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Common Fixed Point Theorems in Fuzzy Metric Spaces Using CLR Property

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Abstract. In this paper, we prove some common fixed point theorems for weakly compatible mappings in Fuzzy metric space using the notion of CLR and JCLR property.

Keywords: Fuzzy metric space, compatible mappings, weakly compatible mappings, CLR property and JCLR property.

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1. Introduction

It proved a turning point in the development of fuzzy mathematics when the notion of fuzzy set was introduced by Zadeh [31]. Fuzzy set theory has many applications in applied science such as neural network theory, stability theory, mathematical programming, modelling theory, engineering sciences, medical sciences (medical genetics, nervous system), image processing, control theory, communication etc. There are many view points of the notion of the metric space in fuzzy topology, see, e.g., Erceg [9], Deng [8], Kaleva and Seikkala [19], Kramosil and Michalek [20], George and Veermani [12]. In this paper, we are considering the Fuzzy metric space in the sense of Kramosil and Michalek [20].

Definition 1.1. A binary operation$*$on $[0,1]$ is a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,
(ii) $a * 1 = a$ for every $a \in [0,1]$,
(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$.

Basics examples of t-norm are t- norm $\Delta_{\mathrm{L}}, \Delta_{\mathrm{L}}(a,b) = max\,(a+b-1, 0)$, t-norm $\Delta_{\mathrm{P}}, \Delta_{\mathrm{P}}(a,b) = ab$ and t- norm $\Delta_{\mathrm{M}}, \Delta_{\mathrm{M}}(a,b) = min\{a,b\}$.

Definition 1.2. The 3- tuple (X, M, Δ) is called a fuzzy metric space (in the sence of Kramosil and Michalek) if X is an arbitrary set, Δ is a continuous t- norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$ and $s, t > 0$

(1) $M(x, y, 0) = 0, M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$, for all t > 0 if and only $if \quad x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$,
(5) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

Note that $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$ and $M(x, y, t) = 0$ with $t = 0$.

Definition 1.3. A sequence $\{x_n\}$ in (X, M, Δ) is said to be

(i) Convergent with limit x if $lim_{n\to\infty} M(x_n, x, t) = 1$for all $t > 0$.
(ii) Cauchy sequence in X if given $\epsilon > 0$ and $\lambda > 0$, there exists a positive Integer $N_{\epsilon,\lambda}$ such that $M(x_n, x_m, \epsilon) > 1 - \lambda$ for all $n, m \geq N_{\epsilon,\lambda}$.
(iii) Complete if every Cauchy sequence in X is convergent in X.

Fixed point theory in fuzzy metric space has been developing since the paper of Grabiec [12]. Subramanyam [21] gave a generalization of Jungck [17] theorem for commuting mapping in the setting of fuzzy metric space.

In 1996, Jungck[17] introduced the notion of weakly compatible as follows :

Definition 1.4. Two maps f and g are said to be weakly compatible if they commute at their coincidence points.

Definition 1.5. The pairs (A, S) and (B, T) on a fuzzy metric space (X, M, Δ) are said to satisfy the common property (E.A) if there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = lim_{n\to\infty} By_n = lim_{n\to\infty} Ty_n = u$, for some $u \in X$.
If $B = A$ and $T = S$ in the definition we get the definition of the property (E.A).

Definition 1.6. A pair of self-mappings A and S of a fuzzy metric space (X, M, Δ) is said to satisfy the common limit range property with respect to the mapping S (briefly CLRs property), if there exists a sequence $\{x_n\}$ in X such that
$lim_{n\to\infty} M(Ax_n, u, t) = lim_{n\to\infty} M(Sx_n, u, t) = 1$, for some $u \in S(X)$ and for all $t > 0$.
Now we give an example of self-mappings A and S satisfying the CLRs property (see [1])

Example 1.1. Let (X, M, Δ) fuzzy metric space with $X = [0, \infty)$ and for all $x, y \in X$ by $M(x, y, t) = \frac{t}{t + |x - y|}, t > 0$ and $M(x, y, 0) = 0$, where $\Delta(a, b) = \min\{a, b\}$ for all $a, b \in [0,1]$. Define self-mappings A and S on X by
$Ax = x + 3, Sx = 4x$. Let a sequence $\{x_n = 1 + \frac{1}{n}\}, n \in N$ in X.
Since $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 4$, then
$\lim_{n\to\infty} M(Ax_n, 4, t) = \lim_{n\to\infty} M(Sx_n, 4, t) = 1$, where $4 \in X$. Therefore, the mappings A and S satisfy the CLRs property.

Remark 1.1. From the example 1.1, it is clear that a pair (A,S) satisfying the property (E.A) property with the closeness of the subspace $S(X)$ always verify the CLRs property.

Definition 1.7. The pairs (A,S) and (B,T) on fuzzy metric space (X,M,Δ) are said to satisfy common limit range property with respect to mappings S and T (briefly CLR_{ST} property) if there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$

$$lim_{n\to\infty}M(Ax_n, u, t) = lim_{n\to\infty}M(Sx_n, u, t) = lim_{n\to\infty}M(By_n, u, t) = lim_{n\to\infty}M(Ty_n, u, t) = 1, \text{where} u \in S(X) \cap T(X).$$

Remark 1.2. If $B = A$ and $T = S$ in the definition we get the definition of CLRs property.

Remark 1.3. The CLR_{ST} property implies the common property (E.A), but the converse is not true in general (see [5]).

Proposition 1.1. [5] If the pairs (A,S) and (B,T) satisfy the common property (E.A) and $S(X)$ and $T(X)$ are closed subsets of X, then the pairs satisfy also the CLR_{ST} property.

Definition 1.8. [14] The pairs (A,S) and (B,T) on a fuzzy metric space (X,M,Δ) are said to satisfy common limit range property with respect to mappings S and T (briefly CLR_{ST} property) if there exists two sequence $\{x_n\}$ and $\{y_n\}$ in X such that for all $t > 0$

$$lim_{n\to\infty}M(Ax_n, u, t) = lim_{n\to\infty}M(Sx_n, u, t) = lim_{n\to\infty}M(By_n, u, t) = lim_{n\to\infty}M(Ty_n, u, t) = 1, \text{where} u = Sz = Tz, \text{ for some } z \in X.$$

Remark 1.4. If $B = A$ and $T = S$in the above definition we get the definition of CLRs property.

