2}, \ldots, v_{d(1-l)}\} \), \( I_3 = \{v_{d(1-l)+1}, v_{d(1-l)+2}, \ldots, v_{dl}\} \), \( I_4 = \{v_{dk+6(d-l)+1}, v_{dk+6(d-l)+2}, \ldots, v_{dl}\} \).

Note that, \( |I_1| = 4k+6 \) for \( i = 1, 2, 3, \ldots, l-1 \) and \( 4k+9 \leq |I_l| \leq 5k+8 \).

Without loss of generality we may assume that \( I_1 \) is dominated by \( \{v_{2k+2}, v_{2k+3}\} \) of \( S \) and \( I_l \) is dominated by \( \{v_{2k+2}, v_{2k+3}\} \) then each of \( I_i \) is by two adjacent vertices of \( S \). Then, vertices \( I_i (4k+9 \leq |I| \leq 5k+8) \) is dominated by three consecutive vertices of \( S \). In

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59
each possibility there exists at least one vertex in I which is not dominated by this three vertices, a contradiction. This completes the claim.

Now it is sufficient to define a total dominating set \( S \) of required cardinality.

We consider the following case:

1. For \( n \equiv 0 \pmod{4k+6} \), \( S = \{ v_{(4k+6)m+2k+2}, v_{(4k+6)m+2k+3} : 0 \leq m < \left\lfloor \frac{n}{4k+6} \right\rfloor \} \).

2. For \( n \equiv 1,2 \pmod{4k+6} \), \( S = \{ v_{(4k+6)m+2k+2}, v_{(4k+6)m+2k+3} : 0 \leq m < \left\lfloor \frac{n}{4k+6} \right\rfloor \} \cup \{ v_n \} \).

3. For \( n \equiv 3,4,5,...,4k+5 \pmod{4k+6} \), \( S = \{ v_{(4k+6)m+2k+1}, v_{(4k+6)m+2k+2} : 0 \leq m < \left\lfloor \frac{n}{4k+6} \right\rfloor \} \cup \{ v_{n-(2k+2)}, v_{n-(2k+1)} \} \).

Lemma 2.3. Let \( S \) be a subset of vertices of \( G = Ciri(n,B) \) with \( k=2 \) and \( G[S] \) has no isolated vertices. If \( |S| \) is even, then \( S \) dominates at most \( 7|S| \) vertices of \( G \).

**Proof:** Let \( S \) be subset of vertices of \( G \) with \( |S|=t \), where \( t \) is even. Any two adjacent vertices of \( S \) dominate 14 vertices of \( G \) including them selves. \( S \) dominates at most \( 14\left( \left\lfloor \frac{|S|}{2} \right\rfloor \right) = 7|S| \) vertices of \( G \).

Lemma 2.4. Let \( S \) be a subset of vertices of \( G = Ciri(n,B) \), \( k = 2 \) and \( G[S] \) has no isolated vertices. If \( |S| \) is odd, then \( S \) dominates at most \( 7|S|-2 \) vertices of \( G \).

**Proof:** Let \( S \) be subset of vertices of \( G \) with \( |S|=t \), where \( t \) is odd. Without loss of generality we may assume that \( G[S] \) has \( d = \left\lfloor \frac{|S|-3}{2} \right\rfloor +1 \) components \( G_1, G_2, ..., G_d \), where \( |V(G_i)|=3 \) and \( |V(G_i)|=2 \) for \( i=2,3,4,...,d \). let \( V(G_1) = \{ x, y, z \} \), then \( \{ x, y, z \} \) dominates at most \( 19 \) vertices of \( G \). \( S \) dominates at most \( 14\left( \left\lfloor \frac{|S|-3}{2} \right\rfloor \right) +19 = 7(|S|-3)+19 = 7|S|-2 \) vertices of \( G \).

Theorem 2.5. For any integer \( n \geq 17 \) and \( k=2 \),

\[
\gamma_t(\text{Ciri}(n,B)) = \begin{cases} 
\left\lfloor \frac{n}{7} \right\rfloor +1 & n \equiv 3,4,5,6,7 \pmod{14} \\
\left\lfloor \frac{n}{7} \right\rfloor & \text{otherwise}
\end{cases}
\]

**Proof:** Let \( S \) be a \( \gamma_t \)-set of \( G = \text{Ciri}(n,B) \). It follows from Lemma 2.3 and Lemma 2.4 that \( |S| \geq \left\lfloor \frac{n}{7} \right\rfloor +1 \).

