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Unicity Theorem for Algebroid Functions Related to Multiple Values and Derivatives on Annuli

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Abstract. In this paper, we first obtain Xiong inequality of algebroid function on annuli and using this result we prove uniqueness theorem of algebroid functions on annuli concerning to their multiple values and derivative.

Keywords: algebroid functions, derivatives on annuli

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1. Introduction

The uniqueness theory of algebroid functions is an interesting problem in the value distribution theory. The uniqueness problem of algebroid functions was firstly considered by Valiron, afterwards some scholars have got several uniqueness theorems of algebroid functions in the complex plane $\mathbb{C}$ (see [2, 3, 5, 6, 9-11, 14]). In 2005, Khrystiyanyn and Kondratyuk have proposed on the Nevanlinna Theory for meromorphic functions on annuli (see [7,8]) and after this work others have done lot of work in this area (see [18,19], etc). In 2009, Cao and Yi [1] investigated the uniqueness of meromorphic functions sharing some values on annuli. In 2015, Tan [12], Tan and Wang [13] proved some interesting results on the multiple values and uniqueness of algebroid functions on annuli. Thus it is interesting to consider the uniqueness problem of algebroid functions in multiply connected domains. By doubly connected mapping theorem [17] each doubly connected domain is conformally equivalent to the annulus

$$\{z : r < |z| < R\}, 0 \le r < R \le +\infty.$$

We consider only two cases : $r = 0, R = +\infty$ simultaneously and $0 \le r < R \le +\infty$. In the latter case the homothety $z \mapsto \frac{z}{rR}$ reduces the given domain to the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right) = \left\{z : \frac{1}{R_0} < |z| < R_0\right\}$, where $R_0 = \sqrt{\frac{R}{r}}$. Thus, in two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$.

2. Basic notations and definitions

We assume that the reader is familiar with the Nevanlinna theory of meromorphic functions and algebroid functions (see [4,15]).

Let $A_\nu(z), A_{\nu-1}(z), ..., A_0(z)$ be a group of analytic functions which have no common zeros and define on the annulus $\mathrm{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$,

$$\psi(z,W) = A_\nu(z)W^\nu + A_{\nu-1}(z)W^{\nu-1} + ... + A_1(z)W + A_0(z) = 0. \qquad (2.1)$$

Then irreducible equation (2.1) defines a ν-valued algebroid function on the annulus $\mathrm{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$.Let $W(z)$ be a ν-valued algebroid function on the annulus $\mathrm{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, we use the notations

$$m(r,W) = \frac{1}{\nu}\sum_{j=1}^{\nu} m(r, w_j) = \frac{1}{\nu}\sum_{j=1}^{\nu}\frac{1}{2\pi}\int_0^{2\pi}\log^+|w_j(re^{i\theta})|\,d\theta,$$

$$N_1(r,W) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{n_1(t,W)}{t}dt, \quad N_2(r,W) = \frac{1}{\nu}\int_1^r \frac{n_2(t,W)}{t}dt,$$

$$\overline{N}_1\left(r,\frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1\left(t,\frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2\left(r,\frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r \frac{\overline{n}_2\left(t,\frac{1}{W-a}\right)}{t}dt,$$

$$\overline{N}_1^{k)}\left(r,\frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1^{k)}\left(t,\frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2^{k)}\left(r,\frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r \frac{\overline{n}_2^{k)}\left(t,\frac{1}{w-a}\right)}{t}dt,$$

$$\overline{N}_1^{(k}\left(r,\frac{1}{W-a}\right) = \frac{1}{\nu}\int_{\frac{1}{r}}^{1}\frac{\overline{n}_1^{(k}\left(t,\frac{1}{W-a}\right)}{t}dt, \quad \overline{N}_2^{(k}\left(r,\frac{1}{W-a}\right) = \frac{1}{\nu}\int_1^r \frac{\overline{n}_2^{(k}\left(t,\frac{1}{W-a}\right)}{t}dt,$$

$$m_0(r,W) = m(r,W) + m\left(\frac{1}{r},W\right) - 2m(1,W), \; N_0(r,W) = N_1(r,W) + N_2(r,W),$$

$$\overline{N}_0\left(r,\frac{1}{W-a}\right) = \overline{N}_1\left(r,\frac{1}{W-a}\right) + \overline{N}_2\left(r,\frac{1}{W-a}\right),$$

$$\overline{N}_0^{k)}\left(r,\frac{1}{W-a}\right) = \overline{N}_1^{k)}\left(r,\frac{1}{W-a}\right) + \overline{N}_2^{k)}\left(r,\frac{1}{W-a}\right),$$