Definition 1.9. [13] Two families of self-mappings $\{A_i\}$ and $\{S_j\}$ are said to be pair wise commuting if

(1) $A_iA_j = A_jA_i$, $i,j \in \{1,2,..,m\}$
(2) $S_kS_l = S_lS_k$, $k,l \in \{1,2,..,n\}$
(3) $A_iS_k = S_kA_i$, $i \in \{1,2,..,m\}$, $k \in \{1,2,..,n\}$

Lemma 1.1. [21] Let $\{x_n\}$ be a sequence in a fuzzy metric space (X,M,Δ) with continuous t-norm Δ and $\Delta(t,t) \geq t$. If there exists a constant $k \in (0,1)$ such that

$$M(x_n, x_{n+1}y, kt) \geq M(x_{n-1}, x_n, t)$$

for all $t > 0$ and $n = 1,2,3.\ldots$. then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 1.2. [21] Let (X,M,Δ) fuzzy metric space . If there exists $k \in (0,1)$ such that $M(x,y,kt) \geq M(x,y,t)$ for all $x,y \in X$ and $t > 0$, then $x = y$.

3. Main results

Now we prove the main results as follows.

Lemma 3.1. Let A,B,S and T be mappings of a complete fuzzy metric space (X,M,Δ) into itself satisfying the following conditions:

(3.1) the pair (A, S) satisfies the CLR_S property or the pair (B, T) satisfies the CLR_T property,
(3.2) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,
(3.3) $T(X)$ or $S(X)$ is a closed subset of X,
(3.4) $B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges

$$\begin{aligned}&\big(1+\alpha M(Sx,Ty,t)\big)M(Ax,By,t)\\&\qquad > \min\{M(Ax,Sx,t)M(By,Ty,t), M(Sx,By,t)M(Ax,Ty,t)\}\\&\qquad + min\left\{\begin{matrix}M(Sx,Ty,t), sup_{t_1+t_2=\frac{2t}{k}}\min\{M(Ax,Sx,t_1), M(By,Ty,t_2)\}\\ sup_{t_3+t_4=2t}\min\{M(Sx,By,t_3), M(Ax,Ty,t_4)\}\end{matrix}\right\} \qquad (3.5)\end{aligned}$$

for all $x, y \in X, t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof: Suppose that the pair (A, S) satisfies the CLR_S property and $T(X)$ is closed subset of X. Then there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z, \text{ where } z \in S(X).$$

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $A(x_n) = T(y_n)$.So

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n = z, \text{ where } z \in S(X) \cap T(X).$$

Thus $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now we show the $By_n \to z$.
Let $\lim_{n\to\infty} M(By_n, l, t_0) = 1$. Now we show that $l = z$. Assume that $l \neq z$. We prove that there exists $t_0 > 0$ such that

$$M(z, l, \tfrac{2}{k} t_0) > M(z, l, t_0). \qquad (3.6)$$

Suppose the contrary. Therefore for all $t > 0$, we have

$$M(z, l, \tfrac{2}{k} t) \leq M(z, l, t). \qquad (3.7)$$

Using repeatedly (3.7), we obtain
$M(z, l, t) \geq M\left(z, l, \frac{2}{k} t\right) \geq \cdots \geq M(z, l, \left(\frac{2}{k}\right)^n t) \to 1$ as $n \to \infty$, this shows that $M(z, l, t) = 1$ for all $t > 0$, which contradicts $l \neq z$.
Without loss of generality, we may assume that t_0 in (3.6) is a continuous point of $M(z, l, t)$. Since every distance distribution function is left-continuous, (3.6) implies that there exists $\epsilon > 0$ such that (3.6) holds for all $t \in (t_0 - \epsilon, t_0)$. Since $M(z, l, t)$ is non-decreasing, the set of all discontinuous points of $M(z, l, t)$ is a countable set at most. Thus when t_0a discontinuous point of is $M(z, l, t)$, we can choose a continuous point t_1 of $M(z, l, t)$ and $M(z, l, t)$ in $(t_0 - \epsilon, t_0)$ to replace t_0. Using the inequality (3.5), with $x = x_n$, y= y_n, we get for some $t_0 > 0$

$$\begin{aligned}&\big(1+\alpha M(Sx_n,Ty_n,t_0)\big)M(Ax_n,By_n,t_0) > \alpha\min\left\{\begin{matrix}M(Ax_n,Sx_n,t_0)M(By_n,Ty_n,t_0),\\ M(Sx_n,By_n,t_0)M(Ax_n,Ty_n,t_0)\end{matrix}\right\}\\&\qquad + min\left\{\begin{matrix}M(Sx_n,Ty_n,t_0),\\ \min\left\{M(Ax_n,Sx_n,\epsilon), M\left(By_n,Ty_n,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\\ \min\{M\big(Sx_n,By_n,(2t_0-\epsilon)\big), M(Ax_n,Ty_n,\epsilon)\}\end{matrix}\right\}\end{aligned}$$

For all $\epsilon \in (0, \frac{2}{k} t_0)$. Letting $n \to \infty$, we have

$$M(z,l,t_0) + \alpha M(z,l,t_0) \geq M(z,l,t_0) + min\left\{M\left(z,l,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(z,l,(2t_0-\epsilon))\right\}$$

As $\epsilon \to 0$, we obtain $M(z,l,t_0) \geq M(z,l,\frac{2}{k}t_0)$, which contradicts (3.6). Thus the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Remark 3.1. The converse of lemma 3.1 is not true.

Theorem 3.1. Let A,B,S and T be mappings of a complete fuzzy metric space (X,M,Δ) into itself satisfying the inequality (3.5) of lemma 3.1. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, the pairs (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B,S and T have a unique common fixed point in X.