We claim that if \( n \equiv 3,4,5,6,7 \pmod{14} \) and \( S \) is a total dominating for \( G \), then \( |S| \geq \left\lfloor \frac{n}{7} \right\rfloor +1 \).

To see this, assume to the contrary that \( |S| = \left\lfloor \frac{n}{7} \right\rfloor \). We have \( n=14l+j \), where \( l \) is a positive integer, \( j \in \{3,4,5,6,7\} \). Then \( |S| = \left\lfloor \frac{14l+j}{7} \right\rfloor = 2l+1 \) is an odd number. So, the induced subgraph \( G[S] \) has a component \( H \) with at least three vertices. We proceed to prove following facts.

i. Any component of \( G[S] \) has at most three vertices.
Assume to the contrary that $H'$ is a component of $G[S]$ with $|V(H')|$ odd. It follows from fact \( i \) that $|V(H')|=3$. Since $|S|$ is odd, there is another component $H'$ with three vertices. Now $S$ dominates at most 
\[ 14 + (14) \left( \frac{|S|-3}{2} \right) + 24 = 14l + 1 \] vertices of $G$, a contradiction.

\[ ii. \] $H$ is the only odd component of $G[S]$.

Assume to the contrary that $H' \neq H$ is a component of $G[S]$ with $|V(H')|$ odd. It follows from fact \( i \) that $|V(H')|=3$. Since $|S|$ is odd, there is another component $H'$ with three vertices. Now $S$ dominates at most 
\[ 14 \left( \frac{|S|-3-3-3}{2} \right) + 3(19) = 14l + 1 \] vertices of $G$, a contradiction.

Hence, we conclude that $H$ is the only odd component of $G[S]$.

For $n \equiv 6,7 \pmod{14}$ and we have $V(H)=\{v_j, v_k, v_l\}$ and $S$ dominates $14 \left( \frac{|S|-3}{2} \right) + 19 = 14l + 5$ vertices of $G$, a contradiction.

For $n \equiv 3, 4, 5 \pmod{14}$. We have $n=14l+j$, where $l$ is a positive integer, $j \in \{3,4,5\}$. It follows from facts that $G[S]$ has $l=\left( \frac{|S|-3}{2} \right) + 1$ components.

For any two adjacent vertices $v_i$ and $v_{i+1}$ of $S$ at most $14$ consecutive vertices of $V(G)$.

For $n \equiv 0 \pmod{14}$, $S = \{v_{14l+j+1}, v_{14l+j+2}, \ldots, v_{14l+j+14} \}$.

Note 1. For any two adjacent vertices $v_a$ and $v_b$ of $G = Cir(n,B)$, $k=1$ and $n \geq 13$. We have the following:

$iv$. If $|v_a, v_b|=1$, then $v_a$ and $v_b$ dominate $10$ vertices of $G$ including themselves.

$ii$. If $|v_a, v_b|=2$, then $v_a$ and $v_b$ dominate $9$ vertices of $G$ including themselves.

$ii$. If $|v_a, v_b|=4$, then $v_a$ and $v_b$ dominate $11$ vertices of $G$ including themselves.

Therefore, any two adjacent vertices of $G$ dominate at most $11$ vertices of $G$ including themselves.

Note 2. Let $G_1$ be a component of $G[S]$ with three vertices $v_a, v_b, v_c$, we have the following:

$iv$. If $|v_a, v_b|=1$, then $G_1$ dominates $11$ vertices of $G$ including themselves.

$ii$. If $|v_a, v_b|=2$, then $G_1$ dominates $11$ vertices of $G$ including themselves.
A.A. Talebi, M. Zameni and Hossein Rashmanlou

iii. If \(|v_{i^*} v_0|=1, |v_{i^*} v_i|=2\) then \(G_1\) dominates 14 vertices of \(G\) including themselves.

iv. If \(|v_{i^*} v_0|=4, |v_{i^*} v_i|=2\) then \(G_1\) dominates 13 vertices of \(G\) including themselves.

v. If \(|v_{i^*} v_0|=|v_{i^*} v_i|=4\) then \(G_1\) dominates 15 vertices of \(G\) including themselves.