$$\overline{N}_0^{(k}\left(r,\frac{1}{W-a}\right) = \overline{N}_1^{(k}\left(r,\frac{1}{W-a}\right) + \overline{N}_2^{(k}\left(r,\frac{1}{W-a}\right),$$

where $w_j(z)(j=1,2,...,\nu)$ is one valued branch of $W(z)$, $n_1(t,W)$ is the counting functions of poles of the function $W(z)$ in $\{z: t<|z|\le 1\}$ and $n_2(t,W)$ is the counting functions of poles of the function $W(z)$ in $\{z: 1<|z|\le t\}$ (both counting multiplicity). $\overline{n}_1\left(t,\frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z: t<|z|\le 1\}$ and $\overline{n}_2\left(t,\frac{1}{W-a}\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ in $\{z: 1<|z|\le t\}$ (both ignoring multiplicity). $\overline{n}_1^{k)}\left(t,\frac{1}{W-a}\right)\left(\overline{n}_1^{(k}\left(t,\frac{1}{W-a}\right)\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ with multiplicity $\le k$ $(or > k)$ in $\{z: t<|z|\le 1\}$, each point count only once; $\overline{n}_2^{k)}\left(t,\frac{1}{W-a}\right)\left(\overline{n}_2^{(k}\left(t,\frac{1}{W-a}\right)\right)$ is the counting functions of poles of the function $\frac{1}{W-a}$ with multiplicity $\le k$ $(or > k)$ in $\{z: 1<|z|\le t\}$, each point count only once, respectively.

Let $W(z)$ be a ν-valued algebroid function which determined by (2.1) on the annulus $A\left(\frac{1}{R_0},R_0\right)(1<R_0\le+\infty)$, when $a\in \mathbb{C}$, $n_0\left(r,\frac{1}{W-a}\right)=n_0\left(r,\frac{1}{\psi(z,a)}\right)$, $N_0\left(r,\frac{1}{W-a}\right)=\frac{1}{\nu}N_0\left(r,\frac{1}{\psi(z,a)}\right)$.

In particular, when $a=0$, $N_0\left(r,\frac{1}{W}\right)=\frac{1}{\nu}N_0\left(r,\frac{1}{A_0}\right)$.

When $a=\infty$, $N_0(r,W)=\frac{1}{\nu}N_0\left(r,\frac{1}{A_\nu}\right)$; where $n_0\left(r,\frac{1}{W-a}\right)$ and $n_0\left(r,\frac{1}{\psi(z,a)}\right)$ are the counting function of zeros of $W(z)-a$ and $\psi(z,a)$ on the annulus $A\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, respectively.

Definition 2.1. [12] *Let* $W(z)$ *be an algebroid function on the annulus* $A\left(\frac{1}{R_0},R_0\right)$ $(1<R_0\le+\infty)$, *the function*

$$T_0(r,W)=m_0(r,W)+N_0(r,W),\ 1\le r<R_0$$

is called Nevanlinna characteristic of $W(z)$.

Definition 2.2. *Let* $W(z)$ *be a* ν*-valued algebroid function determined* (2.1) *on the annulus* $\mathrm{A}\left(\frac{1}{R_0}, R_0\right) 1 < R_0 \le +\infty$ *and* a *be a complex number. The deficiency of* a *with respect to* $W(z)$ *is defined by*

$$\delta_0(a,W) = \underline{\lim}_{r\to\infty} \frac{m_0\left(r,\frac{1}{W-a}\right)}{T_0(r,W)} = 1 - \overline{\lim}_{r\to\infty} \frac{N_0\left(r,\frac{1}{W-a}\right)}{T_0(r,W)}.$$

Definition 2.3. *Let* $W(z)$ *be a* ν*-valued algebroid function determined* (2.1) *on the annulus* $\mathrm{A}\left(\frac{1}{R_0}, R_0\right) 1 < R_0 \le +\infty$ *and* a *be a complex number. We define*

$$\Theta_0(a,W) = 1 - \overline{\lim}_{r\to\infty} \frac{\overline{N}_0\left(r,\frac{1}{W-a}\right)}{T_0(r,W)}.$$

3. Some lemmas

Lemma 3.1. $[7]$ *(Jensen theorem for meromorphic function on annuli) Let* f *be a meromorphic function on the annulus* $\mathrm{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, *then*

$$N_0\left(r,\frac{1}{f}\right) - N_0(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(\frac{1}{r}e^{i\theta}\right)\right| d\theta$$
$$-\frac{1}{2\pi}\int_0^{2\pi} \log|f(e^{i\theta})|\,d\theta,$$

where $1 \le r < R_0$.

