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Hamacher Sum and Hamacher Product of Fuzzy Matrices

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Abstract. In this paper, we define two new operations called Hamacher sum and Hamacher product of fuzzy matrices and investigate the algebraic properties of fuzzy matrices under these operations as well as the properties of fuzzy matrices in the case where these new operations are combined with the well-known operations $\wedge, \vee$, we have proved some new inequalities connected with fuzzy matrices.

Keywords: Fuzzy matrix, Hamacher sum, Hamacher product

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1.Introduction

Among the well-known operations which can be performed on fuzzy matrices are the operations of component wise addition, multiplication, algebraic product, algebraic sum and complement. Much research works are done concerning fuzzy matrices and their applications to medical sciences, engineering, management environment and social sciences. In 1977, Thomason [8] initiated the study on convergence of powers of fuzzy matrix. The theory of fuzzy matrix was developed by Kim and Roush [2] as an extension of Boolean matrix, with max-min operation over the fuzzy algebra. Ragab and Emam [4] presented some properties of the min-max composition of fuzzy matrices. Meenakshi [3] studied the theoretical developments of fuzzy matrices. Shyamal and Pal [5,6] introduced two binary operators on fuzzy matrices and proved several properties on them and these operations are extended to intuitionistic fuzzy matrices and studied its algebraic properties by Sriram and Boobalan [7]. Zhang and Zheng [9] introduced bounded sum and bounded product of fuzzy matrices and presented several properties on these operations.

The paper is organized in three sections. In section 2,we give the basic definitions and operations on fuzzy matrices which will be used in this paper. In section 3, we introduce the hamacher operations on fuzzy matrices and focusing on its properties. The De Morgan's law for the hamacher operations are established in section 4.

2. Preliminaries and definitions

In this section, we give some definitions and preliminaries which are applied in the paper.

Definition 2.1. [3] A fuzzy matrix(FM) of order $m\times n$ is defined as $A=(a_{ij})$, where $a_{ij}\in[0,1]$. Let F_{mn} denote the set of all fuzzy matrices of order $m\times n$.

Definition 2.2. [3] For $A\in F_{mn}$, the transpose is obtained by interchanging its rows and columns and is denoted by A^T.

Definition 2.3. [3] The $m\times n$ zero matrix O is the matrix all of whose entries are zero. The $n\times n$ identity matrix I is the matrix (a_{ij}) such that $a_{ij}=1$ if $i=j$ and $a_{ij}=0$ if $i\neq j$. The $m\times n$ universal matrix J is the matrix all of whose entries are 1.

Definition 2.4. [3] Let $A=(a_{ij})$ and $B=(b_{ij})\in F_{mn}$. we write $A\leq B$ if $a_{ij}\leq b_{ij}$ for all i,j and we say that A is dominated by B (or) B dominates A. A and B are said to be comparable, if either $A\leq B$ (or) $B\leq A$.

We define the following operators for any two matrices $A=(a_{ij})$ and $B=(b_{ij})$ of order $m\times n$,

$(i)\, A\vee B=(\max(a_{ij},b_{ij}))$

$(ii)\, A\wedge B=(\min(a_{ij},b_{ij}))$

$(iii)\, A^c=(1-a_{ij})$ (the complement of A).

3. Some results on Hamacher sum and Hamacher product of fuzzy matrices

Hamacher [1] introduced a generalized $t-norm$ and $t-conorm$ by defining as

$T(x,y)=\left(\dfrac{xy}{\gamma+(1-\gamma)(x+y-xy)}\right)$ and $T^*(x,y)=\left(\dfrac{x+y-xy-(1-\gamma)xy}{1-(1-\gamma)xy}\right)$.

From these operations, when $\gamma=0$ they will reduce to algebraic $t-norm$ and $t-conorm, T(x,y)=\left(\dfrac{xy}{x+y-xy}\right)$ and $T^*(x,y)=\left(\dfrac{x+y-2xy}{1-xy}\right)$.

In this section, based on these operations, Hamacher sum and Hamacher product of fuzzy matrices are definied as $A\oplus_H B=\left(\dfrac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}}\right)$ and $A\odot_H B=\left(\dfrac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}}\right)$

Lemma 3.1. For $a,b\in[0,1], \dfrac{ab}{a+b-ab}\leq\dfrac{a+b-2ab}{1-ab}$.