Therefore, any three vertices of \(G\) belong to a component dominate at most 15 vertices of \(G\) including themselves.

Lemma 2.5. Let \(S\) be a subset of vertices of \(G=\text{Cir}(n, B)\) with \(k = 1\) and \(G[S]\) has no isolated vertices. If \(|S|\) is even, then \(S\) dominates at most \(\left\lceil \frac{|S|}{2} \right\rceil\) vertices of \(G\).

Proof: Let \(S\) be a subset of vertices of \(G\) with \(|S|\equiv t\), where \(t\) is even. Any two adjacent vertices of \(S\) dominate 11 vertices of \(G\) including themselves. So \(S\) dominates at most \(\left\lceil \frac{|S|}{2} \right\rceil\) vertices of \(G\). ■

Lemma 2.6. Let \(S\) be a subset of vertices of \(G=\text{Cir}(n , B)\), \(k = 1\) and \(G[S]\) has no isolated vertices. If \(|S|\) is odd, then \(S\) dominates at most \((\frac{|S|−3}{2})11 + 15\) vertices of \(G\).

Proof: Let \(S\) be a subset of vertices of \(G\) with \(|S|\equiv t\), where \(t\) is odd. Without loss of generality we may assume that \(G[S]\) has \(d = (\frac{|S|−3}{2}) + 1\) components \(G_1, G_2, ..., G_d\), where \(|V(G_i)|=3\) and \(|V(G_i)| = 2\) for \(i = 2, 3, 4, ..., d\). Let \(V(G_i) = \{x, y, z\}\), then \(\{x, y, z\}\) dominates at most 15 vertices of \(G\). So \(S\) dominates at most \((\frac{|S|−3}{2})11 + 15\) vertices of \(G\). ■

Theorem 2.6. For any integer \(n \geq 13\) and \(k = 1\),

\[
\gamma_t(\text{Cir}(n, B)) = \begin{cases} 
\left\lceil \frac{2n}{11} \right\rceil + 1 & n \equiv 3, 5, 10 \text{ (mod 11)} \\
\left\lceil \frac{2n}{11} \right\rceil & \text{otherwise}
\end{cases}
\]

Proof: Let \(S\) be a \(\gamma_t\)-set of \(G = \text{Cir}(n, B)\). It follows from Lemma 2.5 and Lemma 2.6 that \(|S| \geq \left\lceil \frac{2n}{11} \right\rceil\). In the next we prove two claims as following.

Claim 1. If \(n \equiv 3, 5 \text{ (mod 11)}\) and \(S\) is a total dominating set for \(G\), then \(|S| = \left\lceil \frac{2n}{11} \right\rceil + 1\).

Let \(n \equiv 3, 5 \text{ (mod 11)}\) and let \(S\) be a total dominating set for \(G\). Assume to the contrary that \(|S| = \left\lceil \frac{2n}{11} \right\rceil\). We have \(n=11l+j\), where \(l\) is a positive integer, \(j \in \{3, 5\}\). Then \(|S| = \left\lceil \frac{2l+j}{11} \right\rceil = 2l+j\) is an odd number. So, the induced subgraph \(G[S]\) has an odd component \(H\) with at least three vertices. We proceed to following facts.

(i) Any component of \(G[S]\) has at most three vertices.

Assume to the contrary that \(G_1\) is a component of \(G[S]\) and \(G_1\) has at least 4 vertices. Without loss of generality assume that \(G_1\) has 4 vertices. Then \(S\) dominates at most \(15+(11)\left\lceil \frac{|S|−3}{2} \right\rceil + 20 = 11l+2\) vertices of \(G\), a contradiction.

(ii) \(H\) is the only odd component of \(G[S]\).
Vertex Domination Critical in Circulant Graphs

Assume to the contrary that $H'_eH$ is a component of $G[S]$ with $|V(H')|=3$. Since $|S|$ is odd, there is another component $H'$ with three vertices. Now $S$ dominates at most $11\left\lceil\frac{|S|-3-3}{2}\right\rceil + 3(15) = 11l + 1$ vertices of $G$, a contradiction.

For $n\equiv5 \pmod{11}$ and we have $V(H)=\{v_5, v_9, v_{13}\}$ and $S$ dominates $11\left\lceil\frac{|S|-3}{2}\right\rceil + 15 = 11l+4$ vertices of $G$, a contradiction.