Lemma 3.2. $[13]$ *(The first fundamental theorem on annuli) Let* $W(z)$ *be v-valued algebroid function which is determined by* (2.1) *on the annulus*

$$\mathrm{A}\left(\frac{1}{R_0}, R_0\right)\ (1 < R_0 \le +\infty),\ a \in \mathbb{C}$$
$$m_0(r,a) + N_0(r,a) = T_0(r,W) + O(1).$$

Lemma 3.3. $[13]$ *(The second fundamental theorem on annuli). Let* $W(z)$ *be v-valued algebroid function which is determined by* (2.1) *on the annulus* $\mathrm{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, $a_k\ (k = 1,2,..,p)$ *are* p *distinct complex numbers (finite or infinite), then we have*

$$(p-2v)T_0(r,W)\le \sum_{k=1}^{p} N_0\left(r,\frac{1}{W-a_k}\right)-N_1(r,W)+S_0(r,W) \tag{3.1}$$

$N_1(r,W)$ is the density index of all multiple values including finite or infinite, every τ multiple value counts $\tau-1$, and

$$S_0(r,W)=m_0\left(r,\frac{W'}{W}\right)+\sum_{j=1}^{p} m_0\left(r,\frac{W'}{W-a_k}\right)+O(1).$$

The remainder of the second fundamental theorem is the following formula

$$S_0(r,W)=O(\log T_0(r,W))+O(\log r),$$

outside a set of finite linear measure, if $r\to R_0=+\infty$, while

$$S_0(r,W)=O(\log T_0(r,W))+O\left(\log\frac{1}{R_0-r}\right),$$

outside a set E of r such that $\int_E \frac{dr}{R_0-r}<+\infty$, when $r\to R_0<+\infty$.

Remark 3.1. [13] *The second fundamental theorem on annuli has other forms, as the following:*

$$(p-1)T_0(r,W)\le N_0(r,W)+\sum_{k=1}^{p} N_0\left(r,\frac{1}{W-a_k}\right)-N_1(r)+Q_1(r,W), \tag{3.2}$$

$$N_1(r,W)=2N_0(r,W)-N_0(r,W')+N_0\left(r,\frac{1}{W'}\right),$$

$$Q_1(r,W)=\sum_{k=0}^{p} m_0\left(r,\frac{W'}{W-a_k}\right)+O(1),a_0=0.$$

We notice that the following formula is true,

$$\sum_{k=1}^{p} N_0\left(r,\frac{1}{W-a_k}\right)-N_1(r)\le \sum_{k=1}^{p} \overline{N}_0\left(r,\frac{1}{W-a_k}\right). \tag{3.3}$$

$\overline{N}_0\left(r,\frac{1}{W-a_k}\right)$ is the reduced counting function of zeros(ignoring multiplicity). Then the second fundamental theorem can be rewritten as the following

$$(p-2v)T_0(r,W)\le \sum_{k=1}^{p} \overline{N}_0\left(r,\frac{1}{W-a_k}\right)+S_0(r,W). \tag{3.4}$$

Lemma 3.4. [13] *Let $W(z)$ be v-valued algebroid function which is determined by*

(2.1) *on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$ $(1 < R_0 \le +\infty)$, *if the following conditions are satisfied*

$$\underline{\lim}_{r\to\infty} \frac{T_0(r,W)}{\log r} < \infty,\ R_0 = +\infty, \qquad \underline{\lim}_{r\to R_0^-} \frac{T_0(r,W)}{\log \frac{1}{(R_0 - r)}} < \infty,\ R_0 < +\infty,$$

then $W(z)$ is an algebraic function.

Remark 3.2. [13] *Let* $W(z)$ *be a* ν*-valued algebroid functions which is determined by* (2.1) *on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, *where* $1 < R_0 \le +\infty$ *and* $\hat{W}(z)$ *be a* μ*-valued algebroid functions which is determined by the following equation on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, *where* $1 < R_0 \le +\infty$,

$$\varphi(z,\hat{W}) = B_\mu(z)\hat{W}^{\mu} + B_{\mu-1}(z)\hat{W}^{\mu-1} + ... + B_1(z)\hat{W} + B_0(z) = 0.$$

Without loss of generality, let $\mu \le \nu, \overline{n}_{\Delta}^{k)}(r,a)$ denotes the counting function of the common values of $W(z) = a$ and $\hat{W}(z) = a$ with multiplicity $\le k$ on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)(1 < R_0 \le +\infty)$, each point counts only once. And let

$$\overline{N}_{\Delta}(r,a)^{k)} = \frac{\mu+\nu}{2\mu\nu}\int_{\frac{1}{r}}^{1} \frac{\overline{n}_{\Delta_1}^{k)}(t,a)}{t}dt + \frac{\mu+\nu}{2\mu\nu}\int_{1}^{r} \frac{\overline{n}_{\Delta_2}^{k)}(t,a)}{t}dt$$

$$\overline{N}_{12}^{k)}(r,a) = \overline{N}_0^{k)}\left(r, \frac{1}{W-a}\right) + \overline{N}_0^{k)}\left(r, \frac{1}{\hat{W}-a}\right) - 2\overline{N}_{\Delta}^{k)}(r,a). \qquad (3.5)$$