Proof: We know that $(a+b)^2\geq 4ab$ (1.1)

$a+b-ab\leq 1\Rightarrow 1+3(a+b-ab)\leq 4$

$\Rightarrow (1+3(a+b-ab))ab\leq 4ab$ (1.2)

From (1.1) and (1.2)

$(1+3(a+b-ab))ab \le 4ab \le (a+b)^2$

$0 \le (a+b)^2 - (1+3(a+b-ab))ab$

$= (a+b)^2 + 3ab(ab-a-b) - ab$

$ab \le (a+b)^2 + 3a^2b^2 - 3ab(a+b)$

$ab - a^2b^2 \le (a+b-2ab)(a+b-ab)$

$\frac{ab}{a+b-ab} \le \frac{a+b-2ab}{1-ab}$

Hence the result.

Definition 3.2. For any fuzzy matrices A and B of the same size. The Hamacher sum of A and B is defined by $A \oplus_H B = \left(\frac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}} \right)$. The Hamacher product of A and B is defined by $A \odot_H B = \left(\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}} \right)$.

Next, we have prove some results of properties of Hamacher sum and Hamacher product.

Property 3.3. Let A and B be any two fuzzy matrices of same size. Then

$$A \odot_H B \le A \oplus_H B.$$

Proof: By using Lemma 3.1,

$\left(\frac{a_{ij}b_{ij}}{a_{ij}+b_{ij}-a_{ij}b_{ij}} \right) \le \left(\frac{a_{ij}+b_{ij}-2a_{ij}b_{ij}}{1-a_{ij}b_{ij}} \right)$ for all i, j.

Hence, ij^{th} entry of $A \odot_H B \le ij^{th}$ entry of $A \oplus_H B$.

Therefore, $A \odot_H B \le A \oplus_H B.$

Property 3.4. For any fuzzy matrix *A*,

$(i) A \oplus_H A \ge A$

$(ii) A \odot_H A \le A$.

Proof: (i) $A \oplus_H A$ is $\left(\frac{2a_{ij}-2a_{ij}^2}{1-a_{ij}^2} \right) = \left(\frac{2a_{ij}}{1+a_{ij}} \right) \ge a_{ij}$ for all i, j.

Since $a_{ij}^2 - a_{ij} \le 0 \Rightarrow a_{ij}^2 + a_{ij} - 2a_{ij} \le 0 \Rightarrow a_{ij}^2 + a_{ij} \le 2a_{ij}$.

Hence, ij^{th} entry of $A \oplus_H A \ge$ ij^{th} entry of A.

Therefore, $A \oplus_H A \ge A$.

(ii) $A \odot_H A$ is $\left(\dfrac{a_{ij}^2}{2a_{ij} - a_{ij}^2}\right) = \left(\dfrac{a_{ij}}{2 - a_{ij}}\right) \leq a_{ij}$ for all i, j.

Since $1 \leq 2 - a_{ij}$ $(ie)., a_{ij} \leq \dfrac{a_{ij}}{2 - a_{ij}}$.

Hence, ij^{th} entry of $A \odot_H A \leq$ ij^{th} entry of A.

Therefore, $A \odot_H A \leq A$.

The following properties are obvious. The operations $\oplus_H$ and $\odot_H$ are commutative as well as associative. Existence of the identity elements with respect to $\oplus_H$ and $\odot_H$ are determined in the following Theorems.

Property 3.5. Let A, B and C be any three fuzzy matrices. Then

$(i) A \oplus_H B = B \oplus_H A,$

$(ii) (A \oplus_H B) \oplus_H C = A \oplus_H (B \oplus_H C),$

$(iii) A \odot_H B = B \odot_H A,$

$(iv) (A \odot_H B) \odot_H C = A \odot_H (B \odot_H C).$

Property 3.6. For a fuzzy matrix A,

$(i) A \oplus_H O = O \oplus_H A = A,$

$(ii) A \odot_H J = J \odot_H A = A,$

$(iii) A \oplus_H J = J,$

$(iv) A \odot_H O = O.$

Thus $(F_{mn}, \oplus_H)$ and $(F_{mn}, \odot_H)$ form commutative monoids. The operators $\oplus_H$ and $\odot_H$ do not obey the De Morgan's laws over transpose.

Property 3.7. For any two fuzzy matrices A and B of same size,

$(i) (A \oplus_H B)^T = A^T \oplus_H B^T,$

$(ii) (A \odot_H B)^T = A^T \odot_H B^T.$

where A^T is the transpose of A.