Proof: Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST}property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X) \cap T(X)$.
Hence, there exists $u,v \in X$ such that $Su = Tv = z$.
Now we show that $Au = Su = z$.
If$Au \neq Su$, putting $x = u$ and $y = y_n$ in inequality (3.5), we get for some $t_0 > 0$

$$\big(1+\alpha M(Su,Ty_n,t_0)\big)M(Au,By_n,t_0) > \alpha \min\begin{Bmatrix} M(Au,Su,t_0)M(By_n,Ty_n,t_0), \\ M(Su,By_n,t_0)M(Au,Ty_n,t_0) \end{Bmatrix}$$

$$+\,min\begin{Bmatrix} M(Su,Ty_n,t_0), \\ \min\left\{M\left(Au,Sx_n,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(By_n,Ty_n,\epsilon)\right\} \\ \min\{M(Su,By_n,\epsilon), M(Au,Ty_n,(2t_0-\epsilon))\} \end{Bmatrix}$$

$$\big(1+\alpha M(z,z,t_0)\big)M(Au,z,t_0) > \alpha \min\begin{Bmatrix} M(Au,z,t_0)M(z,z,t_0), \\ M(Su,z,t_0)M(Au,z,t_0) \end{Bmatrix}$$

$$+\,min\begin{Bmatrix} M(Su,z,t_0), \\ \min\left\{M\left(Au,z,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(z,z,\epsilon)\right\} \\ \min\{M(z,z,\epsilon), M\big(Au,z,(2t_0-\epsilon)\big)\} \end{Bmatrix}$$

$$M(Au,z,t_0) + \alpha M(Au,z,t_0) \geq \alpha M(Au,z,t_0)$$

$$+min\left\{M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(Az,z,(2t_0-\epsilon))\right\}$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$. Letting $n \to \infty$, we have

As $\epsilon \to 0$, $M(z,l,t_0) \geq M(z,l,\frac{2}{k}t_0)$, which contradicts (3.6) and so $Au = Su = z$. Therefore u is a coincidence point of the pair (A,S).

Now we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.5), we get for some $t_0 > 0$

$$\big(1+\alpha M(Su,Tv,t_0)\big)M(Au,Bv,t_0) > \alpha \min\begin{Bmatrix} M(Au,Su,t_0)M(Bv,Tv,t_0), \\ M(Su,Bv,t_0)M(Au,Tv,t_0) \end{Bmatrix}$$

$$+\,min\begin{Bmatrix} M(Su,Tv,t_0), \\ \min\left\{M(Au,Sv,\epsilon), M\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\{M\big(Su,Bv,(2t_0-\epsilon)\big), M(Au,Tv,\epsilon)\} \end{Bmatrix}$$

$$\left(1+\alpha M(z,z,t_0)\right)M(z,Bv,t_0) > \alpha \min\begin{Bmatrix} M(z,z,t_0)M(Bv,z,t_0), \\ M(z,Bv,t_0)M(z,z,t_0) \end{Bmatrix}$$

$$+ min\begin{Bmatrix} M(z,z,t_0), \\ \min\left\{M(z,z,\epsilon), M\left(Bv,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\{M(Bv,z,(2t_0-\epsilon)), M(z,z,\epsilon)\} \end{Bmatrix}$$

$$M(z,Bv,t_0)+\alpha M(z,Bv,t_0) \geq \alpha M(z,Bv,t_0)$$

$$+min\left\{M\left(Bv,z,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(Bv,z,(2t_0-\epsilon))\right\}$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$. As $\epsilon \to 0$, $M(Bv,z,t_0) \geq M(Bv,z,\frac{2}{k}t_0)$, which contradicts (3.6) and so $Bv = Tv = z$. Therefore v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now we prove that z is a common fixed point of A and S. If $z \neq Az$, applying the inequality (3.5) with $x = z$ and $y = v$, we get for some $t_0 > 0$

$$\left(1+\alpha M(Sz,Tv,t_0)\right)M(Az,Bv,t_0) > \alpha \min\begin{Bmatrix} M(Az,Sz,t_0)M(Bv,Tv,t_0), \\ M(Sz,Bv,t_0)M(Az,Tv,t_0) \end{Bmatrix}$$

$$+ min\begin{Bmatrix} M(Az,z,t_0), \\ \min\left\{M(Az,Sv,\epsilon), M\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\{M(Sz,Bv,(2t_0-\epsilon)), M(Az,Tv,\epsilon)\} \end{Bmatrix}$$

$$\left(1+\alpha M(Az,z,t_0)\right)M(Az,z,t_0) > \alpha \min\begin{Bmatrix} M(z,z,t_0)M(z,z,t_0), \\ M(Az,z,t_0)M(Az,z,t_0) \end{Bmatrix}$$

$$+ min\begin{Bmatrix} M(Az,z,t_0), \\ \min\left\{M(z,z,\epsilon), M\left(z,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\{M(z,z,(2t_0-\epsilon)), M(z,z,\epsilon)\} \end{Bmatrix}$$

$$M(Az,z,t_0)+\alpha(M(Az,z,t_0))^2 \geq \alpha(M(Az,z,t_0))^2 + M(Az,z,t_0)$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$. $M(Az,z,t_0) \geq M(Az,z,t_0)$, which contradicts (3.6) and so $Az = Sz = z$, which shows that z is a common fixed point of A and S.

Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T. Hence z is a common fixed point of A, B, S and T.

We now give an example to illustrate the above theorem see for details [1]

Example 3.1. [1] Let (X,M,Δ) be a fuzzy metric space with $X = [3,11]$ and for all $x,y \in X$ by $M(x,y,t) = \frac{t}{t+|x-y|}, M(x,y,0) = 0, t > 0$, where $\Delta(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$.

Define self maps A, B, S and T on Xas follows:

$$Ax = \begin{cases} 3 & if x \in 3 \cup (5,11) \\ 10 & if x \in (3,5) \end{cases}, \qquad Bx = \begin{cases} 3 & if x \in 3 \cup (5,11) \\ 9 & if x \in (3,5) \end{cases}$$

$$Sx = \begin{cases} 3 & if x = 3 \\ 7 & if x \in (3,5) \\ \dfrac{x+1}{3} & if x \in (5,11) \end{cases}, \qquad Tx = \begin{cases} 3 & if x = 3 \\ x+4 & if x \in (3,5) \\ x-2 & if x \in (5,11) \end{cases},$$

We take $\{x_n = 3\}$, $\{y_n = 5 + \frac{1}{n}\}$ or $\{x_n = 5 + \frac{1}{n}\}$, $\{y_n = 3\}$, since
$lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = lim_{n\to\infty} By_n = lim_{n\to\infty} Ty_n = 3 \in S(X) \cap T(X)$;
Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property . Also
Then all the conditions of Theorem 3.1 are satisfied and 3 is the unique common fixed point of the pairs (A, S) and (B, T).
Remark that all the mappings are even discontinuous at their unique common fixed point 3. In this example $S(X)$ and $T(X)$ are not closed subsets of X.