4. Main results

Let $W(z)$ be an algebroid function on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)$, where $1 < R_0 \le +\infty$, and 'a' be a complex number in the extende complex plane. Write $E(a,W) = \{z \in \mathbb{A} : W(z) - a = 0\}$, where each zero with multiplicity m is counted m times. If we ignore the multiplicity, then the set is denoted by $\overline{E}(a,W)$. We use $\overline{E}_{k)}(a,W)$ to denote the set of zeros of $W - a$ with multiplicities no greater than k, in which each zero is counted only once.

Now we prove the following theorem, which will be used later to prove our main result.

Theorem 4.1. *Let* $W(z)$ *be a* ν*-valued algebroid function determined by* (2.1) *on the annulus* $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)(1 < R_0 \le +\infty)$, *respectively and* k *is a positive integer. Then*

$$N_0\left(r, \frac{1}{W^{(k)}}\right) \le N_0\left(r, \frac{1}{W}\right) + T_0(r, W^{(k)}) - T_0(r, W) + S_0(r, W)$$

which implies

$$N_0\left(r, \frac{1}{W^{(k)}}\right) \le N_0\left(r, \frac{1}{W}\right) + k\overline{N}_0(r, W) + S_0(r, W)$$

Proof: By Lemma 3.3, we have

$$S_0(r, W^{(k)}) = O(\log RT_0(r, W^{(k)})) = O(\log RT_0(r, W)) = S_0(r, W).$$

Since

$$m_0\left(r, \frac{W^{(k)}}{W}\right) = S_0(r, W^{(k)}) = O(\log RT_0(r, W)) = S_0(r, W).$$

We can deduce

$$m_0\left(r, \frac{1}{W}\right) \le m_0\left(r, \frac{1}{W^{(k)}}\right) + m_0\left(r, \frac{W^{(k)}}{W}\right)$$
$$= m_0\left(r, \frac{1}{W^{(k)}}\right) + S_0(r, W).$$

Hence

$$m_0\left(r, \frac{1}{W}\right) + N_0\left(r, \frac{1}{W}\right) - N_0\left(r, \frac{1}{W}\right) \le m_0\left(r, \frac{1}{W^{(k)}}\right) + N_0\left(r, \frac{1}{W^{(k)}}\right)$$
$$- N_0\left(r, \frac{1}{W^{(k)}}\right) + S_0(r, W).$$
$$\Rightarrow T_0\left(r, \frac{1}{W}\right) - N_0\left(r, \frac{1}{W}\right) \le T_0\left(r, \frac{1}{W^{(k)}}\right) - N_0\left(r, \frac{1}{W^{(k)}}\right) + S_0(r, W).$$
$$\Rightarrow T_0(r, W) - N_0\left(r, \frac{1}{W}\right) \le T_0(r, W^{(k)}) - N_0\left(r, \frac{1}{W^{(k)}}\right) + S_0(r, W).$$

Therefore,

$$N_0\left(r, \frac{1}{W^{(k)}}\right) \le T_0(r, W^{(k)}) - T_0(r, W) + N_0\left(r, \frac{1}{W}\right) + S_0(r, W).$$
$$\Rightarrow N_0\left(r, \frac{1}{W^{(k)}}\right) \le k\overline{N}_0(r, W) + N_0\left(r, \frac{1}{W}\right) + S_0(r, W).$$

To prove unicity theorem related to multiple values and derivatives of algebroid functions on the annulus $\mathbb{A}\left(\frac{1}{R_0}, R_0\right)(1 < R_0 \le +\infty)$, we need to get the following

Xiong's inequality of algebroid functions on the annuli.