Property 3.8. For any two fuzzy matrices A and B of same size, if $A \leq B$, then $A \odot_H C \leq B \odot_H C$

Proof: Let $a_{ij} \leq b_{ij}$ for all i, j

$$a_{ij} c_{ij}^2 \leq b_{ij} c_{ij}^2$$

$$a_{ij} c_{ij}^2 + a_{ij} b_{ij} c_{ij} (1 - c_{ij}) \leq b_{ij} c_{ij}^2 + a_{ij} b_{ij} c_{ij} (1 - c_{ij})$$

$a_{ij}c_{ij}^2 + a_{ij}b_{ij}c_{ij} - a_{ij}b_{ij}c_{ij}^2 \le b_{ij}c_{ij}^2 + a_{ij}b_{ij}c_{ij} - a_{ij}b_{ij}c_{ij}^2$

$a_{ij}c_{ij}(c_{ij} + b_{ij} - b_{ij}c_{ij}) \le b_{ij}c_{ij}(c_{ij} + a_{ij} - a_{ij}c_{ij})$

$$\left(\frac{a_{ij}c_{ij}}{a_{ij} + c_{ij} - a_{ij}c_{ij}}\right) \le \left(\frac{b_{ij}c_{ij}}{b_{ij} + c_{ij} - b_{ij}c_{ij}}\right)$$

Therefore, the ij^{th} entry of $A \odot_H C \le ij^{th}$ entry of $B \odot_H C$.
Hencce the result.

Property 3.9. For any two fuzzy matrices A and B of same size, if $A \le B$, then $A \oplus_H C \le B \oplus_H C$

Proof: Let $a_{ij} \le b_{ij}$ for all i, j

$a_{ij}(1 - c_{ij})^2 \le b_{ij}(1 - c_{ij})^2$

$a_{ij}(1 - 2c_{ij} + c_{ij}^2) \le b_{ij}(1 - 2c_{ij} + c_{ij}^2)$

$a_{ij} - 2a_{ij}c_{ij} + a_{ij}c_{ij}^2 \le b_{ij} - 2b_{ij}c_{ij} + b_{ij}c_{ij}^2$

$a_{ij} - 2a_{ij}c_{ij} + a_{ij}c_{ij}^2 + (c_{ij} - a_{ij}b_{ij}c_{ij} + 2a_{ij}b_{ij}c_{ij}^2) \le b_{ij} - 2b_{ij}c_{ij}b_{ij}c_{ij}^2 + (c_{ij} - a_{ij}b_{ij}c_{ij} + 2a_{ij}b_{ij}c_{ij}^2)$

$a_{ij} + c_{ij} - 2a_{ij}c_{ij} - a_{ij}b_{ij}c_{ij} - b_{ij}c_{ij}^2 + 2a_{ij}b_{ij}c_{ij}^2 \le b_{ij} + c_{ij} - 2b_{ij}c_{ij} - a_{ij}b_{ij}c_{ij} - a_{ij}c_{ij}^2 + 2a_{ij}b_{ij}c_{ij}^2$

$a_{ij} + c_{ij} - 2a_{ij}c_{ij} - b_{ij}c_{ij}(a_{ij} + c_{ij} - 2a_{ij}c_{ij}) \le b_{ij} + c_{ij} - 2b_{ij}c_{ij} - a_{ij}c_{ij}(b_{ij} + c_{ij} - 2b_{ij}c_{ij})$

$(a_{ij} + c_{ij} - 2a_{ij}c_{ij})(1 - b_{ij}c_{ij}) \le (b_{ij} + c_{ij} - 2b_{ij}c_{ij})(1 - a_{ij}c_{ij})$

$$\left(\frac{a_{ij} + c_{ij} - 2a_{ij}c_{ij}}{1 - a_{ij}c_{ij}}\right) \le \left(\frac{b_{ij} + c_{ij} - 2b_{ij}c_{ij}}{1 - b_{ij}c_{ij}}\right)$$

Therefore, the ij^{th} entry of $A \oplus_H C \le ij^{th}$ entry of $B \oplus_H C$.
Hence the result.

Property 3.10. For any two fuzzy matrices A and B of same size, Then
$(i)(A \wedge B) \oplus_H (A \vee B) = A \oplus_H B,$
$(ii)(A \wedge B) \odot_H (A \vee B) = A \odot_H B.$

Proof:

$$(i)(A \wedge B) \oplus_H (A \vee B) = \left(min(a_{ij}, b_{ij})\right) \oplus_H \left(max(a_{ij}, b_{ij})\right)$$

$$= \left(\frac{(min(a_{ij}, b_{ij}) + max(a_{ij}, b_{ij}) - 2min(a_{ij}, b_{ij})max(a_{ij}, b_{ij}))}{(1 - min(a_{ij}, b_{ij})max(a_{ij}, b_{ij}))}\right)$$

$$= \left(\frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}}\right)$$

$$= A \oplus_H B.$$

$$(ii)(A \wedge B) \odot_H (A \vee B) = \left(min(a_{ij}, b_{ij})\right) \odot_H \left(max(a_{ij}, b_{ij})\right)$$

$$= \left(\frac{min(a_{ij}, b_{ij}) max(a_{ij}, b_{ij})}{(min(a_{ij}, b_{ij}) + max(a_{ij}, b_{ij}) - min(a_{ij}, b_{ij}) max(a_{ij}, b_{ij}))} \right)$$

$$= \left(\frac{a_{ij} b_{ij}}{a_{ij} + b_{ij} - a_{ij} b_{ij}} \right)$$

$$= A \odot_H B.$$

Hence the result.