When $n\equiv3 \pmod{11}$, we have $n=11l+3$, where $l$ is a positive integer. According to note 1, $\{v_2, v_4\}$ dominate 11 vertices. $N(\{v_6, v_9\})=\{v_6, v_{12}, \ldots, v_{15}\}$ and $v_{12}$ is dominated by $v_{16}$. $N(\{v_{16}, v_{20}\})=\{v_{16}, v_{24}, \ldots, v_{27}\}$ and $v_{24}$ is dominated by $v_{27}$. $N(\{v_{27}, v_{31}\})=\{v_{27}, v_{35}, \ldots, v_{38}\}$, we continue this process and $N(\{v_{l+2}\})=\{v_{l+2}, v_{l+9}, \ldots, v_{l+15}\}$. So $\{v_{l+15}, v_{l+30}, v_{l+36}, \ldots, v_{l+11l}\}=\{v_{l+2}, v_{l+15}\}$ is dominated by three vertices. In each possibility there exits at least one vertex in $\text{Last subset which is not dominated by this 3 vertices}$, a contradiction.

This completes the Claim 1.

Claim 2. If $n\equiv10 \pmod{11}$ and let $S$ be a total dominating set for $G$, then $|S|\geq\frac{2n}{11}+1$.

Assume to the contrary that $|S|=\frac{2n}{11}$. We have $n=11l+10$ where $l$ is a positive integer. Then $|S|=\frac{2n}{11}+2=2l+2$ is an even number. We have any component of $G$ has at least two vertices. Now we are proving any component of $G$ has exactly two vertices.

Assume to the contrary that $G_1$ is a component of $G$ and it has at least 3 vertices. Let $G_1$ has 3 vertices. So $|S|$ is an even number, there exist $G_1\neq G_1'$ is a component of $G[S]$ with $|V(G_1')|=3$, then at least $|V(G_1')|$ is 3. If $|V(G_1')|=3$, then $S$ dominates at most $11\left\lceil\frac{|S|-3-3}{2}\right\rceil + 2(15) = 11l + 8$ vertices of $G$, a contradiction.

So the induced subgraph $G[S]$ has components with two vertices. It follows from Note 1 and process of case 2, $S=\{v_5, v_9, v_{16}, v_{20}, v_{27}, v_{31}, \ldots, v_{l+15}, v_{l+19}, v_{l+23}, v_{l+27}\}$. We have $N(\{v_5, v_9\})=\{v_1, v_2, v_{15}, \ldots, v_{24}\}$, $N(\{v_{15}, v_{20}\})=\{v_{16}, v_{24}, v_{27}, v_{31}\}$, $N(\{v_{27}, v_{31}\})=\{v_{28}, v_{30}, v_{33}, \ldots, v_{37}\}$, $N(\{v_{l+15}, v_{l+19}\})=\{v_{l+16}, v_{l+24}, v_{l+27}, v_{l+31}\}$, $N(\{v_{l+23}, v_{l+27}\})=\{v_{l+24}, v_{l+30}, v_{l+33}, \ldots, v_{l+37}\}$. We have $\{v_5, v_9\}=: \{v_5, v_9\}$ and $\{v_{l+15}, v_{l+19}\}=: \{v_{l+15}, v_{l+19}\}$. We have $\{v_{l+23}, v_{l+27}\}=: \{v_{l+23}, v_{l+27}\}$ and $v_2$ is not dominated by $S$, a contradiction.

This completes the Claim 2.

Now it is sufficient to define a total dominating set $S$ of required cardinality. We consider the following cases:

1. For $n\equiv0 \pmod{11}$, $S=\{v_{l+6k+5}, v_{l+6k+9}: 0 \leq k < \left\lceil\frac{n}{11}\right\rceil\}$.
2. For $n\equiv1,2,4 \pmod{11}$, $S=\{v_{l+6k+5}, v_{l+6k+9}: 0 \leq k < \left\lceil\frac{n}{11}\right\rceil\} \cup\{v_{n-2}\}$.
3. For $n\equiv3,5,6,7,8 \pmod{11}$, $S=\{v_{l+6k+5}, v_{l+6k+9}: 0 \leq k < \left\lceil\frac{n}{11}\right\rceil\} \cup\{v_{n-2}, v_{n-3}\}$.
4. For $n\equiv9 \pmod{11}$, $S=\{v_{l+6k+5}, v_{l+6k+9}: 0 \leq k < \left\lceil\frac{n}{11}\right\rceil\} \cup\{v_{n-2}, v_{n-3}\}$.
5. For $n\equiv0 \pmod{11}$, $S=\{v_{l+6k+5}, v_{l+6k+9}: 0 \leq k < \left\lceil\frac{n}{11}\right\rceil\}$.
A.A.Talebi, M.Zameni and Hossein Rashmanlou