Theorem 4.2. *Let* $W(z)$ *be a* ν*-valued algebroid function determined on the annulus* $A\left(\frac{1}{R_0}, R_0\right)(1 < R_0 \le +\infty)$, *respectively and* $b_j\ (j = 1,2,...,q)$ *be distinct finite non zero complex numbers. Then for any positive integer n, we have*

$$qT_0(r,W) < \overline{N}_0(r,W) + qN_0\left(r,\frac{1}{W}\right) + \sum_{j=1}^{q} N_0\left(r,\frac{1}{W^{(n)} - b_j}\right)$$
$$-\left[(q-1)N_0\left(r,\frac{1}{W^{(n)}}\right) + N_0\left(r,\frac{1}{W^{(n+1)}}\right)\right] + S_0(r,W).$$

Proof: Applying the second fundamental Theorem for algebroid functions on annuli the to function $W^{(n)}(z)$ and three distinct values $0, b_j$ and ∞, we have

$$qT_0(r,W^{(n)}) < \overline{N}_0(r,W) + N_0\left(r,\frac{1}{W^{(n)}}\right) + \sum_{j=1}^{q} N_0\left(r,\frac{1}{W^{(n)} - b_j}\right)$$
$$-N_0\left(r,\frac{1}{W^{(n+1)}}\right) + S_0(r,W).$$

Theorem 4.1 implies

$$T_0(r,W) < T_0(r,W^{(n)}) + N_0\left(r,\frac{1}{W}\right) - N_0\left(r,\frac{1}{W^{(n)}}\right) + S_0(r,W).$$

Thus

$$qT_0(r,W) < qT_0(r,W^{(n)}) + qN_0\left(r,\frac{1}{W}\right) - qN_0\left(r,\frac{1}{W^{(n)}}\right) + S_0(r,W)$$
$$< \overline{N}_0(r,W) + qN_0\left(r,\frac{1}{W}\right) + \sum_{j=1}^{q} N_0\left(r,\frac{1}{W^{(n)} - b_j}\right)$$
$$-\left[(q-1)N_0\left(r,\frac{1}{W^{(n)}}\right) + N_0\left(r,\frac{1}{W^{(n+1)}}\right)\right] + S_0(r,W).$$

which completes the proof of Theorem 4.2.

From Theorem 4.2, we get the following corollary

Corollary 4.1. *Let* $W(z)$ *be a* ν *-valued algebroid function on the annulus* $A\left(\frac{1}{R_0}, R_0\right)(1 < R_0 \le +\infty)$, *respectively and* $b_j\ (j = 1,2\nu,2\nu+1)$ *be three distinct non zero complex numbers. Then for any positive integer n, we have*

$$(2\nu+1)T_0(r,W) < \overline{N}_0(r,W) + (2\nu+1)N_0\left(r,\frac{1}{W}\right) + \sum_{j=1}^{2\nu+1} N_0\left(r,\frac{1}{W^{(n)}-b_j}\right) + S_0(r,W).$$

Now, we prove our main result

Theorem 4.3. *Let* $W(z)$ *and* $\hat{W}(z)$ *be two* ν *-valued and* μ *-valued algebroid functions determined by* (2.1) *on the annulus* $\mathrm{A}\left(\frac{1}{R_0},R_0\right)(1<R_0\le+\infty)$, *respectively and* $\mu\le\nu$ *, let* $b_j(j=1,2\nu,2\nu+1)$ *be* $2\nu+1$ *distinct finite non zero complex numbers,* k *be a positive integer or* ∞ *, and* n *be a positive integer satisfying*

$$\overline{E}_{k)}(b_j,W^{(n)}(z)) = \overline{E}_{k)}(b_j,\hat{W}^{(n)}(z)). \tag{4.1}$$

Set

$$C_1 = (2\nu+1)(k+1)\delta_0(0,W) + (2\nu nk + (2\nu+1)n + k+1)\Theta_0(\infty,W) - (2\nu nk + (2\nu+1)n + (2\nu+1)k + 4\nu),$$

and

$$C_2 = (2\nu+1)(k+1)\delta_0(0,\hat{W}) + (2\nu nk + (2\nu+1)n + k+1)\Theta_0(\infty,\hat{W}) - (2\nu nk + (2\nu+1)n + (2\nu+1)k + 4\nu).$$

If

$$\min\{C_1,C_2\}\ge 0, \tag{4.2}$$

$$\max\{C_1,C_2\} > 0, \tag{4.3}$$

then $W(z)\equiv\hat{W}(z)$.

Proof: By the Corollary 4.1, we get

$$(2\nu+1)T_0(r,W) < \overline{N}_0(r,W) + (2\nu+1)N_0\left(r,\frac{1}{W}\right) + \sum_{j=1}^{(2\nu+1)} N_0\left(r,\frac{1}{W^{(n)}-b_j}\right) + S_0(r,W). \tag{4.4}$$

Note that

$$T_0(r,W^{(n)}) \le T_0(r,W) + n\overline{N}_0(r,W) + S_0(r,W).$$

We deduce that

$$\overline{N}_0\left(r,\frac{1}{W^{(n)}-b_j}\right) < \frac{k}{k+1}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right) + \frac{1}{k+1}T_0(r,W^{(n)}) + O(1)$$
$$< \frac{k}{k+1}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right) + \frac{1}{k+1}T_0(r,W) + \frac{n}{k+1}\overline{N}_0(r,W) + S_0(r,W). \tag{4.5}$$