4. Results on complement of fuzzy matrix

In this section, the complement of a fuzzy matrix is used to analyse the complementary nature of any system. For example, if A represents the crowdness of the roads at a particular time period then its complement A^c represents the clearness of the roads. Using the following results we can study the complement nature of a system with the help of original fuzzy matrix. The operator complement obey the De Morgan's law for the operator $\oplus_H$ and $\odot_H$. This is established in the following property.

Property 4.1. For any two fuzzy matrices A and B of same size,

$(i)(A \oplus_H B)^c = A^c \odot_H B^c$,

$(ii)\ (A \odot_H B)^c = A^c \oplus_H B^c$,

$(iii)\ (A \oplus_H B)^c \le A^c \oplus_H B^c$,

$(iv)\ (A \odot_H B)^c \ge A^c \odot_H B^c$.

Proof:

$$(i) A^c \odot_H B^c = \left((1 - a_{ij}) \odot_H (1 - b_{ij})\right)$$

$$= \left(\frac{(1 - a_{ij})(1 - b_{ij})}{(1 - a_{ij}) + (1 - b_{ij}) - (1 - a_{ij})(1 - b_{ij})} \right)$$

$$= \left(\frac{(1 - b_{ij} - a_{ij} + a_{ij} b_{ij})}{(1 - a_{ij}) + (1 - b_{ij}) - (1 + b_{ij} + a_{ij} - a_{ij} b_{ij})} \right)$$

$$= \left(\frac{1 - b_{ij} - a_{ij} + a_{ij} b_{ij}}{2 - a_{ij} - b_{ij} - 1 + b_{ij} + a_{ij} - a_{ij} b_{ij}} \right)$$

$$= \left(\frac{1 + a_{ij}b_{ij} - a_{ij} - b_{ij}}{1 - a_{ij}b_{ij}} \right)$$

$$= \left(\frac{(1 - a_{ij})(1 - b_{ij})}{1 - a_{ij}b_{ij}} \right)$$

$$= \left(1 - \frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{1 - a_{ij}b_{ij}} \right)$$

$$= (A \oplus_H B)^c.$$

$$(ii) A^c \oplus_H B^c = \left((1 - a_{ij}) \oplus_H (1 - b_{ij}) \right)$$

$$= \left(\frac{(1 - a_{ij}) + (1 - b_{ij}) - 2(1 - a_{ij})(1 - b_{ij})}{1 - (1 - a_{ij})(1 - b_{ij})} \right)$$

$$= \left(\frac{2 - a_{ij} - b_{ij} - 2(1 - b_{ij} - a_{ij} + a_{ij}b_{ij})}{1 - (1 - b_{ij} - a_{ij} + a_{ij}b_{ij})} \right)$$

$$= \left(\frac{2 - a_{ij} - b_{ij} - 2 + 2b_{ij} + 2a_{ij} - 2a_{ij}b_{ij}}{1 - 1 + b_{ij} + a_{ij} - a_{ij}b_{ij}} \right)$$

$$= \left(\frac{a_{ij} + b_{ij} - 2a_{ij}b_{ij}}{a_{ij} + b_{ij} - a_{ij}b_{ij}} \right)$$

$$= \left(1 - \frac{a_{ij}b_{ij}}{a_{ij} + b_{ij} - a_{ij}b_{ij}} \right)$$

$$= (A \odot_H B)^c.$$

(iii) From Property 1. $A \oplus_H B \geq A \odot_H B.$

Then $(A \oplus_H B)^c \leq (A \odot_H B)^c$

$$= A^c \oplus_H B^c.$$

$$(iv) (A \odot_H B)^c \geq (A \oplus_H B)^c.$$

$$= A^c \odot_H B^c.$$

Hence the result.

5. Conclusions

In this article, Hamacher sum and Hamacher product of fuzzy matrices are defined and some properties are proved. The set of all fuzzy matrices form a commutative monoids with respect to these operations. Thus the hamacher sum and hamacher product are very useful to further works.

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