Lemma 3.2. Let A, B, S and T be mappings of a completefuzzy metric space (X, M, Δ) into itself satisfying the conditions (3.1), (3.2), (3.3) and (3.4) of lemma 3.1 an

$$\big(1 + \alpha M(Sx, Ty, t)\big)M(Ax, By, t) > \alpha \min \begin{Bmatrix} M(Ax, Sx, t)M(By, Ty, t), \\ M(Sx, By, t)M(Ax, Ty, t) \end{Bmatrix} + min \left\{ \begin{cases} M(Sx, Ty, t) \\ sup_{t_1+t_2=\frac{2t}{k}} \min\{M(Ax, Sx, t_1), M(Sx, By, t_2)\} \\ sup_{t_3+t_4=\frac{2t}{k}} \min\{M(By, Ty, t_3), M(Ax, Ty, t_4)\} \end{cases} \right\} \quad (3.8)$$

for all $x, y \in X$,$t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A, S) and (B, T) satisfy the CLR_{ST} property.

Proof: As in the proof of Lemma 3.1, there exists $t_0 > 0$ such that (3.6) holds. Using the inequality (3.8), with $x = x_n, y = y_n$, we have

$$\big(1 + \alpha M(Sx_n, Ty_n, t_0)\big)M(Ax_n, By_n, t_0) > \alpha\min \begin{Bmatrix} M(Ax_n, Sx_n, t_0)M(By_n, Ty_n, t_0), \\ M(Sx_n, By_n, t_0)M(Ax_n, Ty_n, t_0) \end{Bmatrix} + min \left\{ \begin{cases} M(Sx_n, Ty_n, t_0) \\ \min\left\{M(Ax_n, Sx_n, \epsilon), M\left(Sx_n, By_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \\ \min\left\{M\left(By_n, Ty_n, \left(\frac{2}{k}t_0 - \epsilon\right)\right), M(Ax_n, Ty_n, \epsilon)\right\} \end{cases} \right\}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \to \infty$, we have

$$\big(1 + \alpha M(z, z, t_0)\big)M(z, l, t_0) > \alpha\min \begin{Bmatrix} M(z, z, t_0)M(l, z, t_0), \\ M(z, l, t_0)M(z, z, t_0) \end{Bmatrix} + min \left\{ \begin{cases} M(z, z, t_0) \\ \min\left\{M(z, z, \epsilon), M\left(z, l, \left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \\ \min\left\{M\left(l, z, \left(\frac{2}{k}t_0 - \epsilon\right)\right), M(z, z, \epsilon)\right\} \end{cases} \right\}$$

$$M(z,l,t_0)+\alpha M(z,z,t_0)$$

$$> \alpha M(z,z,t_0)\ +\ min\left\{\begin{Bmatrix} M(z,z,t_0) \\ \min\left\{M\big(z,z,(\epsilon)\big), M\left(z,l,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\} \\ \min\left\{M\left(l,z,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right), M\big(z,z,(\epsilon)\big)\right\}\end{Bmatrix}\right\}$$

$$M(z,l,t_0) > min\left\{1, M\left(z,l,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}$$

As $\epsilon \to 0$, we obtain, $M(z,l,t_0) \geq M(z,l,\frac{2}{k}t_0)$, which contradicts (3.6) and so we have $z = l$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Theorem 3.2. Let A, B, S and T be mappings of a complete fuzzy metric space (X, M, Δ) into itself satisfying the inequality (3.8) of lemma 3.2. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, the pairs (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.

Proof: Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X) \cap T(X)$.

Hence there exists $u, v \in X$ such that $Su = Tv = z$.

Now we show that $Au = Su = z$.

If $Au \neq Su$, putting $x = u$ and $y = y_n$ in inequality (3.8), we get for some $t_0 > 0$

$$\big(1+\alpha M(Su,Ty_n,t_0)\big)M(Au,By_n,t_0) > \alpha \min\begin{Bmatrix} M(Au,Su,t_0)M(By_n,Ty_n,t_0), \\ M(Su,By_n,t_0)M(Au,Ty_n,t_0)\end{Bmatrix}$$

$$+\ min\begin{Bmatrix} M(Su,Ty_n,t_0), \\ \min\left\{M\left(Au,Su,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(Su,By_n,\epsilon)\right\} \\ \min\left\{M(By_n,Ty_n,\epsilon), M\left(Au,Ty_n,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\end{Bmatrix}$$

$$\big(1+\alpha M(z,z,t_0)\big)M(Au,z,t_0) > \alpha \min\begin{Bmatrix} M(Au,z,t_0)M(z,z,t_0), \\ M(z,z,t_0)M(Au,z,t_0)\end{Bmatrix}$$

$$+\ min\begin{Bmatrix} M(z,z,t_0), \\ \min\left\{M\left(Au,z,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(z,z,\epsilon)\right\} \\ \min\left\{M(z,z,\epsilon), M\left(Au,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\end{Bmatrix}$$

$$M(Au,z,t_0)+\alpha M(Au,z,t_0) \geq \alpha M(Au,z,t_0)$$

$$+min\left\{M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right), M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. Letting $n \to \infty$, we have
As $\epsilon \to 0$, $M(z,l,t_0) \geq M(z,l,\frac{2}{k}t_0)$, which contradicts (3.6) and so $Au = Su = z$. Therefore u is a coincidence point of the pair (A,S).

Now we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.8), we get for some $t_0 > 0$

$$\left(1 + \alpha M(Su,Tv,t_0)\right)M(Au,Bv,t_0) > \alpha \min\left\{\begin{matrix} M(Au,Su,t_0)M(Bv,Tv,t_0), \\ M(Su,Bv,t_0)M(Au,Tv,t_0) \end{matrix}\right\}$$
$$+ min\left\{\begin{matrix} M(Su,Tv,t_0), \\ \min\left\{M(Au,Sv,\epsilon), M\left(Su,Bv,\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \\ \min\left\{M\left(Bv,Tv,\left(\frac{2}{k}t_0 - \epsilon\right)\right), M(Au,Tv,\epsilon)\right\} \end{matrix}\right\}$$
$$\left(1 + \alpha M(z,z,t_0)\right)M(z,Bv,t_0) > \alpha \min\left\{\begin{matrix} M(z,z,t_0)M(Bv,z,t_0), \\ M(z,Bv,t_0)M(z,z,t_0) \end{matrix}\right\}$$
$$+ min\left\{\begin{matrix} M(z,z,t_0), \\ \min\left\{M(z,z,\epsilon), M\left(Bv,z,\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \\ \min\left\{M\left(Bv,z,\left(\frac{2}{k}t_0 - \epsilon\right)\right), M(z,z,\epsilon)\right\} \end{matrix}\right\}$$
$$M(z,Bv,t_0) + \alpha M(z,Bv,t_0) \geq \alpha M(z,Bv,t_0)$$
$$+min\left\{M\left(Bv,z,\left(\frac{2}{k}t_0 - \epsilon\right)\right), M\left(Bv,z,\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\}$$

