Domination in Fuzzy Soft Graphs

R. Jahir Hussain and S. Satham Hussain

P.G and Research Department of Mathematics, Jamal Mohamed College
Trichy, Tamil Nadu, India.

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Abstract. Fuzzy soft set are introduced by author Molodtsov, which is solve imprecise problems in the field of engineering, social science, economic, medical science and environment. This paper addresses the study of domination in fuzzy soft graphs. By using the concept of strength of a path, strength of connectedness and strong arc, domination set is established. The necessary and sufficient condition for the minimum domination set of FSG is investigated. Further some properties of independent domination number of FSG are obtained and the proposed concepts are described with suitable examples. Finally, we state and prove some results related to these concepts.

Keywords: Fuzzy soft set; fuzzy soft domination; independent domination fuzzy soft graph; total domination fuzzy soft graph.

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1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfield in 1975. Though it is very young it has been growing very fast and has hemeous applications in varies fields. Fuzzy set was introduced by Zadeh [8] whose basic components is only a membership function. The generalization of Zadeh’s fuzzy set, called fuzzy soft set was introduced by Molodtsov [2]. Molodtsov [1] applied this theory to several direction such as smoothness of function, game theory, operation research, probability and measurement were more active doing research on soft set. A. Somasundaram and S. Somasundaram [9] presented more concept of independent domination, connected domination in fuzzy graphs. In 2006, Nagoorgani and Chandrasekaran [7] define µ complement of fuzzy graph, which slightly differs from the definition from the definition of complement of fuzzy graph discussed by Sunitha and Vijayakumar[6]

irregular fuzzy soft graphs and described applications of fuzzy soft graphs in social network and road network. Akram and Zafar [13] introduced notions of fuzzy soft cycles, fuzzy soft bridge, fuzzy soft cut node, fuzzy soft trees, and investigate some of their fundamental properties. They also studied some types of arcs in fuzzy soft graphs.

In this paper, we introduced dominating set, domination number, independent set, independent number, total dominating set and total dominating number in fuzzy soft graph. The necessary and sufficient condition for the minimum domination set of FSG is investigated. Further some properties of independent domination number of FSG are obtained and the proposed concepts are described with suitable examples.

2. Preliminaries

Definition 2.1. Let $U$ be an initial universe set and $E$ be the set of parameters. Let $P(U)$ denotes the power set of $U$. A pair $(F, E)$ is called a soft set over $U$ where $F$ is a mapping given by $F : E \rightarrow P(U)$.

Definition 2.2. Let $U$ be an initial universe set and $E$ be the set of parameters. Let $A \subseteq E$. A pair $(F, A)$ is called fuzzy soft set over $U$ where $F$ is mapping given by $F : A \rightarrow I^U$ where $I^U$ denotes the collection of all fuzzy subsets of $U$.

Definition 2.3. Let $V$ be a nonempty finite set and $\sigma(x, y) \leq \sigma(x) \land \sigma(y)$ for all $(x, y) \in V \times V$. Then the pair $G = (\sigma, \mu)$ is called a fuzzy graph over the set $V$. Here $\sigma$ and $\mu$ are respectively called fuzzy vertex and fuzzy edge of the fuzzy graph $G = (\sigma, \mu)$.

Definition 2.4. Let $G = (\sigma, \mu)$ be a fuzzy graph. The order of $G = (\sigma, \mu)$ is defined as:

$$O(G) = \sum_{u \in V} \sigma(u)$$

and the size of $G = (\sigma, \mu)$ is defined as:

$$S(G) = \sum_{u, v \in V} \mu(u, v).$$

Definition 2.5. Let $G = (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex $u$ is defined as:

$$d(u) = \sum_{v \in V, v \neq u} \mu(u, v).$$

Definition 2.6. Given $G = (\sigma, \mu)$ be a fuzzy graph. The complement of $G$ is defined as $\overline{G} = (\sigma, \overline{\mu})$, where $\overline{\mu}(x, y) = \sigma(x) \land \sigma(y) - \mu(x, y)$ for all $x, y \in V$. When $G$ is a fuzzy graph, $\overline{G} = (\sigma, \overline{\mu})$ is also a fuzzy graph.
Definition 2.7. Given $G = (\sigma, \mu)$ to be a fuzzy graph. The $\mu$-complement of $G$ is defined as $G^\mu = (\sigma^\mu, \mu^\mu)$, where $\mu^\mu(x, y) = \sigma(x) \land \sigma(y) - \mu(x, y)$ if $\mu(x, y) > 0$ and $\mu^\mu(x, y) = 0$ if $\mu(x, y) = 0$.