Substituting equation (4.5) into (4.4) yields

$$(2\nu+1)T_0(r,W) < \overline{N}_0(r,W)+(2\nu+1)N_0\left(r,\frac{1}{W}\right)+\frac{(2\nu+1)}{k+1}T_0(r,W)$$
$$+\frac{k}{k+1}\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)+\frac{(2\nu+1)n}{k+1}\overline{N}_0(r,W)+S_0(r,W)$$
$$=\frac{(2\nu+1)n+k+1}{k+1}\overline{N}_0(r,W)+(2\nu+1)N_0\left(r,\frac{1}{W}\right)+\frac{(2\nu+1)}{k+1}T_0(r,W)$$
$$+\frac{k}{k+1}\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)+S_0(r,W)$$

$$=\frac{(2\nu+1)n+k+1}{k+1}(1-\Theta_0(\infty,W))T_0(r,W)+(2\nu+1)(1-\delta_0(0,W))T_0(r,W)$$
$$+\frac{(2\nu+1)}{k+1}T_0(r,W)+\frac{k}{k+1}\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)+S_0(r,W)$$

Therefore

$$\left((2\nu+1)\delta_0(0,W)+\frac{(2\nu+1)n+k+1}{k+1}\Theta_0(\infty,W)-\frac{(2\nu+1)n+k+4\nu}{k+1}\right)T_0(r,W)$$
$$<\frac{k}{k+1}\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)+S_0(r,W)$$
$$\Rightarrow\left((2\nu+1)(k+1)\delta_0(0,W)+((2\nu+1)n+k+1)\Theta_0(\infty,W)-((2\nu+1)n+k+4\nu)\right)T_0(r,W)$$
$$<k\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)+S_0(r,W).$$

Hence

$$\{2\nu k+2\nu nk(1-\Theta_0(\infty,W))+C_1\}T_0(r,W) < k\sum_{j=1}^{2\nu+1}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)+S_0(r,W). \quad (4.6)$$

Similarly

$$\{2\nu k+2\nu nk(1-\Theta_0(\infty,\hat{W}))+C_1\}T_0(r,\hat{W}) < k\sum_{j=1}^{2\nu+1}\overline{N}_0^{k)}\left(r,\frac{1}{\hat{W}^{(n)}-b_j}\right)+S_0(r,\hat{W}). \quad (4.7)$$

It follows from (4.1) that

$$\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)=\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{\hat{W}^{(n)}-b_j}\right)$$
$$\le (2\nu+1)T_0(r,\hat{W}^{(n)})+O(1)$$

$$\leq (2\nu+1)(n+1)T_0(r,\hat{W})+S_0(r,\hat{W}). \quad (4.8)$$

Note that $C_1 \geq 0$. (4.6) and (4.8) give

$$T_0(r,W)=O(T_0(r,\hat{W})) \quad (r\to\infty, r\notin E). \quad (4.9)$$

Similarly, we have

$$T_0(r,\hat{W})=O(T_0(r,W)) \quad (r\to\infty, r\notin E). \quad (4.10)$$

If $W(z)\not\equiv \hat{W}(z)$, then we have

$$\sum \overline{n}_{\Delta}^{(k))}(r,a) \leq n_0\left(r, \frac{1}{R(\varphi^{(n)},\psi^{(n)})}\right),$$

$R(\varphi^{(n)},\psi^{(n)})$ denotes the resultant of $\varphi(z,W^{(n)})$ and $\psi(z,W^{(n)})$, it can be written as the following

$$R(\varphi^{(n)},\psi^{(n)})=[A_\nu(z)]^\mu [B_\mu(z)]^\nu \prod_{\substack{1\leq j\leq \nu \\ 1\leq k\leq \mu}} [w_j^{(n)}(z)-\hat{w}_k^{(n)}(z)].$$

It can be written in the another form

$$R(\varphi^{(n)},\psi^{(n)})=\begin{vmatrix} A_\nu(z) & A_{\nu-1}(z) & \ldots & \ldots & A_0(z) & 0 & \ldots & 0 \\ 0 & A_\nu(z) & A_{\nu-1}(z) & \ldots & A_1(z) & A_0(z) & \ldots & 0 \\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & A_\nu(z) & A_{\nu-1}(z) & \ldots & \ldots & A_0(z) \\ B_\mu(z) & B_{\mu-1}(z) & \ldots & \ldots & B_0(z) & 0 & \ldots & 0 \\ 0 & B_\mu(z) & B_{\mu-1}(z) & \ldots & B_1(z) & B_0(z) & \ldots & 0 \\ \vdots & \vdots & & & \vdots & & & \\ 0 & 0 & 0 & B_\mu(z) & B_{\mu-1}(z) & \ldots & \ldots & B_0(z) \end{vmatrix}$$