for all $\epsilon \in (0, \frac{2}{k}t_0)$. As $\epsilon \to 0$, $M(Bv,z,t_0) \geq M(Bv,z,\frac{2}{k}t_0)$, which contradicts (3.6) and so $Bv = Tv = z$. Therefore v is a coincidence point of the pair (B,T).
Since the pair (A,S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now we prove that z is a common fixed point of $A$and S. If$z \neq Az$, applying the inequality (3.8) with $x = z$ and $y = v$, we get for some $t_0 > 0$

$$\left(1 + \alpha M(Sz,Tv,t_0)\right)M(Az,Bv,t_0) > \alpha \min\left\{\begin{matrix} M(Az,Sz,t_0)M(Bv,Tv,t_0), \\ M(Sz,Bv,t_0)M(Az,Tv,t_0) \end{matrix}\right\}$$
$$+ min\left\{\begin{matrix} M(Az,z,t_0), \\ \min\left\{M(Az,Sz,\epsilon), M\left(Sz,Bv,\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \\ \min\left\{M(Bv,Tv,\epsilon), M\left(Az,Tv,\left(\frac{2}{k}t_0 - \epsilon\right)\right)\right\} \end{matrix}\right\}$$
$$\left(1 + \alpha M(Az,z,t_0)\right)M(Az,z,t_0) > \alpha \min\left\{\begin{matrix} M(z,z,t_0)M(z,z,t_0), \\ M(Az,z,t_0)M(Az,z,t_0) \end{matrix}\right\}$$

$$+\,min\begin{Bmatrix} M(Az,z,t_0), \\ \min\left\{M(z,z,\epsilon),M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\left\{M(z,z,\epsilon),M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\end{Bmatrix}$$

$$M(Az,z,t_0)+\alpha\big(M(Az,z,t_0)\big)^2 \geq \alpha\big(M(Az,z,t_0)\big)^2 + \min\left\{M(Az,z,t_0),M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}$$

$$M(Az,z,t_0) > \min\left\{M(Az,z,t_0),M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} = M(Az,z,t_0)$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$, we get $M(Az,z,t_0) \geq M(Az,z,t_0)$, which contradicts (3.6) and so $Az = Sz = z$, which shows that z is a common fixed point of A and S.

Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that z is a common fixed point of B and T. Hence, z is a common fixed point of A,B,S and T.

If $B = A$ and $T = S$ in Theorems 3.1 and 3.2, we obtain a common fixed point for a set of self-mappings.

In the proof of the following lemma, we do not need to prove the inequality (3.6).

Lemma 3.3. Let A,B,S and T be mappings of a complete fuzzy metric space (X,M,Δ) into itself satisfying the conditions (3.1),(3.2),(3.3) and (3.4) of lemma 3.1 and

$$\big(1+\alpha M(Sx,Ty,t)\big)M(Ax,By,t) > \alpha\min\begin{Bmatrix}M(Ax,Sx,t)M(By,Ty,t), \\ M(Sx,By,t)M(Ax,Ty,t)\end{Bmatrix}$$
$$+\,min\left\{\begin{Bmatrix} M(Sx,Ty,t) \\ sup_{t_1+t_2=\frac{2t}{k}}\max\{M(Ax,Sx,t_1),M(By,Ty,t_3)\} \\ sup_{t_3+t_4=2t}\max\{M(Sx,By,t_2),M(Ax,Ty,t_4)\}\end{Bmatrix}\right\} \quad (3.9)$$

for all $x,y \in X$,$t > 0$, for some $\alpha \geq 0$ and $1 \leq k < 2$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Proof: Suppose that the pair (A,S) satisfies the CLR_S property and $T(X)$ is closed subset of X. Then there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$=z, where $z \in S(X)$.

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $A(x_n) = T(y_n)$.So

$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ax_n$= z, where $z \in S(X) \cap T(X)$.

Thus $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now we show the $By_n \to z$.

Let $\lim_{n\to\infty} M(By_n,l,t_0) = 1$.

Now we show that $l = z$. Assume that $l \neq z$. Using the inequality (3.9), with $x = x_n$, $y = y_n$, we get for some $t_0 > 0$.

$$\big(1+\alpha M(Sx_n,Ty_n,t_0)\big)M(Ax_n,By_n,t_0) > \alpha\min\begin{Bmatrix}M(Ax_n,Sx_n,t_0)M(By_n,Ty_n,t_0), \\ M(Sx_n,By_n,t_0)M(Ax_n,Ty_n,t_0)\end{Bmatrix}$$

$$+\,min\left\{\begin{Bmatrix} M(Sx_n,Ty_n,t_0) \\ \max\left\{M(Ax_n,Sx_n,\epsilon),M\left(By_n,Ty_n,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\{M(Sx_n,By_n,(2t_0-\epsilon)),M(Ax_n,Ty_n,\epsilon)\} \end{Bmatrix}\right\}$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$. Letting $n \to \infty$, we have

$$\big(1+\alpha M(z,z,t_0)\big)M(z,l,t_0) > \alpha\min\begin{Bmatrix} M(z,z,t_0)M(l,z,t_0), \\ M(z,l,t_0)M(z,z,t_0) \end{Bmatrix}$$

$$+\,min\left\{\begin{Bmatrix} M(z,z,t_0) \\ \max\left\{M(z,z,\epsilon),M\left(z,l,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \max\{M(z,l,(2t_0-\epsilon)),M(z,z,\epsilon)\} \end{Bmatrix}\right\}$$

$M(z,l,t_0) + \alpha M(z,z,t_0)$

$$> \alpha M(z,z,t_0)\ +\ min\left\{\begin{Bmatrix} M(z,z,t_0) \\ \max\left\{M(z,z,(\epsilon)),M\left(z,l,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\} \\ \max\left\{M\left(l,z,((2t_0-\epsilon))\right),M(z,z,(\epsilon))\right\} \end{Bmatrix}\right\}$$

$M(z,l,t_0) > 1$ for some $t_0 > 0$ and so we have $z = l$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Theorem 3.3. Let A,B,S and T be mappings of a complete fuzzy metric space (X,M,Δ) into itself satisfying the inequality (3.9) of lemma 3.3. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, the pairs (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then $A,\ B,S$ and T have a unique common fixed point in X.