Definition 2.8. Let $V = \{x_1, x_2, \ldots, x_n\}$ is non empty set, $E$ (Parameters Set) and $A \subseteq E$. Also let,

(i) $\rho : A \to F(V)$ (Collection of all fuzzy subsets in $V$)

$e \mapsto \rho(e) = \rho_e$ (say) and

$\rho_e : V \to [0,1]$

$x \mapsto \rho_e(x)$

$(A, \rho)$: Fuzzy soft vertex.

(ii) $\mu : A \to F(V \times V)$ (Collection of all fuzzy subsets in $V \times V$)

$e \mapsto \mu(e) = \mu_e$ (say)

$\mu_e : V \times V \to [0,1]$

$(x_i, x_j) \mapsto \mu_e(x_i, x_j)$

$(A, \mu)$: Fuzzy soft edge.

Then $((A, \rho), (A, \mu))$ is called a fuzzy soft graph if and only if $\mu(x_i, x_j) \leq \rho_e(x_i) \land \rho_e(x_j)$ for all $e \in A$ and for all $i, j = 1, 2, \ldots$, and this fuzzy soft graphs is denoted by $G_{AV}$.

Definition 2.9. The underlying crisp graph of a fuzzy soft graph $G_{AV} = ((A, \rho), (A, \mu))$ is denoted by $G^* = (\rho^*, \mu^*)$, where $\rho = x \in V : \rho_e(x) > 0$ for some $e \in E$, $\mu^* = (x, x) \in V \times V : \mu_e(x, x) > 0$ for some $e \in E$.

Definition 2.10. A fuzzy soft graph $G_{AV} = ((A, \rho), (A, \mu))$ is called strong fuzzy soft graph if $\mu(x, x) = \rho_e(x) \land \rho_e(x)$ for all $(x, x) \in \mu^*$, $e \in A$ and is complete fuzzy soft graph if $\mu_e(x, x) = \rho_e(x) \land \rho_e(x)$ for all $(x, x) \in \rho^*$, $e \in A$.

Definition 2.11. Let $G_{AV} = ((A, \rho), (A, \mu))$ be a fuzzy soft graph Then the order of $G_{AV}$ is defined as:

$$O(G_{AV}) = \sum_{e \in A} \sum_{x \in V} \rho_e(x_i)$$

and the size of $G_{AV}$ is defined as:
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\[ O(G_{AV}) = \sum_{e \in A} \left( \sum_{x_i, x_j} \mu_e(x_i, x_j) \right) \]

**Definition 2.12.** Let \( G_{AV} = ((A, \rho), (A, \mu)) \) be a fuzzy soft graph. The order of a vertex \( u \) is defined as:

\[ d_{G_{AV}}(u) = \sum_{e \in A} \left( \sum_{x \in V, u \neq v} \mu_e(u, v) \right). \]

**Definition 2.13.** A fuzzy soft edge joining a fuzzy soft vertex to itself is called a fuzzy soft loop.

**Definition 2.14.** Let \( G_{AV} = ((A, \rho), (A, \mu)) \) be a fuzzy soft graph. If for all \( e \in A \) there is more than one fuzzy soft edge joining two soft vertices, then the fuzzy soft graph is called a fuzzy soft pseudo graph and these edges are called fuzzy soft multiple edges.

**Definition 2.15.** \( G_{AV} = ((A, \rho), (A, \mu)) \) is called a fuzzy soft simple graph if it has neither fuzzy soft loops nor fuzzy soft multiple edges for all \( e \in A \).

**Definition 2.16.** Let \( G_{AV} = ((A, \rho), (A, \mu)) \) be a fuzzy soft graph. Then the \( G_{AV} \) is called isolated fuzzy soft graph if \( \mu_e(x_i, x_j) = 0 \) for all \( x_i, x_j \in V \times V, e \in A \).

**Definition 2.17.** An edge in a fuzzy soft graph \( G_{AV} = ((A, \rho), (A, \mu)) \) is said to be an effective fuzzy soft edge if \( \mu_e(x_i, x_j) = \rho_e(x_i) \wedge \rho_e(x_j) \) where \( (x_i, x_j) \in V \times V, e \in A \).

**Definition 2.18.** If two fuzzy soft vertices have a fuzzy soft edge joining them, then they are called fuzzy soft adjacent vertices. And if two fuzzy soft edges are incident with a common fuzzy soft vertex, then they are called fuzzy soft adjacent edges.