**Theorem 2.7.** For $n \geq 7$, $Cir(n, A)$ is $\gamma$-critical if and only if $n \equiv 4 \pmod{2k+3}$.

**Proof.** First we show that if $n \equiv 4 \pmod{2k+3}$ that $G$ is $\gamma$-critical. Let $x$ be a vertex of $G = Cir((2k+3)l+4, A)$, for some positive integer $l$. Since $G$ is transitive, we assume that $x = v_{n-2}$. It is easy to see that $S = \{v_{(2k+3)i}: 0 \leq i \leq \lceil \frac{n}{2k+3} \rceil \}$ is a dominating set for $G - x$. It follows that $\gamma(G - x) \leq \lfloor \frac{n}{2k+3} \rfloor \leq \lfloor \frac{n}{2k+3} \rfloor + 1 = \gamma(G)$. Hence, $G$ is $\gamma$-critical.

Suppose now that $n \equiv 4 \pmod{2k+3}$, we show that $G$ is not $\gamma$-critical. Let $T$ be a subset of vertices with $|T| < \gamma(G)$. Without loss of generality we let $|T| = \gamma(G)-1$. We show that any $|T|$ vertices of $G$ dominate at most $n - 2$ vertices of $G$.

We consider the following cases:

1. For $n \equiv 4, 6, 8, \ldots, 2k+2 \pmod{2k+3}$, by Theorem 2.1 $\gamma(G) = \lfloor \frac{n}{2k+3} \rfloor$.

   If $n \equiv 0 \pmod{2k+3}$, then $n = (2k+3)l$ for some integer $l$. It follows that $\gamma(G) = l$. Now $T$ dominates at most $(2k+3)(l-1) \leq n - 2$ vertices of $G$. Similarly, for $n \equiv 2, 3, 5, 7, 9, 11, \ldots, 2k+1 \pmod{2k+3}$, $T$ dominates at most $(2k+3)(l-1) \leq n - 2$ vertices of $G$.

   We assume that $n \equiv 1 \pmod{2k+3}$. There is an integer $l$ such that $n = (2k+3)l + 1$, $|T| \equiv \lfloor \frac{n}{2k+3} \rfloor - 1 = l$.

   If there are two consecutive vertices $x, y$ in $T$ such that $|x - y| < 2k+3$, then $N_{d}(x) \cap N_{d}(y) \not= \emptyset$. Hence, $\{x, y\}$ dominates at most $4k+5$ vertices of $G$ and $T \setminus \{x, y\}$ dominates at most $(2k+3)(l-2)$ vertices of $G$. So, $T$ dominates at most $n - 2$ vertices of $G$.

   It remains to assume that for any two consecutive vertices $a, b$ in $T$, $|a - b| \geq 2k+3$. In this case, there are two consecutive vertices $x, y$ in $T$ such that $|x - y| > 2k+3$. Then there exist two vertices $u, v$ lie between $x$ and $y$ in $G$, and $T$ does not dominate $\{u, v\}$. So, $T$ dominates at most $n - 2$ vertices of $G$, which is a contradiction.

2. For $n \equiv 2t \pmod{2k+3}$, $t$ is an integer with $3 \leq t \leq k+1$ by Theorem 2.1, $\gamma(G) = \lfloor \frac{n}{2k+3} \rfloor + 1$. There are two consecutive vertices $v_i, v_{i+1} \in S$ such that $|i - i'| < 2k+3$. Let $v_i' \not= v_i$ be a consecutive vertex of $v_i$. Without loss of generality we assume that $|v_i' - v_i| = 2k+3+2t$. Then there are $2k+2+2t$ possibilities for $v_i'$ to lie between $v_i$ and $v_i''$. In each possibility there exists at least two vertex between $v_i$ and $v_i''$ which is not dominated by $\{v_0, v_i', v_i''\}$.