Proof: Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X)\cap T(X)$. Hence, there exists $u,v \in X$ such that $Su = Tv = z$.

Now we show that $Au = Su = z$.

If $Au \neq Su$, putting $x = u$ and $y = y_n$ in inequality (3.9), we get for some $t_0 > 0$

$$\big(1+\alpha M(Su,Ty_n,t_0)\big)M(Au,By_n,t_0) > \alpha\min\begin{Bmatrix} M(Au,Su,t_0)M(By_n,Ty_n,t_0), \\ M(Su,By_n,t_0)M(Au,Ty_n,t_0) \end{Bmatrix}$$

$$+\,min\begin{Bmatrix} M(Su,Ty_n,t_0), \\ \max\left\{M\left(Au,Su,\left(\frac{2}{k}t_0-\epsilon\right)\right),M(By_n,Ty_n,\epsilon)\right\} \\ \max\{M(Su,By_n,\epsilon),M(Au,Ty_n,(2t_0-\epsilon))\} \end{Bmatrix}$$

$$\big(1+\alpha M(z,z,t_0)\big)M(Au,z,t_0) > \alpha\min\begin{Bmatrix} M(Au,z,t_0)M(z,z,t_0), \\ M(z,z,t_0)M(Au,z,t_0) \end{Bmatrix}$$

$$+\,min\left\{\begin{matrix} M(z,z,t_0), \\ \max\left\{M\left(Au,z,\left(\frac{2}{k}t_0-\epsilon\right)\right),M(z,z,\epsilon)\right\} \\ \max\{M(z,z,\epsilon),M(Au,z,(2t_0-\epsilon))\}\end{matrix}\right\}$$

$M(Au,z,t_0)>1$, and so $Au=\ Su=z$. Therefore u is a coincidence point of the pair (A,S).

Now we assert that $Bv=Tv=z$. If $z\neq Bv$, putting $x=u$ and $y=v$ in the inequality (3.9), we get for some $t_0>0$

$$\left(1+\alpha M(Su,Tv,t_0)\right)M(Au,Bv,t_0)>\alpha\min\begin{Bmatrix} M(Au,Su,t_0)M(Bv,Tv,t_0), \\ M(Su,Bv,t_0)M(Au,Tv,t_0)\end{Bmatrix}$$

$$+\,min\left\{\begin{matrix} M(Su,Tv,t_0), \\ \max\left\{M(Au,Sv,\epsilon),M\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \max\{M(Su,Bv,(2t_0-\epsilon)),M(Au,Tv,\epsilon)\}\end{matrix}\right\}$$

$$\left(1+\alpha M(z,z,t_0)\right)M(z,Bv,t_0)>\alpha\min\begin{Bmatrix} M(z,z,t_0)M(Bv,z,t_0), \\ M(z,Bv,t_0)M(z,z,t_0)\end{Bmatrix}$$

$$+\,min\left\{\begin{matrix} F(z,z,t_0), \\ \max\left\{M(z,z,\epsilon),M\left(Bv,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \max\{M(Bv,z,(2t_0-\epsilon)),M(z,z,\epsilon)\}\end{matrix}\right\}$$

$F(Bv,z,t_0)>1$and so $Bv=\ Tv=z$. Therefore v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au=\ Su$ we obtain $Az=\ Sz$. Now we prove that z is a common fixed point of $A$and S. If$z\neq Az$, applying the inequality (3.9) with $x=z$ and $y=v$, we get for some $t_0>0$

$$\left(1+\alpha M(Sz,Tv,t_0)\right)M(Az,Bv,t_0)>\alpha\min\begin{Bmatrix} M(Az,Sz,t_0)M(Bv,Tv,t_0), \\ M(Sz,Bv,t_0)M(Az,Tv,t_0)\end{Bmatrix}$$

$$+\,min\left\{\begin{matrix} M(Az,z,t_0), \\ \max\left\{M(Az,Sz,\epsilon),M\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \max\{M(Sz,Bv,\epsilon),M(Az,Tv,(2t_0-\epsilon))\}\end{matrix}\right\}$$

$$\left(1+\alpha M(Az,z,t_0)\right)M(Az,z,t_0)>\alpha\min\begin{Bmatrix} M(z,z,t_0)M(z,z,t_0), \\ ,M(Az,z,t_0)M(Az,z,t_0)\end{Bmatrix}$$

$$+\,min\left\{\begin{matrix} M(Az,z,t_0), \\ \max\left\{M(z,z,\epsilon),M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \max\{M(z,z,\epsilon),M(Az,z,(2t_0-\epsilon))\}\end{matrix}\right\}$$

$$M(Az,z,t_0)+\alpha\left(M(Az,z,t_0)\right)^2\geq\alpha\left(M(Az,z,t_0)\right)^2+$$

$$\min\{M(Az,z,t_0),M(Az,z,t_0)$$

$$M(Az,z,t_0)>M(Az,z,t_0)$$

which is impossible and so $Az=\ Sz=z$, which shows that z is a common fixed point of A and S.

Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that $z$is a common fixed point of B and T. Hence z is a common fixed point of A, B, S and T.
Let $\boldsymbol{\varphi}$ be the set of all non-decreasing and continuous functions $\varphi: (0,1] \to (0,1]$ such that $\varphi(\mathrm{t}) > t$ for all $t \in (0,1]$.

Lemma 3.4. Let A, B, S and T be mappings of a complete fuzzy metric space (X, M, Δ) into itself satisfying the conditions (3.1), (3.2), (3.3) and (3.4) of lemma 3.1 and

$$\left(1+\alpha M(Sx,Ty,t)\right)M(Ax,By,t) > \alpha\min\begin{Bmatrix} M(Ax,Sx,t)M(By,Ty,t), \\ M(Sx,By,t)M(Ax,Ty,t)\end{Bmatrix}$$
$$+\varphi\left\{min\left\{\begin{array}{c} M(Sx,Ty,t) \\ sup_{t_1+t_2=\frac{2t}{k}}\min\{M(Ax,Sx,t_1),M(By,Ty,t_3)\} \\ sup_{t_3+t_4=\frac{2t}{k}}\min\{M(Sx,By,t_2),M(Ax,Ty,t_4)\}\end{array}\right.\right\} \quad (3.10)$$

for all $x, y \in X$,$t > 0$, for some $\alpha \ge 0$ and $1 \le k < 2$, $\varphi \in \boldsymbol{\varphi}$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Proof: From the lemma3.3 we have $Ax_n \to z$, $Sx_n \to z$ and $Ty_n \to z$. Now we show the $By_n \to z$.