So we know that $R(\varphi^{(n)},\psi^{(n)})$ is a holomorphic function and using Jensen Theorem for meromorphic function on annuli, we have

$$\begin{aligned} & N_0\left(r, \frac{1}{R(\varphi^{(n)},\psi^{(n)})}\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(re^{i\theta},W^{(n)}),\varphi(re^{i\theta},\hat{W}^{(n)})]|\,d\theta \\ &+ \frac{1}{2\pi}\int_0^{2\pi} \log\left|R\left[\psi\left(\frac{1}{r}e^{i\theta},W^{(n)}\right),\varphi\left(\frac{1}{r}e^{i\theta},\hat{W}^{(n)}\right)\right]\right|d\theta \\ &+ 2.\frac{1}{2\pi}\int_0^{2\pi} \log|R[\psi(e^{i\theta},W^{(n)}),\varphi(e^{i\theta},\hat{W}^{(n)})]|\,d\theta \\ &= \frac{\mu}{2\pi}\int_0^{2\pi} \log|A_\nu(re^{i\theta})|\,d\theta + \frac{\nu}{2\pi}\int_0^{2\pi} \log|B_\mu(re^{i\theta})|\,d\theta \end{aligned}$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(re^{i\theta})-\hat{w}_k^{(n)}(re^{i\theta})]\right|d\theta$$

$$+\frac{\mu}{2\pi}\int_0^{2\pi}\log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta+\frac{\nu}{2\pi}\int_0^{2\pi}\log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}\left[w_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)-\hat{w}_k^{(n)}\left(\frac{1}{r}e^{i\theta}\right)\right]\right|d\theta-2.\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(e^{i\theta})|\,d\theta$$

$$-2.\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|\,d\theta-2.\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(e^{i\theta})-\hat{w}_k^{(n)}(e^{i\theta})]\right|d\theta$$

$$=\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(re^{i\theta})|\,d\theta+\frac{\mu}{2\pi}\int_0^{2\pi}\log\left|A_\nu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta-2.\frac{\mu}{2\pi}\int_0^{2\pi}\log|A_\nu(e^{i\theta})|\,d\theta$$

$$+\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(re^{i\theta})|\,d\theta+\frac{\nu}{2\pi}\int_0^{2\pi}\log\left|B_\mu\left(\frac{1}{r}e^{i\theta}\right)\right|d\theta-2.\frac{\nu}{2\pi}\int_0^{2\pi}\log|B_\mu(e^{i\theta})|\,d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(re^{i\theta})-\hat{w}_k^{(n)}(re^{i\theta})]\right|d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}\left[w_j^{(n)}\left(\frac{1}{r}e^{i\theta}\right)-\hat{w}_k^{(n)}\left(\frac{1}{r}e^{i\theta}\right)\right]\right|d\theta$$

$$-2.\frac{1}{2\pi}\int_0^{2\pi}\log\left|\prod_{\substack{1\le j\le\nu\\1\le k\le\mu}}[w_j^{(n)}(e^{i\theta})-\hat{w}_k^{(n)}(e^{i\theta})]\right|d\theta$$

$$\le\mu\left[m_0(r,A_\nu)-m_0\left(r,\frac{1}{A_\nu}\right)\right]+\nu\left[m_0(r,B_\mu)-m_0\left(r,\frac{1}{B_\mu}\right)\right]$$
$$+\mu\nu[m_0(r,W^{(n)})+m_0(r,\hat{W}^{(n)})]+O(1)$$
$$=\mu\nu[T_0(r,W^{(n)})+T_0(r,\hat{W}^{(n)})]+O(1).$$

Then we get

$$\sum\overline{N}_{\Delta}^{k)}(r,a_j)\le\frac{2\mu\nu}{\mu+\nu}[T_0(r,W^{(n)})+T_0(r,\hat{W}^{(n)})]+O(1)$$
$$\le\nu[T_0(r,W^{(n)})+T_0(r,\hat{W}^{(n)})]+O(1). \qquad (4.11)$$

By the condition of Theorem 4.3, we know that the set of zeros of $W(z)$ and $\hat{W}(z)$

take the same values with multiplicity $<k$ about q distinct b_j, each point counts only once, at the same time we get $\overline{N}_{12}^{k)}(r,a_j)=0$.