   So, $T$ dominates at most $n - 2$ vertices of $G$, which is a contradiction.

**Theorem 2.8.** For $n \geq 9$, $Cir(n, B)$ is $\gamma$-critical if and only if $n \equiv 6 \pmod{2k+5}$.

**Proof:** First we show that if $n \equiv 6 \pmod{2k+5}$ that $G$ is $\gamma$-critical. Let $x$ be a vertex of $G = Cir((2k+5)l+6, A)$ for some positive integer $l$. Since $G$ is transitive, we assume that $x = v_{n-3}$. It is easy to see that $S = \{v_{(2k+5)i}: 0 \leq i \leq \lceil \frac{n}{2k+5} \rceil \}$ is a dominating set for $G - x$. It follows that $\gamma(G - x) \leq \lfloor \frac{n}{2k+5} \rfloor \leq \lfloor \frac{n}{2k+5} \rfloor + 1 = \gamma(G)$. Hence, $G$ is $\gamma$-critical.

Suppose now, that $n \equiv 6 \pmod{2k+5}$. We show that $G$ is not $\gamma$-critical. Let $T$ be a subset of vertices with $|T| < \gamma(G)$. Without loss of generality we let $|T| = \gamma(G)-1$. We show that any $|T|$ vertices of $G$ dominate at most $n - 2$ vertices of $G$.

We consider the following cases:

1. For $n \equiv 8, \ldots, 2k+2, 2k+4 \pmod{2k+5}$, by Theorem 2.2, $\gamma(G) = \lfloor \frac{n}{2k+5} \rfloor$. 

64
Vertex Domination Critical in Circulant Graphs

If \( n \equiv 0 \pmod{2k+5} \), then \( n = (2k+5)l \) for some integer \( l \). It follows that \( \gamma(G) = l \). Now, \( T \) dominates at most \((2k+5)(i-1) \leq n - 2\) vertices of \( G \). Similarly for \( n \equiv 2,3,4,5,7,9,11, \ldots,2k+3 \pmod{2k+5} \), \( T \) dominates at most \((2k+5)(i-1) \leq n - 2\) vertices of \( G \). We assume that
\( n \equiv 1 \pmod{2k+5} \). There is an integer \( l \) such that \( n = (2k+5)l+1 \). Without loss of generality we let \( |T| = \left[ \frac{n}{2k+5} \right] = 1 = l \).

If there are two consecutive vertices \( x, y \) in \( T \) such that \( |x - y| < 2k+5 \), then \( N_G(x) \cap N_G(y) \neq \emptyset \). Hence, \( \{x,y\} \) dominates at most \( 4k+9 \) vertices of \( G \) and \( T \setminus \{x,y\} \) dominates at most \((2k+5)(l-2)\) vertices of \( G \). So, \( T \) dominates at most \( n - 2 \) vertices of \( G \).

It remains to assume that for any two consecutive vertices \( a,b \) in \( T \), \( |a-b| \geq 2k+5 \). In this case there are two consecutive vertices \( x,y \) in \( T \) such that \( |x - y| > 2k+5 \). Then there exist two vertices \( u, v \) lie between \( x \) and \( y \) in \( C \), and \( T \) does not dominate \( \{u,v\} \). So, \( T \) dominates at most \( n - 2 \) vertices of \( G \), which is a contradiction. □

2. For \( n \equiv 2t \pmod{2k+5} \), \( t \) is an integer with \( 4 \leq t \leq k+2 \), by Theorem 2.2, \( \gamma(G) = \left[ \frac{n}{2k+5} \right] + 1 \). There are two consecutive vertices \( v_i, v_j \in S \) such that \( |l - l'| < 2k+5 \). Let \( v_l'' \neq v_j \) be a consecutive vertex of \( v_i \). Without loss of generality we assume that \( |v_l'' - v_j| = 2k+5+2t \). Then there are \( 2k+4+2t \) possibilities for \( v_l'' \) to lies between \( v_i \) and \( v_j \). In each possibly there exists at least two vertex between \( v_i \) and \( v_j '' \) which is not dominated by \( \{v_i,v_j,'',v_j''\} \).

So, \( T \) dominates at most \( n - 2 \) vertices of \( G \), which is a contradiction. □

REFERENCES