Let $\lim_{n\to\infty} F(By_n, l, t_0) = 1$. Now we show that $l = z$. Assume that $l \ne z$. Using the inequality (3.10) with $x = x_n,\ y = y_n$, we get for some $t_0 > 0$

$$\left(1+\alpha M(Sx_n,Ty_n,t_0)\right)M(Ax_n,By_n,t_0) > \alpha\min\begin{Bmatrix} M(Ax_n,Sx_n,t_0)M(By_n,Ty_n,t_0), \\ M(Sx_n,By_n,t_0)M(Ax_n,Ty_n,t_0)\end{Bmatrix}$$
$$+\varphi\left\{min\left\{\begin{array}{c} M(Sx_n,Ty_n,t_0) \\ \min\left\{M(Ax_n,Sx_n,\epsilon),M\left(By_n,Ty_n,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\left\{M\left(Sx_n,By_n,\left(\frac{2}{k}t_0-\epsilon\right)\right),M(Ax_n,Ty_n,\epsilon)\right\}\end{array}\right\}\right\}$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$. Letting $n \to \infty$, we have

$$\left(1+\alpha M(z,z,t_0)\right)M(z,l,t_0) > \alpha\min\begin{Bmatrix} M(z,z,t_0)M(l,z,t_0), \\ M(z,l,t_0)M(z,z,t_0)\end{Bmatrix}$$
$$+\varphi\left\{min\left\{\begin{array}{c} M(z,z,t_0) \\ \min\left\{M(z,z,\epsilon),M\left(z,l,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\left\{M\left(z,l,\left(\frac{2}{k}t_0-\epsilon\right)\right),M(z,z,\epsilon)\right\}\end{array}\right\}\right\}$$

$$M(z,l,t_0)+\alpha M(z,z,t_0)$$
$$> \alpha M(z,z,t_0)$$
$$+\ \varphi\left\{min\left\{\begin{array}{c} M(z,z,t_0) \\ \min\left\{M\big(z,z,(\epsilon)\big),M\left(z,l,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\} \\ \min\left\{M\left(l,z,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right),M\big(z,z,(\epsilon)\big)\right\}\end{array}\right\}\right\}$$

$$M(z,l,t_0) > \varphi\left(M\left(l,z,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right)$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$, $as\ \epsilon \to 0, M(z,l,t_0) > M\left(l,z,\left(\left(\frac{2}{k}t_0\right)\right)\right)$ for some $t_0 > 0$.

By (3.7) so we have $z = l$. Then the pairs (A,S) and (B,T) satisfy the CLR_{ST} property.

Remark 3.2. Lemmas 3.1, 3.2, 3.3 and 3.4 remain true if we assume that the pair (B,T) satisfies the CLR_T property, $B(X) \subseteq S(X)$ and $S(X)$ is a closed subset of X.

Theorem 3.4. Let A,B,S and T be mappings of a completefuzzy metric space (X,M,Δ) into itself satisfying the inequality (3.10) of lemma 3.4. If the pairs (A,S) and (B,T) satisfy the CLR_{ST} property, the pairs (A,S) and (B,T) have coincidence points. Moreover, if (A,S) and (B,T) are weakly compatible, then A, B,S and T have a unique common fixed point in X.

Proof: Suppose that the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X)\cap T(X)$.
Hence, there exists $u,v \in X$ such that $Su = Tv = z$.
Now we show that $Au = Su = z$.
If$Au \neq Su$, putting $x = u$ and $y = y_n$ in inequality (3.10), we get for some $t_0 > 0$

$$\left(1+\alpha M(Su,Ty_n,t_0)\right)M(Au,By_n,t_0) > \alpha \min\begin{Bmatrix} M(Au,Su,t_0)M(By_n,Ty_n,t_0), \\ M(Su,By_n,t_0)M(Au,Ty_n,t_0) \end{Bmatrix}$$

$$+\varphi\left\{min\left\{\begin{matrix} M(Su,Ty_n,t_0), \\ \min\left\{M\left(Au,Su,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(By_n,Ty_n,\epsilon)\right\} \\ \min\left\{M(Su,By_n,\epsilon), M\left(Au,Ty_n,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\} \end{matrix}\right\}\right\}$$

$$\left(1+\alpha M(z,z,t_0)\right)M(Au,z,t_0) > \alpha \min\begin{Bmatrix} M(Au,z,t_0)M(z,z,t_0), \\ M(z,z,t_0)M(Au,z,t_0) \end{Bmatrix}$$

$$+\varphi\left\{min\left\{\begin{matrix} M(z,z,t_0), \\ \min\left\{M(Au,z,\epsilon), M\left(z,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\left\{M(z,z,\epsilon), M\left(Au,z,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\} \end{matrix}\right\}\right\}$$

$$M(Au,z,t_0) \geq \varphi\left\{\min\left\{M(Au,z,\epsilon), M\left(Au,z,\left(\left(\frac{2}{k}t_0-\epsilon\right)\right)\right)\right\}\right\}$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$, as $\epsilon \to 0, M(Au,z,t_0) \geq \varphi\left(M(Au,z,t_0)\right) > M(Au,z,t_0)$,
which is impossible and so $Au = Su = z$. Therefore u is a coincidence point of the pair (A,S).

Now we assert that $Bv = Tv = z$. If $z \neq Bv$, putting $x = u$ and $y = v$ in the inequality (3.10), we get for some $t_0 > 0$

$$\left(1+\alpha M(Su,Tv,t_0)\right)M(Au,Bv,t_0) > \alpha \min\begin{Bmatrix} M(Au,Su,t_0)M(Bv,Tv,t_0), \\ M(Su,Bv,t_0)M(Au,Tv,t_0) \end{Bmatrix}$$

$$+\varphi\left\{min\left\{\begin{matrix} M(Su,Tv,t_0), \\ \min\left\{F(Au,Sv,\epsilon), M\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\{M(Su,Bv,\epsilon), M(Au,Tv,)\} \end{matrix}\right\}\right\}$$

$$\left(1+\alpha M(z,z,t_0)\right)M(z,Bv,t_0) > \alpha \min\begin{Bmatrix} M(z,z,t_0)M(Bv,z,t_0), \\ M(z,Bv,t_0)M(z,z,t_0) \end{Bmatrix}$$

$$+\varphi\left\{min\left\{\begin{matrix} M(z,z,t_0), \\ \min\left\{M(z,z,\epsilon), M\left(Bv,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\left\{M(Bv,z,\epsilon), M\left(z,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \end{matrix}\right\}\right\}$$

$$M(Bv,z,t_0) \geq \varphi\left\{\min\left\{M(Bv,z,\epsilon), M\left(Bv,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\right\}$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$,As $\epsilon \to 0, M(Bv,z,t_0) \geq \varphi\left(M(Bv,z,t_0)\right) > M(Bv,z,t_0)$, which is impossible and so $Bv = Tv = z$. Therefore v is a coincidence point of the pair (B,T).