3. Main results

**Definition 3.1.** The strength of connectedness between two nodes \( u, v \) in a fuzzy soft graph \( G_{AV} \) is \( \mu^e_v(x_i, x_j) = \sup \{ \mu^k_e(x_i, x_j) : k = 1, 2, 3, \ldots \} \) where

\[ \mu^k_e(x_i, x_j) = \sup \{ \mu_e(x_i, x_{j+1}) \wedge \mu_e(x_{j+2}, x_j) \wedge \cdots \wedge \mu_e(x_{k-1}, x_j) \} \]

**Definition 3.2.** An arc \( (x_i, x_j) \) is said to be a strong arc or strong edge, if \( \mu_e(u, v) > \mu^e_v(x_i, x_j) \) and the node \( x_j \) is said to be strong neighbour of \( x_i \). If \( (x_i, x_j) \) is not strong arc then \( x_j \) is called isolated node or isolated vertex. In a fuzzy soft graph, every arc is a strong arc then the graph is called strong arc fuzzy soft graph.
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Let \( x_i \) be a node in fuzzy soft graph \( G_{A,v} \) then \( N(x_i) = \{ (x_i, x_j) \} \) is strong arc is called neighbourhood of \( x_i \) and \( N(x_i) = N(x_i) \cup \{ x_i \} \) is closed neighbourhood of \( x_i \).

**Definition 3.3.** A vertex \( x_i, x_j \in \mu, e \in A \) of a fuzzy soft graph \( G_{A,v} = ((A, \rho), (A, \mu)) \) is said to be an isolated vertex if \( x_i, x_j \in \mu^*, \mu^*(x_i, x_j) = 0 \) for all \( x_i, x_j \neq x_i, x_j \). That is \( N(x_i) = \emptyset \). Thus an isolated vertex does not dominated any other vertex of \( G_{A,v} \).

**Definition 3.4.** Let \( G_{A,v} = (A, \rho), (A, \mu) \) be a fuzzy soft graph on \( \rho^*, e \in A \). Let \( (x_i, x_j), (x_k, x_l) \in \rho^*, e \in A \) we say that \( x_i, x_j \) dominates \( x_k, x_l \) in \( G_{A,v} \) if there exists a strong edges between them.

(i) For any \( (x_i, x_j), (x_k, x_l) \in \rho^*, e \in A \) if \( (x_i, x_j) \) dominates \( (x_k, x_l) \) then \( (x_k, x_l) \) dominates \( (x_i, x_j) \) and hence domination is a symmetric relation on \( \rho^*, e \in A \).

(ii) For any \( x_i, x_j \in \rho^*, N(x_i, x_j) \) is precisely the set of all vertices in \( \rho^* \) which dominated by \( x_i, x_j \).

(iii) If \( \mu^*(x_i, x_j) < \mu^*(x_k, x_l) \) for all \( (x_i, x_j), (x_k, x_l) \in \rho^*, e \in A \) then the dominating set of \( G_{A,v} \) is \( \rho^* \).

**Definition 3.5.** Let \( G_{A,v} = (\rho^*, \mu^*) \) be a fuzzy soft graph and \( (x_i, x_j) \) be a node in \( G_{A,v} \) then there exists a node \( (x_k, x_l) \) such that \( ((x_i, x_j), (x_k, x_l)) \) is a strong arc then we say that \( (x_k, x_l) \) dominates \( x_i, x_j \).

**Definition 3.6.** A subset \( D \) of \( V \) is called a dominating set in \( G_{A,v} \) if for every vertex \( x_{k-1} \in V - D \), there exists a vertex \( x_i, x_j \in D \) such that \( x_i, x_j \) dominates \( x_{k-1}, x_i, x_j \in D \).

**Definition 3.7.** A dominating set \( D \) of \( G_{A,v} \) is said to be minimal dominating set if no proper subset of \( D \) is a dominating set.

**Definition 3.8.** Minimum cardinality among all dominating set is called lower domination number of \( G_{A,v} \), and is denoted by \( d_b(G_{A,v}) \)

\[
d_b(G_{A,v}) = \sum_{x \in A} \sum_{x \in V} \rho_x(D)
\]

**Definition 3.9.** Maximum cardinality among all dominating set is called upper domination number of \( G_{A,v} \), and is denoted by \( D_b(G_{A,v}) \)

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**Definition 3.10.** Consider a fuzzy soft graph $G_{A,\rho}$, where $V = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Here $G_{A,\rho}$ described by table and

$\mu_{e_i}(x_i, x_j) \in V \times V \{ (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_1, x_4) \}$ and for all $e \in E$.