From (3.5), we have

$$\overline{N}_0^{k)}\left(r,\frac{1}{W-a}\right)+\overline{N}_0^{k)}\left(r,\frac{1}{\hat{W}-a}\right)=2\sum\overline{N}_{\Delta}^{k)}(r,a_j)+\sum\overline{N}_{12}^{k)}(r,a_j)$$
$$\leq \nu[T_0(r,W^{(n)})+T_0(r,\hat{W}^{(n)})]+O(1). \quad (4.12)$$

Then from (4.1), we get

$$\sum_{j=1}^{(2\nu+1)}\overline{N}_0^{k)}\left(r,\frac{1}{W^{(n)}-b_j}\right)\leq\sum_{j=1}^{(2\nu+1)}N_0\left(r,\frac{1}{W^{(n)}-\hat{W}^{(n)}}\right)$$
$$\leq \nu[T_0(r,W^{(n)})+T_0(r,\hat{W}^{(n)})]+O(1)$$
$$\leq \nu[T_0(r,W)+n\overline{N}(r,W)+T_0(r,\hat{W})+n\overline{N}_0(r,\hat{W})]$$
$$+S_0(r,W)+S_0(r,\hat{W})$$
$$\leq \nu[T_0(r,W)+n(1-\Theta_0(\infty,W))T_0(r,W)]$$
$$+\nu[T(r,\hat{W})+n(1-\Theta(\infty,\hat{W}))T_0(r,\hat{W})]+S_0(r,W)+S_0(r,\hat{W}). \quad (4.13)$$

Substituting the equation (4.13) into (4.6) gives

$$[k\nu+nk\nu(1-\Theta_0(\infty,W))+C_1]T_0(r,W)$$
$$<[k\nu+nk\nu(1-\Theta_0(\infty,\hat{W}))]T_0(r,\hat{W})+S_0(r,W)+S_0(r,\hat{W}). \quad (4.14)$$

Similarly, we have

$$[k\nu+nk\nu(1-\Theta_0(\infty,\hat{W}))+C_2]T(r,\hat{W})$$
$$<[k\nu+nk\nu(1-\Theta_0(\infty,W))]T_0(r,W)+S_0(r,W)+S_0(r,\hat{W}). \quad (4.15)$$

From the equations (4.14) and (4.15), we get

$$C_1T_0(r,W)+C_2T_0(r,\hat{W})<S_0(r,W)+S_0(r,\hat{W}). \quad (4.16)$$

By (4.2), (4.3), (4.9) and (4.10), we see that the above inequality can not hold. So $W^{(n)}(z)\equiv\hat{W}^{(n)}(z)$, and thus $W(z)\equiv\hat{W}(z)+p(z)$, where $p(z)$ is a polynomial of at most degree $n-1$.

(4.2) means that $\delta_0(0,W)>0$, $\delta_0(0,\widehat{W})>0$, $\Theta_0(\infty,W)>0$, and $\Theta_0(\infty,W)>0$. Therefore $W(z)$ and $\hat{W}(z)$ must be algebroid functions. Hence $T(r,p)=o(T(r,W))$ and $T(r,p)=o(T(r,\hat{W}))$. If $p(z)\not\equiv 0$, then

$$\Theta_0(0,W)+\Theta_0(p,W)+\Theta(\infty,W)\ \geq\delta_0(0,W)+\delta_0(p,W)+\Theta_0(\infty,W)$$
$$=\delta_0(0,W)+\delta_0(0,\hat{W})+\Theta_0(\infty,W)$$

$$\geq \frac{2\nu nk+(2\nu+1)n+(2\nu+1)k+4\nu}{(2\nu+1)(k+1)} - \left(\frac{2\nu nk+(2\nu+1)n+k+1}{(2\nu+1)(k+1)} - 1\right)\Theta_0(\infty, W)$$
$$+\frac{2\nu nk+(2\nu+1)n+(2\nu+1)k+4\nu}{(2\nu+1)(k+1)} - \frac{2\nu nk+(2\nu+1)n+k+1}{(2\nu+1)(k+1)}\Theta_0(\infty, \hat{W})$$

$$\geq \frac{1}{(2\nu+1)(k+1)}[(2\nu nk+(2\nu+1)n+(2\nu+1)k+4\nu)-(2\nu nk+(2\nu+1)n-2\nu k-2\nu)]$$
$$+\frac{1}{(2\nu+1)(k+1)}[(2\nu nk+(2\nu+1)n+(2\nu+1)k+4\nu)-(2\nu nk+(2\nu+1)n+k+1)]$$
$$=\frac{(6\nu+1)k+8\nu+1}{(2\nu+1)(k+1)} > 2\nu.$$

This, however is impossible. Hence $p(z) \not\equiv 0$, and thus $W(z) = \hat{W}(z)$.

Open problems

Can we establish Theorem 4.3 for differential polynomials of algebroid functions on annuli.

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