Since the pair (A,S) is weakly compatible and $Au = Su$ we obtain $Az = Sz$. Now we prove that z is a common fixed point of $A$and S. If$z \neq Az$, applying the inequality (3.10) with $x = z$ and $y = v$, we get for some $t_0 > 0$

$$\left(1+\alpha M(Sz,Tv,t_0)\right)M(Az,Bv,t_0) > \alpha \min\begin{Bmatrix} M(Az,Sz,t_0)M(Bv,Tv,t_0), \\ M(Sz,Bv,t_0)M(Az,Tv,t_0) \end{Bmatrix}$$

$$+\varphi\left\{min\left\{\begin{matrix} M(Az,z,t_0), \\ \min\left\{M(Az,Sz,\epsilon), M\left(Bv,Tv,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \\ \min\left\{M(Sz,Bv,\epsilon), M\left(Az,Tv,(2t_0-\epsilon)\right)\right\} \end{matrix}\right\}\right\}$$

$$\left(1+\alpha M(Az,z,t_0)\right)M(Az,z,t_0) > \alpha \min\begin{Bmatrix} M(z,z,t_0)M(z,z,t_0), \\ M(Az,z,t_0)M(Az,z,t_0) \end{Bmatrix}$$

$$+\varphi\left\{min\left\{\begin{matrix} M(Az,z,t_0), \\ \min\left\{M\left(z,z,\left(\frac{2}{k}t_0-\epsilon\right)\right), M(Az,z,\epsilon)\right\} \\ \min\left\{M(z,z,\epsilon), M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\} \end{matrix}\right\}\right\}$$

$$M(Az,z,t_0) + \alpha\left(M(Az,z,t_0)\right)^2 \geq \alpha\left(M(Az,z,t_0)\right)^2 +$$

$$\varphi\left(\min\left\{M(Az,z,t_0), min\left\{M(Az,z,\epsilon), M\left(Az,z,\left(\frac{2}{k}t_0-\epsilon\right)\right)\right\}\right\}\right)$$

for all $\epsilon \in (0,\frac{2}{k}t_0)$, as $\epsilon \to 0$, we get $M(Az,z,t_0) \geq \varphi\big(M(Az,z,t_0)\big) > M(Az,z,t_0)$, which is impossible and so $Az = Sz = z$, which shows that z is a common fixed point of A and S.
Since the pair (B,T) is weakly compatible, we get $Bz = Tz$. Similarly, we can prove that $z$is a common fixed point of B and T. Hence z is a common fixed point of A, B, S and T.

Remark 3.3. In the Theorems 3.1, 3.2, 3.3 and 3.4 by a similar manner, we can prove that A, B, S and T have a unique common fixed point in X if we assume that the pairs (A,S) and (B,T) verify $JCLR_{ST}$ property or CLR_{AB} property instead of CLR_{ST} property.

Theorem 3.5. Let A, B, S and T be mappings of a completefuzzy metric space (X, M, Δ) into itself satisfying the conditions of lemma 3.1, or lemma 3.2, lemma 3.3, lemma 3.4,. If the pairs (A,S) and (B,T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.
Proof: In view of lemma 3.1, lemma 3.2, lemma 3.3 and lemma 3.4, the pairs (A,S) and (B,T) satisfies the CLR_{ST} property, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = z$, where $z \in S(X) \cap T(X)$.

The rest of the proof follows as in the proof of theorems 3.1, 3.2, 3.3 and 3.4.

Example 3.2. We retain A and B and replace S and T in the example 3.1 by the following mappings

$$Sx = \begin{cases} 3 & if x = 3 \\ 6 & if x \in (3,5) \\ \dfrac{x+1}{3} & if x \in [5,11) \end{cases}, \qquad Tx = \begin{cases} 3 & if x = 3 \\ 9 & if x \in (3,5) \\ x-2 & if x \in [5,11), \end{cases}$$

Therefore,
$A(X) = \{3,4\} \subset [3,9] = T(X)$and $B(X) = \{3,5\} \subset [3,6] = S(X)$.
Thus all the conditions of Theorem3.3 are satisfied and 3 is the unique common fixed point of the pairs (A,S) and (B,T). Also it is noted that theorem 3.1 cannot be used in the context of this example as $S(X)$ and $T(X)$ are closed subsets of X.
Applying theorems 3.1, 3.2, 3.3 and 3.4, we deduce a common fixed point for four finite families of self-mappings given by the following corollary.

Corollary 3.1. Let $\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ be four finite families of self-mappings of afuzzy metric space (X, M, Δ) where Δ is a continuous t-norm with $A = A_1A_2 \dots A_m$,$B = B_1B_2 \dots B_n$,$S = S_1S_2 \dots S_p$,$T = T_1T_2 \dots T_q$ satisfies the inequality (3.5) of lemma 3.1 or the inequality (3.8) of lemma 3.2. Suppose that the pairs (A,S) and (B,T) verify the CLR_{ST} property. Then
$\{A_i\}_{i=1}^m$, $\{B_r\}_{r=1}^n$, $\{S_k\}_{k=1}^p$ and $\{T_h\}_{h=1}^q$ a unique common fixed point in X provided that the pairs of families $(\{A_i\}_{i=1}^m, \{S_k\}_{k=1}^p)$, $(\{B_r\}_{r=1}^n, \{T_h\}_{h=1}^q)$ commute pair wise.
By setting $A = A_1 = A_2 = \cdots = A_m$, $B = B_1 = B_2 = \cdots = B_n$,$S = S_1 = S_2 \dots = S_p$, $T = T_1 = T_2 = \cdots = T_q$ in the Corollary 3.1, we get that A, B, S and T have a unique common fixed point in X, provided that the pairs (A^m, S^p) and (B^n, T^q) commute pair wise.

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