**Table 1:** Tabular representation of a fuzzy soft graph

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>0.5</td>
<td>0.7</td>
<td>0.8</td>
<td>0.1</td>
</tr>
<tr>
<td>$e_2$</td>
<td>0.4</td>
<td>0.9</td>
<td>0.8</td>
<td>0.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$(x_1, x_2)$</th>
<th>$(x_2, x_3)$</th>
<th>$(x_3, x_4)$</th>
<th>$(x_1, x_4)$</th>
<th>$(x_1, x_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>0.4</td>
<td>0.4</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.2</td>
<td>0.5</td>
<td>0.6</td>
<td>0.2</td>
<td>0.2</td>
</tr>
</tbody>
</table>

**Figure 1:** Fuzzy soft domination

Here for corresponding parameter $e_1$, $\{ \{x_1\}, \{x_1, x_1\}, \{x_1, x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_4\}\}$ are dominating set, for corresponding parameter $e_2$,  

$\{ \{x_1\}, \{x_1, x_1\}, \{x_1, x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_4\} \}$ are dominating set.

for corresponding parameter $e_1$, minimum dominating set is $\{x_1\}$.

for corresponding parameter $e_2$, minimum dominating set is $\{x_1\}$.

Fuzzy soft graph minimum dominating number is $d_{\rho} G_{A,\rho} = 0.9$

for corresponding parameter $e_1$, maximum dominating set is $\{x_1\}$.

for corresponding parameter $e_2$, maximum dominating set is $\{x_2\}$.

Fuzzy soft graph maximum dominating number is $D_{\rho} G_{A,\rho} = 1.5$

**Definition 3.11.** Two vertices in a fuzzy soft graph $G_{A,\rho} = ((A, \rho), (A, \mu))$ are said to be independent if there is no strong edge between them.
Definition 3.12. A subset $D$ of fuzzy soft graph, $G_{A,v}$, is said to be independent set if $\mu_\rho(x_i, x_j) < \mu_\rho^-(x_i, x_j)$ for all $x_i, x_j \in \rho, e \in A$.

Definition 3.13. An independent set $D$ of fuzzy soft graph, $G_{A,v} = ((A, \rho), (A, \mu))$ is said to be maximal independent, if for every vertex $x_i, x_j \in V - D$ the set $D \cap \{x_i, x_j\}$ is not independent for all $x_i, x_j \in V$.

Definition 3.14. The minimum cardinality among all minimum independent set is called lower independent number of $G_{A,v}$, and is denoted by $i_B(G_{A,v}) = \sum_{e \in A} \sum_{D \notin V} \rho_\rho(d)$.

Definition 3.15. The maximum cardinality among all maximum independent set is called upper independent number of $G_{A,v}$, and is denoted by $I_d(G_{A,v}) = \sum_{e \in A} \sum_{D \notin V} \rho_\rho(d)$.

Definition 3.16. Consider a fuzzy soft graph $G_{A,v}$, where $V = \{a,b,c,d\}$ and $E = \{e_1, e_2\}$. Here $G_{A,v}$ described by table and $\mu_\rho(x_i, x_j) \in V \times V \setminus \{(a,b), (b,c), (c,d), (d,e), (a,e), (b,d)\}$ and for all $e \in E$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>0.5</td>
<td>0.6</td>
<td>0.7</td>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>$e_2$</td>
<td>0.5</td>
<td>0.7</td>
<td>0.8</td>
<td>0.5</td>
<td>0.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$(a,b)$</th>
<th>$(b,c)$</th>
<th>$(c,d)$</th>
<th>$(d,e)$</th>
<th>$(a,e)$</th>
<th>$(b,d)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu$</td>
<td>0.5</td>
<td>0.6</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.5</td>
<td>0.7</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Figure 2:
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For corresponding parameter $e_1$ minimum independent dominating set is $\{a, d\}$.

Fuzzy soft graph minimum independent domination number $i_bG_{A,v} = 2$

for corresponding parameter $e_2$ minimum independent dominating set is $\{e, c\}$.

Fuzzy soft graph maximum independent dominating number $I_bG_{A,v} = 3.2$

**Definition 3.17.** Let $G_{A,v} = ((A, \rho), (A, \mu))$ be a fuzzy soft graph without isolated vertices. A set $D$ is a total dominating set if for every vertex $x_i, x_m \in V$, there exists a vertex $x_i, x_j \in D, x_i, x_j \neq x_i, x_m$ such that $x_i, x_j$ dominates $x_i, x_m$ for all $e \in A, x_i, x_j \in V$.

**Definition 3.18.** The minimum cardinality among all minimum total dominating set is called lower total domination number of $G_{A,v}$ and is denoted by $t_b(G_{A,v}) = \sum_{e \in A} \left( \sum_{D \in V} \rho_x(d) \right)$.

**Definition 3.19.** The maximum cardinality among all maximum total dominating set is called upper total domination number of $G_{A,v}$ and is denoted by $I_b(G_{A,v}) = \sum_{e \in A} \left( \sum_{D \in V} \rho_x(d) \right)$.

**Theorem 3.20.** A dominating set $D$ of an FSG, $G_{A,v} = ((A, \rho), (A, \mu))$ is a minimal dominating set if and only if for each $d \in D$ one of the following conditions holds.

(i) $d$ is not a strong neighbor of any vertex in $D$.

(ii) There is a vertex $v \in V - \{D\}$ such that $N(u) \cap D = d$.

**Proof:** Assume that $D$ is a minimal dominating set of $G_{A,v}$. Then for every vertex $d \in D, D - \{d\}$ is not a dominating set and hence there exists $v \in V - (D - \{d\})$ which is not dominated by any vertex in $D - \{d\}$.

If $v = d$, we get $v$ is not a strong neighbor of any vertex in $D$. If $v \neq d$, $v$ is not dominated by $D - \{v\}$, but is dominated by $D$, then the vertex $v$ is strong neighbor only to $d$ in $D$. That is, $N(v) \cap D = d$.

Conversely, assume that $D$ is a dominating set and for each vertex $d \in D$, one of the two conditions holds, suppose $D$ is not a minimal dominating set, then there exists a vertex $d \in D, D - \{d\}$ is a dominating set. Hence $d$ is a strong neighbor to at least one vertex in $D - \{d\}$, the condition one does not hold. If $D - \{d\}$ is a dominating set then every vertex in $V - D$ is a strong neighbor at least one vertex in $D - \{d\}$, the
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second condition does not hold which contradicts our assumption that at least one of these conditions holds. So \( D \) is a minimal dominating set.

**Theorem 3.21.** Let \( G_{A,\nu} = ((A, \rho),(A, \mu)) \) be an FSG without isolated vertices and \( D \) is a minimal dominating set. Then \( V - D \) is a dominating set of \( G_{A,\nu} \).

**Proof:** \( D \) be a minimal dominating set. Let \( v \) be any vertex of \( D \). Since \( G_{A,\nu} \) has no isolated vertices, there is a vertex \( d \in N(v) \). \( v \) must be dominated by at least one vertex in \( D - v \) that is \( D - v \) is a dominating set. By above theorem, it follows that \( d \in V - D \). Thus every vertex in \( D \) is dominated by at least one vertex in \( V - D \), and \( V - D \) is a dominating set.

**Theorem 3.22.** An independent set is a maximal independent set of FSG, \( G = (V, E) \) if and only if it is independent and dominating set.

**Proof:** Let \( D \) be a maximal independent set in an FSG, and hence for every vertex \( v \in V - D \), the set \( D \cup v \) is not dependent. For every vertex \( v \in V - D \), there is a vertex \( u \in D \) such that \( u \) is a strong neighbor to \( v \). Thus \( D \) is a dominating set. Hence \( D \) is both dominating and independent set.

Conversely, assume \( D \) is both independent and dominating. Suppose \( D \) is not maximal independent, then there exists a vertex \( v \in V - D \), the set \( D \cup v \) is independent. If \( D \cup v \) is independent then no vertex in \( D \) is a strong neighbor to \( v \). Hence \( D \) cannot be a dominating set, which is contradiction. Hence \( D \) is a maximal independent set.

**Theorem 3.23.** Every maximal independent set in an FSG, \( G = (V, E) \) is a minimal dominating set.

**Proof:** Let \( S \) be a maximal independent set in a FSG, by previous theorem, \( S \) is a dominating set. Suppose \( S \) is not a minimal dominating set, then there exists at least one vertex \( v \in S \) for which \( S - v \) is a dominating set. But if \( S - v \) dominates \( V - S - (v) \), then at least one vertex in \( S - v \) must be a strong neighbor to \( v \). This contradicts the fact that \( S \) is an independent set of \( G \). Therefore, \( S \) must be a minimal dominating set.

4. Conclusion

In this paper, we have investigated the domination in fuzzy soft graph which will give new ideas in this field. Also regular fuzzy soft graph have been discussed. Further these results can be extended to the field of intuitionistic fuzzy soft graph and in bipolar soft graphs.

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REFERENCES